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On the limiting distribution of a generalized divisor problem for the case $-1 \le a < -1/2$

by

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1. Introduction. Let $\sigma_a(n) = \sum_{d|n} d^a$ and define

$$\Delta_a(t) = \sum_{n\le t} \sigma_a(n) - \zeta(1-a)t - \frac{\zeta(1+a)}{1+a} t^{1+a} + \frac{1}{2}\zeta(-a).$$

We are concerned with the case $-1 \le a < -1/2$. The case $a = -1$ is defined by taking limit. It should be noted that the definition in this case ($-1 \le a < -1/2$) is slightly different from the case $-1/2 < a < 0$ in [6]. The difference is that the last term is not halved even if x is an integer. It will not have any influence on our results.

Unlike the case $-1/2 < a < 0$, our discussion is not based on the Voronoi-type formula. Such a formula also exists in the case $-1 \le a < -1/2$. A truncated form with an explicit error term was obtained by Meurman [7] with a delicate method. However, by means of the Voronoi-type formula, one can only prove

$$\int_1^T \Delta_a(t)^2\,dt = O(T) \qquad (-1 < a < -1/2),$$

which is superseded by an old result of Chowla [2] who proved that

$$\int_1^T \Delta_a(t)^2\,dt = \frac{1}{12} \cdot \frac{\zeta(-2a)\zeta^2(1-a)}{\zeta(2-2a)} T + O(T^{3/2+a}\log T). \tag{1.1}$$

This gives us evidence that an initial section of the Voronoi-type formula cannot provide a good approximation to $\Delta_a(t)$ when $-1 \le a < -1/2$. Moreover, it seems that [4, Theorem 5] and [1, Theorem 4.1] cannot yield results on its limiting distribution.

As was shown in [5], if we define

$$D_{a,T}(u) = T^{-1}\mu\{t \in [1,T] : \Delta_a(t) \le u\},$$

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then the limiting distribution $D_a(u) = \lim_{T\to\infty} D_{a,T}(u)$ exists for $-1 \le a < -1/2$. However we do not have any further information about its properties. In the following, we shall show that $D_a(u)$ is continuous and symmetric (i.e. $1 - D_a(u) = D_a(-u)$). In addition, we shall discuss its rate of convergence. To study the rate of convergence, we adopt the argument in [5] (i.e. based on the Berry–Esseen Theorem), and so we have to know the modulus of continuity of $D_a(u)$.

When $-1/2 < a < 0$, Meurman [7] proved the following mean square formula with a "sharp" remainder term:

$$\int_2^T \Delta_a(t)^2\,dt = c_2 T^{3/2+a} + O(T)$$

where $c_2 = (6+4a)^{-1}\pi^{-2}\zeta(3/2-a)\zeta(3/2+a)\zeta(3/2)^2\zeta(3)^{-1}$. In view of (1.1), one expects that the behaviour of $\Delta_a(t)$ is very different in these two regions. The property of $D_a(u)$ being symmetric (when $-1 \le a < -1/2$) supports the change in behaviour of $\Delta_a(t)$ because it is known that $D_0(u)$ is not symmetric (see Heath-Brown [4]). In fact, we expect that $D_a(u)$ is also non-symmetric for $-1/2 < a < 0$ and we already know it is partly true. Let us state our results. Bear in mind that the value of a in the following theorems lies in $[-1, -1/2)$.

THEOREM 1. *For any $0 < \varepsilon < 1/4$ and any $y \in \mathbb{R}$, we have $D_a(y+\varepsilon) - D_a(y) \ll_a \sqrt{\varepsilon}$ uniformly in y. In particular, $D_a(u)$ is a continuous function.*

THEOREM 2. *We have*

$$D_{a,T}(u) - D_a(u) \ll_a \left(\frac{\log T}{\log\log T}\right)^{(1+2a)/6}$$

where the implied constant depends only on a.

THEOREM 3. *D_a is symmetric (i.e. $1 - D_a(u) = D_a(-u)$).*

2. Proof of Theorem 1. Let $y \in \mathbb{R}$ be fixed and define

$$p(\alpha) = \begin{cases} 2 - |y+\varepsilon-\alpha|/\varepsilon, & y-\varepsilon < \alpha < y+3\varepsilon,\\ 0, & \text{otherwise.}\end{cases}$$

Denoting the characteristic function of the interval $(y, y+\varepsilon]$ by $\chi_{(y,y+\varepsilon]}(u)$, we see that

$$(2.1)\qquad D_a(y+\varepsilon) - D_a(y) = \int_{-\infty}^{\infty} \chi_{(y,y+\varepsilon]}(u)\,dD_a(u) \le \int_{-\infty}^{\infty} p(u)\,dD_a(u) = \lim_{T\to\infty}\int_{-\infty}^{\infty} p(u)\,dD_{a,T}(u),$$

since $D_{a,T}$ converges weakly to D_a and p is continuous; moreover, we have

$$(2.2)\quad \int_1^T p(\Delta_a(u))\,du = \int_1^T \int_{-\infty}^{\infty} \widehat{p}(\alpha)e(-\alpha\Delta_a(u))\,d\alpha\,du = \int_1^T \int_{-\infty}^{\infty} \frac{\sin^2(2\pi\alpha)}{\pi^2\alpha^2} e\biggl(\frac{\alpha}{\varepsilon}(-\Delta_a(u)+y+\varepsilon)\biggr)\,d\alpha\,du.$$

Define $S_T = \{u \in [1,T] : |y+\varepsilon-\Delta_a(u)| \le \sqrt{\varepsilon}\}$. Since $\Delta_a(u)-\Delta_a(v) = -\zeta(1-a)(u-v)+O(m^a)$ for $m \le v \le u < m+1$, we have $\mu(S_T) \ll \sqrt{\varepsilon}\,T$ where the implied constant is independent of y but depends on a. If we set $S_T^{\mathrm{c}} = [1,T] \setminus S_T$, integration by parts yields that

$$\int_1^T p(\Delta_a(u))\,du = \varepsilon \int_{S_T^{\mathrm{c}}} \biggl\{ \frac{e\bigl(\frac{\alpha}{\varepsilon}(-\Delta_a(u)+y+\varepsilon)\bigr)}{-\Delta_a(u)+y+\varepsilon} \cdot \frac{\sin^2(2\pi\alpha)}{\pi^2\alpha^2}\bigg|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{e\bigl(\frac{\alpha}{\varepsilon}(-\Delta_a(u)+y+\varepsilon)\bigr)}{-\Delta_a(u)+y+\varepsilon} \cdot \frac{d}{d\alpha}\biggl(\frac{\sin^2(2\pi\alpha)}{\pi^2\alpha^2}\biggr)\,d\alpha \biggr\}\,du + \int_{S_T} O(1)\,du \ll \sqrt{\varepsilon}\,T.$$

Since $\int_{-\infty}^{\infty} p(u)\,dD_{a,T}(u) = T^{-1}\int_1^T p(\Delta_a(u))\,du$, our assertion follows from (2.1) and (2.2).

3. Proof of Theorem 2. Let $\psi(u) = u-[u]-1/2$ where $[u]$ is the integral part of u. From Chowla [2, Lemma 15], we have

$$\Delta_a(t) = -\sum_{n\le\sqrt{t}} n^a\psi\biggl(\frac{t}{n}\biggr) - t^a \sum_{n\le\sqrt{t}} n^{|a|}\psi\biggl(\frac{t}{n}\biggr) + O(t^{a/2}).$$

(Chowla proved the case $-1<a<-1/2$ only but the case $a=-1$ can be proved in a similar way.) Following Chowla's argument in [2], we obtain the following result.

LEMMA 3.1. *Let $-1 \le a < -1/2$ and $1 \le N \le \sqrt{T}$. We have*

$$\int_T^{2T} \biggl|\Delta_a(t) + \sum_{n\le N} n^a\psi\biggl(\frac{t}{n}\biggr)\biggr|^2 dt \ll_a TN^{1+2a} + T^{3/2+a}\log T$$

where the implied constant depends on a.

Proof. We firstly note that

(a) the Fourier series $-\pi^{-1}\sum l^{-1}\sin(2\pi lu)$ is square integrable on any bounded interval and converges to $\psi(u)$ in L^2-norm;

(b) from [2, Lemma 7],

$$\sum_{a\le b\le x}\sum_{\substack{m,n=1\\ mb\ne na}}^{\infty}\frac{1}{mn|mb-na|}\ll x\log x; \tag{3.1}$$

(c) from [2, Lemma 8],

$$\sum_{a\le b\le x}\sum_{m,n=1}^{\infty}\frac{1}{mn(mb+na)}\ll x. \tag{3.2}$$

Then we split the integral into three parts:

$$\begin{aligned}\int_T^{2T}\Big|\Delta_a(t)+\sum_{n\le N}n^a\psi\Big(\frac{t}{n}\Big)\Big|^2\,dt \tag{3.3}\\ \ll\int_T^{2T}\Big|\sum_{N<n\le\sqrt{t}}n^a\psi\Big(\frac{t}{n}\Big)\Big|^2\,dt+\int_T^{2T}t^{2a}\Big|\sum_{n\le\sqrt{t}}n^{|a|}\psi\Big(\frac{t}{n}\Big)\Big|^2\,dt+T^{1+a}.\end{aligned}$$

The second integral on the right hand side of (3.3) is

$$\begin{aligned}&\ll T^{2a}\int_T^{2T}\Big|\sum_{n\le\sqrt{t}}n^{|a|}\psi\Big(\frac{t}{n}\Big)\Big|^2\,dt \tag{3.4}\\
&=T^{2a}\sum_{m,n\le\sqrt{2T}}(mn)^{|a|}\int_{\max(m^2,n^2,T)}^{2T}\psi\Big(\frac{t}{m}\Big)\psi\Big(\frac{t}{n}\Big)\,dt\\
&=\pi^{-2}T^{2a}\lim_{M\to\infty}\sum_{m,n\le\sqrt{2T}}(mn)^{|a|}\sum_{k,l=1}^{M}\frac{1}{kl}\\
&\quad\times\int_{\max(m^2,n^2,T)}^{2T}\sin\Big(\frac{2\pi k}{m}t\Big)\sin\Big(\frac{2\pi l}{n}t\Big)\,dt\\
&\ll T^{1+2a}\sum_{m,n\le\sqrt{2T}}(mn)^{|a|}\sum_{kn=lm}\frac{1}{kl}\\
&\quad+O\Big(T^{2a}\sum_{m,n\le\sqrt{2T}}(mn)^{|a|}\sum_{kn\ne lm}\Big(kl\Big|\frac{k}{m}-\frac{l}{n}\Big|\Big)^{-1}\Big)\\
&\quad+O\Big(T^{2a}\sum_{m,n\le\sqrt{2T}}(mn)^{|a|}\sum_{k,l}\Big(kl\Big(\frac{k}{m}+\frac{l}{n}\Big)\Big)^{-1}\Big).\end{aligned}$$

The O-terms in (3.4) are $\ll T^{3/2+a}\log T$ by (3.1) and (3.2), while the first sum in (3.4) is

$$\begin{aligned}
&\ll T^{1+2a}\sum_{m,n\le\sqrt{2T}}(mn)^{|a|}\sum_{kn=lm}\frac{1}{kl}\\
&\ll T^{1+2a}\sum_{m,n\le\sqrt{2T}}(mn)^{|a|-1}(m,n)^2\\
&\ll T^{1+2a}\sum_{d\le\sqrt{2T}}d^{2|a|}\sum_{\substack{u,v\le\sqrt{2T}/d\\(u,v)=1}}(uv)^{|a|-1}\\
&\ll T^{3/2+a}
\end{aligned}$$

where (m,n) denotes the greatest common divisor of m and n. Hence, the last integral in (3.3) is absorbed by $O(T^{3/2+a}\log T)$. With the same argument, the first integral on the right hand side of (3.3) is equal to

$$(2\pi)^{-2}\sum_{N<n,m\le\sqrt{2T}}(mn)^a\sum_{kn=lm}\frac{1}{kl}\int\limits_{\max(n^2,m^2,T)}^{2T}dt+O(T^{3/2+a}\log T).$$

The sum here is

$$\begin{aligned}
&\ll T\sum_{N<n,m\le\sqrt{2T}}\frac{(m,n)^2}{(mn)^{1-a}}\\
&\ll T\sum_{d\le N}d^{2a}\sum_{N/d<u,v\le\sqrt{2T}/d}\frac{1}{(uv)^{1-a}}+T\sum_{N<d\le\sqrt{2T}}d^{2a}\sum_{u,v\le\sqrt{2T}/d}\frac{1}{(uv)^{1-a}}\\
&\ll TN^{1+2a}.
\end{aligned}$$

Lemma 3.1 then follows, with (3.3).

Now we prove Theorem 2. Let $\chi_{a,T}(\alpha)$ be the characteristic function of $D_{a,T}(u)$. Then $\chi_{a,T}(\alpha)=T^{-1}\int_1^T e(\alpha\Delta_a(t))\,dt$. Choose

(3.5) $$R=N^{-(1+2a)/3}\quad\text{and}\quad N=[\log T/(4\log\log T)].$$

By the Berry–Esseen Theorem and Theorem 1, we have

(3.6) $$\begin{aligned}
&|D_{a,T}(u)-D_a(u)|\\
&\ll R\int\limits_0^{1/R}(D_a(u+\alpha)-D_a(u-\alpha))\,d\alpha+\int\limits_{-R}^{R}\left|\frac{\chi_{a,T}(\alpha)-\chi_a(\alpha)}{\alpha}\right|d\alpha\\
&\ll\frac{1}{\sqrt{R}}+\int\limits_{-R}^{R}\left|\frac{\chi_{a,T}(\alpha)-\chi_a(\alpha)}{\alpha}\right|d\alpha
\end{aligned}$$

where $\chi_a(\alpha)$ is the characteristic function of $D_a(u)$. Define

$$\chi_{N,T}(\alpha) = \frac{1}{T}\int_1^T e\Big(-\alpha\sum_{n\le N} n^a\psi\Big(\frac{t}{n}\Big)\Big)\,dt, \tag{3.7}$$
$$\chi_N(\alpha) = \lim_{T\to\infty}\frac{1}{T}\int_1^T e\Big(-\alpha\sum_{n\le N} n^a\psi\Big(\frac{t}{n}\Big)\Big)\,dt.$$

Note that the limit exists. Then the last integral in (3.6) is

$$\begin{aligned}(3.8)\qquad &\le \int_{-R}^{R}|\chi_{a,T}(\alpha)-\chi_{N,T}(\alpha)|\,\frac{d\alpha}{|\alpha|} + \int_{-R}^{R}|\chi_{N,T}(\alpha)-\chi_N(\alpha)|\,\frac{d\alpha}{|\alpha|}\\ &\quad + \int_{-R}^{R}|\chi_N(\alpha)-\chi_a(\alpha)|\,\frac{d\alpha}{|\alpha|}\\ &= I_1+I_2+I_3,\ \text{say.}\end{aligned}$$

From (3.7), Lemma 3.1 and the fact that $e(u)-1 \ll \min(1,|u|)$, we have (recall that $\chi_{a,T}(\alpha) = T^{-1}\int_1^T e(\alpha\Delta_a(t))\,dt$)

$$\begin{aligned}(3.9)\qquad I_1 &\ll \frac{R}{T}\int_1^T \Big|\Delta_a(t)+\sum_{n\le N} n^a\psi\Big(\frac{t}{n}\Big)\Big|\,dt\\ &\ll R(N^{1/2+a}+T^{(1+2a)/4}\sqrt{\log T}) \ll RN^{1/2+a}\end{aligned}$$

by (3.5), and so

$$I_3 \ll RN^{1/2+a}. \tag{3.10}$$

To evaluate I_2, we first note that from the periodicity of $\psi(u)$,

$$\chi_N(\alpha) = \frac{1}{N!}\int_1^{N!+1} e\Big(-\alpha\sum_{n\le N} n^a\psi\Big(\frac{t}{n}\Big)\Big)\,dt.$$

Write $A=\log^2 T$ and $T = N!q+r$ $(0\le r<N!)$, and split the integral I_2 into two parts,

$$I_2 = \int_{|\alpha|\le 1/A} + \int_{1/A<|\alpha|\le R}. \tag{3.11}$$

Using $e(u)-1\ll|u|$ shows that the first part $\int_{|\alpha|\le 1/A}$ is

$$\begin{aligned}(3.12)\qquad &\ll \int_{|\alpha|\le 1/A}\Big(\frac{1}{T}\int_1^T\Big|\sum_{n\le N} n^a\psi\Big(\frac{t}{n}\Big)\Big|\,dt + \frac{1}{N!}\int_1^{N!+1}\Big|\sum_{n\le N} n^a\psi\Big(\frac{t}{n}\Big)\Big|\,dt\Big)\,d\alpha\\ &\ll A^{-1}\end{aligned}$$

since $T^{-1}\int_1^T |\sum_{n\le N} n^a\psi(t/n)|^2\,dt \ll 1$ (see the proof of Lemma 3.1). The integrand in the second part $\int_{1/A<|\alpha|\le R}$ can be expressed as

$$\frac{q}{T}\int_1^{N!+1} e\biggl(-\alpha\sum_{n\le N} n^a\psi\Bigl(\frac{t}{n}\Bigr)\biggr)\,dt - \frac{1}{N!}\int_1^{N!+1} e\biggl(-\alpha\sum_{n\le N} n^a\psi\Bigl(\frac{t}{n}\Bigr)\biggr)\,dt$$
$$+\frac{1}{T}\int_1^{r} e\biggl(-\alpha\sum_{n\le N} n^a\psi\Bigl(\frac{t}{n}\Bigr)\biggr)\,dt \ll N!T^{-1}.$$

Again, we have used the fact that $\sum_{n\le N} n^a\psi(t/n)$ is periodic and its period divides $N!$. Hence $\int_{1/A<|\alpha|\le R} \ll (\log\log T + \log R)N!T^{-1}$. Together with (3.12) and (3.11), we get $I_2 \ll (\log T)^{-2} + (\log\log T + \log R)N!T^{-1}$. Putting this estimate, (3.10) and (3.9) into (3.8) and then (3.6), we get

$$D_{a,T}(u) - D_a(u) \ll R^{-1/2} + RN^{1/2+a} + (\log T)^{-2} + (\log\log T + \log R)N!T^{-1}.$$

Theorem 2 follows with the choice (3.5) and Stirling's formula.

4. Proof of Theorem 3. It is known that a distribution function is symmetric if and only if its characteristic function is a real-valued function. (One direction follows from the definition and the other can be seen by the inversion formula, see [3, Lemma 1.10].)

Let $\chi_N(\alpha)$ be defined as in (3.7). Then

$$\chi_a(\alpha) = \lim_{N\to\infty}\chi_N(\alpha).$$

It suffices to show $\chi_N(\alpha)$ is real-valued. Since $\psi(u)$ is periodic and $\psi(-u) = -\psi(u)$ for $u\in(0,1)$, we have

$$\begin{aligned}\chi_N(\alpha) &= \frac{1}{N!}\int_0^{N!} e\biggl(-\alpha\sum_{n\le N} n^a\psi\Bigl(\frac{t}{n}\Bigr)\biggr)\,dt\\ &= \frac{1}{N!}\int_{-N!/2}^{N!/2} e\biggl(-\alpha\sum_{n\le N} n^a\psi\Bigl(\frac{t}{n}\Bigr)\biggr)\,dt\\ &= \frac{2}{N!}\int_0^{N!/2} \cos\biggl(2\pi\alpha\sum_{n\le N} n^a\psi\Bigl(\frac{t}{n}\Bigr)\biggr)\,dt.\end{aligned}$$

This completes the proof.

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