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On value-relations, functional relations and singularities of Mordell–Tornheim and related triple zeta-functions

by

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1. Introduction. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the ring of rational integers, $\mathbb{Q}$ the field of rational numbers, $\mathbb{R}$ the field of real numbers and $\mathbb{C}$ the field of complex numbers.

The Mordell–Tornheim r-ple zeta-function

$$(1.1)\quad \zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1}) = \sum_{m_1,\ldots,m_r=1}^{\infty} \frac{1}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_r)^{s_{r+1}}}$$

was defined by the first-named author (see [5, 8]). It can be meromorphically continued to the whole space $\mathbb{C}^{r+1}$, and possible singularities of (1.1) can be explicitly determined (see [8, Theorem 1]).

In the 1950's, Tornheim considered the double series $\zeta_{MT,2}(p,q;r)$ $(p,q,r \in \mathbb{N})$, and gave some fascinating formulas (see [16]). A little later, Mordell independently studied the values $\zeta_{MT,2}(k,k;k)$ $(k\in\mathbb{N})$, and showed that $\zeta_{MT,2}(2p,2p;2p)$ can be written as $M_p\cdot\pi^{6p}$ for some constant $M_p \in \mathbb{Q}$ $(p\in\mathbb{N})$ (see [11]).

About 30 years later, Subbarao and Sitaramachandrarao gave an evaluation formula for $\zeta_{MT,2}(2p,2p;2p)$ ([14]). Then Zagier [22] proved the following simple formula:

$$(1.2)\quad \zeta_{MT,2}(2p,2p;2p) = \frac{4}{3}\sum_{j=0}^{p}\binom{4p-2j-1}{2p-1}\zeta(2j)\zeta(6p-2j) \qquad (p\in\mathbb{N}),$$

which is much simpler than the Subbarao–Sitaramachandrarao formula. As an analogue of (1.2), Huard, Williams and Zhang [3] gave an evaluation

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formula for $\zeta_{MT,2}(2p+1,2p+1;2p+1)$ ($p\in\mathbb{N}_0$):

$$\zeta_{MT,2}(2p+1,2p+1;2p+1) = -4\sum_{j=0}^{p}\binom{4p-2j+1}{2p}\zeta(2j)\zeta(6p-2j+3). \tag{1.3}$$

Recently, as interpolations of these formulas, the fourth-named author gave some functional relations for $\zeta_{MT,2}(s_1,s_2;s_3)$ (see [21, Theorem 4.5]). More recently the second-named author proved functional relations for $\zeta_{MT,2}(s_1,s_2;s_3)$ by a different method ([12, Theorem 1.2]). His relations are

$$\begin{aligned}\zeta_{MT,2}(a,b;s)+(-1)^b\zeta_{MT,2}(b,s;a)+(-1)^a\zeta_{MT,2}(s,a;b) &= \frac{2}{a!b!}\sum_{k=0}^{\max([a/2],[b/2])}\left\{a\binom{b}{2k}+b\binom{a}{2k}\right\}\\ &\quad\times(a+b-2k-1)!(2k)!\zeta(2k)\zeta(a+b+s-2k)\end{aligned} \tag{1.4}$$

for $a,b\in\mathbb{N}$, where $[x]$ is the integer part of x. These have simpler forms than those in [21]. Note that (1.4) holds for all $s\in\mathbb{C}$ except the singularities of both sides.

Furthermore, triple and more general multiple zeta values of the Mordell–Tornheim type have been studied. Actually Mordell considered the multiple series

$$\sum_{m_1,\dots,m_r=1}^{\infty}\frac{1}{m_1\cdots m_r(m_1+\cdots+m_r+a)}$$

for $a>-r$, which can be regarded as a prototype of (1.1). Based on his work, Hoffman studied $\zeta_{MT,r}(1,1,\dots,1;k)$ for $k\in\mathbb{N}$ and gave some relations between these values and the Euler–Zagier type of multiple zeta values (see [2, Section 4]). Markett independently expressed $\zeta_{MT,3}(1,1,1;k)$ ($k\in\mathbb{N}$) as a polynomial in the values of $\zeta(s)$ at positive integers with $\mathbb{Q}$-coefficients (see [4]). Recently the fourth-named author proved a certain property of the values of $\zeta_{MT,r}$ ([20, Theorem 1.1]), which is called the "parity result" (for details, see Remark 4.8 in Section 4).

In the present paper, we mainly study the Mordell–Tornheim double and triple zeta-functions. In Section 2, we prove a key lemma (Lemma 2.1) for the study of double and triple series. As applications, we confirm that the functional relations for $\zeta_{MT,2}$ given by the fourth-named author coincide with (1.4) (see Proposition 2.2), and consider some alternating double series. In Section 3, we give some relation formulas for the values of $\zeta_{MT,3}$ which can be regarded as triple analogues of (1.2) and (1.3) (see Theorem 3.1). In Section 4, we give some functional relations among triple zeta-functions, double zeta-functions and the Riemann zeta-function, which can be regarded as triple analogues of (1.4) (see Theorem 4.5). In Sections 5 and 6, we discuss analytic properties of triple zeta-functions appearing in Section 4. Actually, in Section 6, we study more general $\zeta_{MT,r}$ and determine their true singularities (see Theorem 6.1).

2. The key lemma and its applications. Let $\zeta(s)$ be the Riemann zeta-function and

$$\phi(s)=\sum_{m=1}^{\infty}\frac{(-1)^m}{m^s}=(2^{1-s}-1)\zeta(s). \tag{2.1}$$

We recall that

$$\sum_{m=1}^{\infty}\frac{(-1)^m\cos(m\theta)}{m^{2k}}=\sum_{\nu=0}^{k}\phi(2k-2\nu)\,\frac{(-1)^{\nu}\theta^{2\nu}}{(2\nu)!}, \tag{2.2}$$

$$\sum_{m=1}^{\infty}\frac{(-1)^m\sin(m\theta)}{m^{2l+1}}=\sum_{\nu=0}^{l}\phi(2l-2\nu)\,\frac{(-1)^{\nu}\theta^{2\nu+1}}{(2\nu+1)!}, \tag{2.3}$$

for $k\in\mathbb{N}$, $l\in\mathbb{N}_0$ and $\theta\in(-\pi,\pi)\subset\mathbb{R}$ (see, for example, [17, Lemma 2]). Note that $\phi(0)=\zeta(0)=-1/2$. Since both sides of (2.2) and of (2.3) are continuous for $\theta\in[-\pi,\pi]$ and $k,l\in\mathbb{N}$, we can let $\theta\to\pi$ in (2.2) and (2.3), to obtain, $\cos(n\pi)=(-1)^n$ and $\sin(n\pi)=0$ $(n\in\mathbb{Z})$,

$$\zeta(2k)=\sum_{\nu=0}^{k}\phi(2k-2\nu)\,\frac{(-1)^{\nu}\pi^{2\nu}}{(2\nu)!}, \tag{2.4}$$

$$0=\sum_{\nu=0}^{l}\phi(2l-2\nu)\,\frac{(-1)^{\nu}\pi^{2\nu+1}}{(2\nu+1)!}, \tag{2.5}$$

for $k,l\in\mathbb{N}$.

Now we prove the following lemma which is a key to considering rearrangements of sums appearing in relation formulas for double and triple zeta values.

Lemma 2.1. *For arbitrary functions $f,g:\mathbb{N}_0\to\mathbb{C}$ and $a\in\mathbb{N}$, we have*

$$\sum_{\substack{j=0\\ j\equiv a\,(\mathrm{mod}\,2)}}^{a}\phi(a-j)\sum_{\mu=0}^{[j/2]}f(j-2\mu)\,\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu)!}=\sum_{\varrho=0}^{[a/2]}\zeta(2\varrho)f(a-2\varrho), \tag{2.6}$$

$$\sum_{\substack{j=0\\ j\equiv a\,(\mathrm{mod}\,2)}}^{a}\phi(a-j)\sum_{\mu=0}^{[(j-1)/2]}g(j-2\mu)\,\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu+1)!}=-\frac{1}{2}\,g(a). \tag{2.7}$$

Proof. On the left-hand side of (2.6), we change the running indices j and μ to ϱ and ν defined by $j=a+2\mu-2\varrho$ and $\mu=\nu$ $(\le \varrho)$. Since $0\le j\le a$ and $0\le\mu\le[j/2]$, we have $\varrho=\mu+(a-j)/2\ge\mu=\nu$ and $0\le 2\varrho\le a$, so $0\le\varrho\le[a/2]$. Hence, as $a-j=2\varrho-2\nu$ and $j-2\mu=a-2\varrho$, we see that the left-hand side of (2.6) is

$$\sum_{\varrho=1}^{[a/2]}\sum_{\nu=0}^{\varrho}\phi(2\varrho-2\nu)f(a-2\varrho)\,\frac{(-1)^\nu\pi^{2\nu}}{(2\nu)!}+\phi(0)f(a).$$

From (2.4) and $\phi(0)=\zeta(0)=-1/2$, this is equal to the right-hand side of (2.6). Similarly, changing the running indices j and μ to ϱ and ν defined by $j=a+2\mu-2\varrho$ and $\nu=\mu\le\varrho$ on the left-hand side of (2.7), and using (2.5), we can see that (2.7) holds.

As a direct application of this lemma, we obtain the following proposition which implies that (1.4) essentially coincides with the result of [21].

PROPOSITION 2.2. *For* $a,b\in\mathbb{N}$,

$$\begin{aligned}(2.8)\qquad &\zeta_{MT,2}(a,b;s)+(-1)^a\zeta_{MT,2}(a,s;b)+(-1)^b\zeta_{MT,2}(b,s;a)\\ &\qquad=2\sum_{\varrho=0}^{\max([a/2],[b/2])}\left\{\binom{a+b-2\varrho-1}{a-1}+\binom{a+b-2\varrho-1}{b-1}\right\}\\ &\qquad\qquad\times\zeta(2\varrho)\zeta(s+a+b-2\varrho)\end{aligned}$$

for all $s\in\mathbb{C}$ *except the singularities of both sides.*

Proof. The fourth-named author gave the functional relations

$$\begin{aligned}(2.9)\qquad &\zeta_{MT,2}(a,b;s)+(-1)^a\zeta_{MT,2}(a,s;b)+(-1)^b\zeta_{MT,2}(b,s;a)\\ &=2\sum_{\substack{j=0\\ j\equiv a\,(\mathrm{mod}\,2)}}^{a}\phi(a-j)\sum_{\mu=0}^{[j/2]}\frac{(i\pi)^{2\mu}}{(2\mu)!}\binom{b-1+j-2\mu}{j-2\mu}\zeta(b+j+s-2\mu)\\ &\quad-4\sum_{\substack{j=0\\ j\equiv a\,(\mathrm{mod}\,2)}}^{a}\phi(a-j)\sum_{\mu=0}^{[(j-1)/2]}\frac{(i\pi)^{2\mu}}{(2\mu+1)!}\sum_{\substack{\nu=0\\ \nu\equiv b\,(\mathrm{mod}\,2)}}^{b}\zeta(b-\nu)\\ &\quad\times\binom{\nu-1+j-2\mu}{j-2\mu-1}\zeta(\nu+j+s-2\mu)\end{aligned}$$

for $a,b\in\mathbb{N}$ and $s\in\mathbb{C}$ (see [21, Theorem 4.5]). Applying (2.6) and (2.7) to (2.9) with

$$f(X)=\binom{b-1+X}{X}\zeta(b+s+X),$$

$$g(X)=\sum_{\substack{\nu=0\\ \nu\equiv b\,(\mathrm{mod}\,2)}}^{b}\zeta(b-\nu)\binom{\nu-1+X}{X-1}\zeta(\nu+s+X),$$

we have

$$\begin{aligned}\zeta_{MT,2}(a,b;s)&+(-1)^a\zeta_{MT,2}(a,s;b)+(-1)^b\zeta_{MT,2}(b,s;a)\\ &=2\sum_{\varrho=0}^{[a/2]}\zeta(2\varrho)\binom{a+b-2\varrho-1}{a-2\varrho}\zeta(s+a+b-2\varrho)\\ &\quad+2\sum_{\substack{\nu=0\\ \nu\equiv b\,(\mathrm{mod}\,2)}}^{b}\zeta(b-\nu)\binom{\nu-1+a}{a-1}\zeta(s+\nu+a).\end{aligned}$$

By putting $\nu=b-2\varrho$ in the second sum, we obtain (2.8).

We can easily check that

$$\frac{2}{m!n!}\,m\binom{n}{2k}(m+n-2k-1)!(2k)!=2\binom{m+n-2k-1}{m-1}\tag{2.10}$$

for $k,m,n\in\mathbb{N}_0$. Substituting (2.10) in the cases $(m,n)=(a,b)$ and (b,a) into (1.4), we obtain (2.8). Thus we showed that (1.4), (2.8) and (2.9) are all equivalent.

Remark 2.3. Putting $a=b=2p$ and $2p+1$ in (2.8), we can obtain (1.2) and (1.3). Hence (2.8) can be regarded as a continuous interpolation of both (1.2) and (1.3).

Lemma 2.1 is also useful for the study of

$$\phi_2(s_1,s_2;s_3)=\sum_{m,n=1}^{\infty}\frac{(-1)^{m+n}}{m^{s_1}n^{s_2}(m+n)^{s_3}},\tag{2.11}$$

$$\psi_2(s_1,s_2;s_3)=\sum_{m,n=1}^{\infty}\frac{(-1)^{n}}{m^{s_1}n^{s_2}(m+n)^{s_3}},\tag{2.12}$$

for $s_1,s_2,s_3\in\mathbb{C}$. The fourth-named author proved that

$$\begin{aligned}\phi_2(2p+1,2p+1;2p+1)&=2\sum_{\varrho=0}^{p}\binom{4p+1-2\varrho}{2p}\phi(2\varrho)\phi(6p+3-2\varrho)\\ &\quad-4\sum_{j=0}^{p}\phi(2j)\Bigg\{\sum_{\nu=0}^{p}\phi(2p-2\nu)\sum_{\mu=0}^{\nu}\binom{2p+1-2j+2\nu-2\mu}{2\nu-2\mu}\\ &\qquad\times\zeta(4p+3-2j+2\nu-2\mu)\,\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu+1)!}\Bigg\}\end{aligned}\tag{2.13}$$

for $p \in \mathbb{N}_0$ (see [19, Theorem 3.4]). Applying (2.7) to (2.13) with $a = 2p+1$ and

$$g(t) = \binom{2p-2j+t}{t-1}\zeta(4p+2-2j+t),$$

we have

$$\begin{aligned}\phi_2(2p+1, 2p+1; 2p+1) = 2\sum_{\varrho=0}^{p}\binom{4p+1-2\varrho}{2p}\phi(2\varrho)\phi(6p+3-2\varrho)\\ + 2\sum_{j=0}^{p}\phi(2j)\binom{4p+1-2j}{2p}\zeta(6p+3-2j).\end{aligned}$$

Since $\phi(s) = (2^{1-s}-1)\zeta(s)$, we can rewrite (2.13) as follows.

PROPOSITION 2.4. *For* $p \in \mathbb{N}_0$,

$$\begin{aligned}(2.14)\qquad &\phi_2(2p+1, 2p+1; 2p+1)\\ &= 2^{-6p}\sum_{j=0}^{p}\binom{4p+1-2j}{2p}(2^{2j-1}-1)\zeta(2j)\zeta(6p+3-2j).\end{aligned}$$

On the other hand, from [13, Theorem 3.1] and (2.10), we have

PROPOSITION 2.5. *For* $p \in \mathbb{N}_0$,

$$\begin{aligned}(2.15)\qquad &\phi_2(2p+1, 2p+1; 2p+1) - 2\psi_2(2p+1, 2p+1; 2p+1)\\ &= 4\sum_{j=0}^{p}\binom{4p+1-2j}{2p}(2^{2j-2-6p}-1)\zeta(2j)\zeta(6p+3-2j).\end{aligned}$$

Hence, combining (2.14) and (2.15), we have

PROPOSITION 2.6. *For* $p \in \mathbb{N}_0$,

$$\begin{aligned}(2.16)\qquad &\psi_2(2p+1, 2p+1; 2p+1)\\ &= 2^{-6p-1}\sum_{j=0}^{p}\binom{4p+1-2j}{2p}(2^{6p+2}-2^{2j-1}-1)\zeta(2j)\zeta(6p+3-2j).\end{aligned}$$

EXAMPLE 2.7. By (2.16), for example, we obtain

$$\begin{aligned}\psi_2(5,5;5) &= -\frac{2064195}{16384}\zeta(15) + \frac{573335}{8192}\zeta(2)\zeta(13) + \frac{81875}{8192}\zeta(4)\zeta(11),\\ \psi_2(7,7;7) &= -\frac{899676921}{524288}\zeta(21) + \frac{242220363}{262144}\zeta(2)\zeta(19)\\ &\quad + \frac{22019907}{131072}\zeta(4)\zeta(17) + \frac{7339801}{524288}\zeta(6)\zeta(15).\end{aligned}$$

These formulas correct those in [18, Example 3.7].

3. Relation formulas for triple zeta values. In this section, we prove relation formulas for $\zeta_{MT,3}(k,k,k;k)$ for $k\in\mathbb{N}$, which are the triple analogues of (1.2) and (1.3). The method of proof is similar to that in [17, 20]. Combining that method with Lemma 2.1, we can obtain the following simple expressions like (1.2) and (1.3).

THEOREM 3.1. *For* $p\in\mathbb{N}$,

$$\zeta_{MT,3}(2p,2p,2p;2p)=4\sum_{\nu=1}^{p}\binom{2\nu+2p-1}{2p-1}\zeta(2p-2\nu)\times\{\zeta_{MT,2}(2p,2p;2p+2\nu)-\zeta_{MT,2}(2p+2\nu,2p;2p)\}-\zeta(4p)^2, \tag{3.1}$$

and for $p\in\mathbb{N}_0$,

$$\zeta_{MT,3}(2p+1,2p+1,2p+1;2p+1)=-4\sum_{\nu=0}^{p}\binom{2\nu+2p-1}{2p}\zeta(2p-2\nu)\times\{\zeta_{MT,2}(2p+1,2p+1;2p+2\nu+2)+\zeta_{MT,2}(2p+2\nu+2,2p+1;2p+1)\}+\zeta(4p+2)^2. \tag{3.2}$$

EXAMPLE 3.2. From Theorem 3.1, for example, we can obtain

$$\begin{aligned}
\zeta_{MT,3}(1,1,1;1)&=\frac{12}{5}\zeta(2)^2=\frac{1}{15}\pi^4,\\
\zeta_{MT,3}(2,2,2;2)&=6\{\zeta_{MT,2}(4,2;2)-\zeta_{MT,2}(2,2;4)\}-\zeta(4)^2,\\
\zeta_{MT,3}(3,3,3;3)&=-12\zeta(2)\{\zeta_{MT,2}(3,3;4)+\zeta_{MT,2}(4,3;3)\}\\
&\quad+20\{\zeta_{MT,2}(3,3;6)+\zeta_{MT,2}(6,3;3)\}+\zeta(6)^2.
\end{aligned}$$

Using (2.8), we can rewrite

$$\zeta_{MT,3}(2,2,2;2)=\frac{1}{11340}\pi^8-9\zeta_{MT,2}(2,2;4).$$

Note that it has not been proven yet that $\zeta_{MT,2}(2,2;4)$ can be expressed by means of the values of $\zeta(s)$.

Now we give some preparations for the proof of Theorem 3.1. Fix any $p\in\mathbb{N}$. By (2.2), we have

$$\begin{aligned}
&\sum_{l,m=1}^{\infty}\frac{(-1)^{l+m}\cos((l+m)\theta)}{l^{2p}m^{2p}}\\
&\times\Big\{\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\theta)}{n^{2p}}-\sum_{j=0}^{p}\phi(2p-2j)\,\frac{(-1)^j\theta^{2j}}{(2j)!}\Big\}\\
&-\sum_{l=1}^{\infty}\frac{(-1)^l\sin(l\theta)}{l^{2p}}\Big\{\sum_{m=1}^{\infty}\frac{(-1)^m\cos(m\theta)}{m^{2p}}-\sum_{j=0}^{p}\phi(2p-2j)\,\frac{(-1)^j\theta^{2j}}{(2j)!}\Big\}
\end{aligned} \tag{3.3}$$

$$\times \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\theta)}{n^{2p}}$$

$$- \left\{ \sum_{l=1}^{\infty} \frac{(-1)^l \cos(l\theta)}{l^{2p}} - \sum_{j=0}^{p} \phi(2p-2j) \frac{(-1)^j \theta^{2j}}{(2j)!} \right\}$$

$$\times \sum_{m=1}^{\infty} \frac{(-1)^m \sin(m\theta)}{m^{2p}} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\theta)}{n^{2p}} = 0$$

for $\theta \in (-\pi, \pi)$. Using the addition formulas for $\sin x$ and $\cos x$, we can rewrite (3.3) as

$$(3.4) \quad \sum_{l,m,n=1}^{\infty} \frac{(-1)^{l+m+n} \cos((l+m+n)\theta)}{l^{2p} m^{2p} n^{2p}} + \sum_{j=0}^{p} \phi(2p-2j) \frac{(-1)^j \theta^{2j}}{(2j)!}$$

$$\times \left\{ \sum_{l,m=1}^{\infty} \frac{(-1)^{l+m} \cos((l-m)\theta)}{l^{2p} m^{2p}} - 2 \sum_{l,m=1}^{\infty} \frac{(-1)^{l+m} \cos((l+m)\theta)}{l^{2p} m^{2p}} \right\}$$

$$= \sum_{l,m,n=1}^{\infty} \frac{(-1)^{l+m+n} \cos((l+m+n)\theta)}{l^{2p} m^{2p} n^{2p}} + \sum_{j=0}^{p} \phi(2p-2j) \frac{(-1)^j \theta^{2j}}{(2j)!}$$

$$\times \left\{ \sum_{\substack{l,m=1 \\ l \neq m}}^{\infty} \frac{(-1)^{l+m} \cos((l-m)\theta)}{l^{2p} m^{2p}} - 2 \sum_{l,m=1}^{\infty} \frac{(-1)^{l+m} \cos((l+m)\theta)}{l^{2p} m^{2p}} \right\}$$

$$+ \zeta(4p) \sum_{l=1}^{\infty} \frac{(-1)^l \cos(l\theta)}{l^{2p}} = 0$$

for $\theta \in (-\pi, \pi)$, using (2.2) again. This implies, by integrating both sides by parts repeatedly, that

$$(3.5) \quad \sum_{l,m,n=1}^{\infty} \frac{(-1)^{l+m+n} \sin((l+m+n)\theta)}{l^{2p} m^{2p} n^{2p} (l+m+n)^{2d+1}}$$

$$+ \sum_{j=0}^{p} \phi(2p-2j) \sum_{\nu=0}^{2j} \binom{2d+2j-\nu}{2j-\nu} \frac{(-1)^\nu \theta^\nu}{\nu!}$$

$$\times \left\{ \sum_{\substack{l,m=1 \\ l \neq m}}^{\infty} \frac{(-1)^{l+m} \sin^{(\nu)}((l-m)\theta)}{l^{2p} m^{2p} (l-m)^{2d+2j+1-\nu}} - 2 \sum_{l,m=1}^{\infty} \frac{(-1)^{l+m} \sin^{(\nu)}((l+m)\theta)}{l^{2p} m^{2p} (l+m)^{2d+2j+1-\nu}} \right\}$$

$$+ \zeta(4p) \sum_{l=1}^{\infty} \frac{(-1)^l \sin(l\theta)}{l^{2p+2d+1}} = \sum_{\varrho=0}^{d} C_{2d-2\varrho}(2p) \frac{(-1)^\varrho \theta^{2\varrho+1}}{(2\varrho+1)!} \quad (\theta \in (-\pi, \pi))$$

for $d \in \mathbb{N}_0$, and

$$(3.6)\qquad \sum_{l,m,n=1}^{\infty} \frac{(-1)^{l+m+n}\cos((l+m+n)\theta)}{l^{2p}m^{2p}n^{2p}(l+m+n)^{2e}}$$
$$+\sum_{j=0}^{p}\phi(2p-2j)\sum_{\nu=0}^{2j}\binom{2e-1+2j-\nu}{2j-\nu}\frac{(-1)^{\nu}\theta^{\nu}}{\nu!}$$
$$\times\Biggl\{\sum_{\substack{l,m=1\\ l\neq m}}^{\infty}\frac{(-1)^{l+m}\cos^{(\nu)}((l-m)\theta)}{l^{2p}m^{2p}(l-m)^{2e+2j-\nu}}-2\sum_{l,m=1}^{\infty}\frac{(-1)^{l+m}\cos^{(\nu)}((l+m)\theta)}{l^{2p}m^{2p}(l+m)^{2e+2j-\nu}}\Biggr\}$$
$$+\zeta(4p)\sum_{l=1}^{\infty}\frac{(-1)^{l}\cos(l\theta)}{l^{2p+2e}}=\sum_{\varrho=0}^{e}C_{2e-2\varrho}(2p)\,\frac{(-1)^{\varrho}\theta^{2\varrho}}{(2\varrho)!}\qquad(\theta\in(-\pi,\pi))$$

for $e \in \mathbb{N}_0$, where $\{C_{2\nu}(2p) \mid \nu \in \mathbb{N}_0\}$ are constants which are determined inductively, $f^{(\nu)}(x)$ denotes the νth derivative of $f(x)$ and $f^{(\nu)}(\alpha) := f^{(\nu)}(x)\big|_{x=\alpha}$ for $f(x) = \sin x, \cos x$. Note that the left-hand side of (3.5) (resp. (3.6)) is an odd (resp. even) function, hence each coefficient of $\theta^{2\varrho}$ (resp. $\theta^{2\varrho+1}$) on the right-hand side of (3.5) (resp. (3.6)) is equal to 0.

Since both sides of (3.5) and of (3.6) are continuous for $\theta \in [-\pi,\pi]$, (3.5) and (3.6) hold for $\theta = \pi$. Note that, by putting $h = l-m$ (resp. $k = m-l$) if $l > m$ (resp. $l < m$), we have, for example,

$$(3.7)\qquad \sum_{\substack{l,m=1\\ l\neq m}}^{\infty}\frac{1}{l^{2p}m^{2p}(l-m)^{2d+2j-2\mu}}$$
$$=\sum_{h,m=1}^{\infty}\frac{1}{h^{2d+2j-2\mu}m^{2p}(h+m)^{2p}}+\sum_{k,l=1}^{\infty}\frac{1}{k^{2d+2j-2\mu}l^{2p}(k+l)^{2p}}$$
$$=2\zeta_{MT,2}(2d+2j-2\mu,2p;2p).$$

Hence, letting $\theta \to \pi$ on both sides of (3.5) and of (3.6), we have

$$(3.8)\qquad \sum_{j=0}^{p}\phi(2p-2j)\sum_{\mu=0}^{j-1}\binom{2d+2j-2\mu-1}{2j-2\mu-1}\frac{(-1)^{\mu}\pi^{2\mu+1}}{(2\mu+1)!}$$
$$\times\{\zeta_{MT,2}(2d+2j-2\mu,2p;2p)-\zeta_{MT,2}(2p,2p;2d+2j-2\mu)\}$$
$$=\sum_{\mu=0}^{d}C_{2d-2\mu}(2p)\,\frac{(-1)^{\mu}\pi^{2\mu+1}}{(2\mu+1)!}$$

and

$$\begin{aligned}(3.9)\qquad & \zeta_{MT,3}(2p,2p,2p;2e) \\ & + 2\sum_{j=0}^{p}\phi(2p-2j)\sum_{\mu=0}^{j}\binom{2e+2j-2\mu-1}{2j-2\mu}\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu)!} \\ & \times\{\zeta_{MT,2}(2e+2j-2\mu,2p;2p)-\zeta_{MT,2}(2p,2p;2e+2j-2\mu)\} \\ & + \zeta(4p)\zeta(2p+2e)=\sum_{\mu=0}^{e}C_{2e-2\mu}(2p)\,\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu)!}.\end{aligned}$$

Applying Lemma 2.1 to (3.8) and (3.9) with $a=2p$ and

$$\begin{aligned}f(x)&=\binom{2e+x-1}{x}\{\zeta_{MT,2}(2e+x,2p;2p)-\zeta_{MT,2}(2p,2p;2e+x)\},\\ g(x)&=\binom{2d+x-1}{x-1}\{\zeta_{MT,2}(2d+x,2p;2p)-\zeta_{MT,2}(2p,2p;2d+x)\},\end{aligned}$$

we can rewrite (3.8) and (3.9) as

$$\begin{aligned}(3.10)\qquad & \binom{2d+2p-1}{2p-1}\{\zeta_{MT,2}(2d+2p,2p;2p)-\zeta_{MT,2}(2p,2p;2d+2p)\} \\ & =\sum_{\mu=0}^{d}C_{2d-2\mu}(2p)\,\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu+1)!}\end{aligned}$$

and

$$\begin{aligned}(3.11)\qquad & \zeta_{MT,3}(2p,2p,2p;2e)+2\sum_{\xi=0}^{p}\zeta(2\xi)\binom{2e+2p-2\xi-1}{2p-2\xi} \\ & \times\{\zeta_{MT,2}(2e+2p-2\xi,2p;2p)-\zeta_{MT,2}(2p,2p;2e+2p-2\xi)\} \\ & + \zeta(4p)\zeta(2p+2e)=\sum_{\mu=0}^{e}C_{2e-2\mu}(2p)\,\frac{(-1)^{\mu}\pi^{2\mu}}{(2\mu)!}\end{aligned}$$

for $d,e\in\mathbb{N}_0$.

Now we recall the following.

Lemma 3.3 ([21, Lemma 4.4]). *Let $\{\alpha_{2d}\}_{d\in\mathbb{N}_0}$, $\{\beta_{2d}\}_{d\in\mathbb{N}_0}$, $\{\gamma_{2d}\}_{d\in\mathbb{N}_0}$ be sequences such that*

$$\alpha_{2d}=\sum_{j=0}^{d}\gamma_{2d-2j}\,\frac{(-1)^{j}\pi^{2j}}{(2j)!},\qquad \beta_{2d}=\sum_{j=0}^{d}\gamma_{2d-2j}\,\frac{(-1)^{j}\pi^{2j}}{(2j+1)!}$$

for any $d\in\mathbb{N}_0$. Then

$$\alpha_{2d}=-2\sum_{\nu=0}^{d}\beta_{2\nu}\zeta(2d-2\nu)$$

for any $d\in\mathbb{N}_0$.

Proof of Theorem 3.1. Applying Lemma 3.3 to (3.10) and (3.11) with $d=e$, and putting $\nu=p-\xi$ in (3.11), we have

$$\begin{aligned}(3.12)\qquad &\zeta_{MT,3}(2p,2p,2p;2d)\\ &+2\sum_{\nu=0}^{p}\zeta(2p-2\nu)\binom{2d+2\nu-1}{2\nu}\\ &\times\{\zeta_{MT,2}(2d+2\nu,2p;2p)-\zeta_{MT,2}(2p,2p;2\nu+2p)\}+\zeta(4p)\zeta(2p+2d)\\ &=4\sum_{\nu=0}^{d}\zeta(2d-2\nu)\binom{2\nu+2p-1}{2p-1}\\ &\times\{\zeta_{MT,2}(2\nu+2p,2p;2p)-\zeta_{MT,2}(2p,2p;2\nu+2p)\}.\end{aligned}$$

In particular when $d=p$, we obtain (3.1).

Recall that we deduced (3.3) from (2.2). Similarly, we now deduce the following relation from (2.3):

$$\begin{aligned}(3.13)\qquad &\sum_{l,m=1}^{\infty}\frac{(-1)^{l+m}\cos((l+m)\theta)}{l^{2p+1}m^{2p+1}}\\ &\times\left\{\sum_{n=1}^{\infty}\frac{(-1)^n\sin(n\theta)}{n^{2p+1}}-\sum_{j=0}^{p}\phi(2p-2j)\,\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\right\}\\ &+\sum_{l=1}^{\infty}\frac{(-1)^l\cos(l\theta)}{l^{2p+1}}\left\{\sum_{m=1}^{\infty}\frac{(-1)^m\sin(m\theta)}{m^{2p+1}}-\sum_{j=0}^{p}\phi(2p-2j)\,\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\right\}\\ &\times\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\theta)}{n^{2p+1}}\\ &-\left\{\sum_{l=1}^{\infty}\frac{(-1)^l\sin(l\theta)}{l^{2p+1}}-\sum_{j=0}^{p}\phi(2p-2j)\,\frac{(-1)^j\theta^{2j+1}}{(2j+1)!}\right\}\\ &\times\sum_{m=1}^{\infty}\frac{(-1)^m\cos(m\theta)}{m^{2p+1}}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\theta)}{n^{2p+1}}=0\end{aligned}$$

for $\theta\in(-\pi,\pi)$. Then, by the same argument as mentioned above, we can prove (3.2). This completes the proof of Theorem 3.1.

4. Functional relations for triple zeta-functions. The aim of this section is to give some functional relations for triple zeta-functions. These can be regarded as triple analogues of (2.8). To this end, we consider analytic properties of $\zeta_{MT,3}(s_1,s_2,s_3;s_4)$ and

$$G(s_1,s_2,s_3,s_4):=\sum_{\substack{k,l,m,n=1\\k+l=m+n}}^{\infty}\frac{1}{k^{s_1}l^{s_2}m^{s_3}n^{s_4}}. \tag{4.1}$$

First we state the following two theorems. Their proofs will be given in the following sections. In fact, we first give the proof of Theorem 4.2 in the next section. Next we generalize Theorem 4.1 to a result on $\zeta_{MT,r}$ for any $r\geq 3$ (see Theorem 6.1) and give the proof of this generalized result in Section 6.

THEOREM 4.1. *$\zeta_{MT,3}(s_1,s_2,s_3;s_4)$ can be continued meromorphically to $\mathbb{C}^4$, and the singularities lie on the subsets of $\mathbb{C}^4$ defined by one of the following equations*:

$$s_j+s_4=1-l \qquad (1\leq j\leq 3;\ l\in\mathbb{N}_0), \tag{4.2}$$

$$s_j+s_k+s_4=2-l \qquad (1\leq j<k\leq 3;\ l\in\mathbb{N}_0), \tag{4.3}$$

$$s_1+s_2+s_3+s_4=3, \tag{4.4}$$

all points of which are true singularities.

THEOREM 4.2. *$G(s_1,s_2,s_3,s_4)$ can be continued meromorphically to $\mathbb{C}^4$, and the singularities lie on the subsets of $\mathbb{C}^4$ defined by one of the following equations*:

$$s_j+s_k=1-l \qquad (j=1,2;\ k=3,4;\ l\in\mathbb{N}_0), \tag{4.5}$$

$$s_h+s_j+s_k=2-l \qquad (1\leq h<j<k\leq 4;\ l\in\mathbb{N}_0), \tag{4.6}$$

$$s_1+s_2+s_3+s_4=3, \tag{4.7}$$

all points of which are true singularities.

Based on these results, we give some functional relations for triple zeta-functions mentioned above. In the rest of this section, we use the same notation as in [12] and generalize Proposition 2.2 to the case of triple zeta-functions. The method used in this section can be regarded as a triple analogue of that in [12].

We denote by $B_j(x)$ the jth Bernoulli polynomial defined by

$$\frac{te^{xt}}{e^t-1}=\sum_{j=0}^{\infty}B_j(x)\,\frac{t^j}{j!} \qquad (|t|<2\pi).$$

It is known (see [1, p. 266, (22) and p. 267, (24)]) that

$$B_{2j}:=B_{2j}(0)=(-1)^{j+1}2(2j)!(2\pi)^{-2j}\zeta(2j) \qquad (j\in\mathbb{N}), \tag{4.8}$$

$$B_j(x-[x])=-\frac{j!}{(2\pi i)^j}\lim_{K\to\infty}\sum_{\substack{k=-K\\k\neq 0}}^{K}\frac{e^{2\pi ikx}}{k^j} \qquad (j\in\mathbb{N}). \tag{4.9}$$

Hence, for $k \in \mathbb{Z}$, $j \in \mathbb{N}$ we have

$$(4.10)\qquad \int_0^1 e^{-2\pi ikx} B_j(x)\,dx = \begin{cases} 0 & (k=0),\\ -(2\pi ik)^{-j} j! & (k\neq 0).\end{cases}$$

It follows from [1, p. 276, 19.(b)] that for $p+q\geq 2$,

$$(4.11)\qquad B_p(x)B_q(x)$$
$$= \sum_{k=0}^{\max([p/2],[q/2])} \left\{ p\binom{q}{2k} + q\binom{p}{2k}\right\} \frac{B_{2k}B_{p+q-2k}(x)}{p+q-2k} - (-1)^p \frac{p!q!}{(p+q)!} B_{p+q}.$$

Using these facts, we obtain the following theorems.

Theorem 4.3. *For $a, b \in \mathbb{N}$,*

$$(4.12)\qquad (-1)^b \zeta_{MT,3}(b, s_3, s_4; a) + (-1)^a \zeta_{MT,3}(s_3, s_4, a; b) + G(a, b, s_3, s_4)$$
$$= \frac{2}{a!b!} \sum_{k=0}^{\max([a/2],[b/2])} \left\{ a\binom{b}{2k} + b\binom{a}{2k}\right\} (a+b-2k-1)!(2k)!$$
$$\times \zeta(2k)\zeta_{MT,2}(s_3, s_4; a+b-2k)$$

for all $s_3, s_4 \in \mathbb{C}$ except the singularities of both sides.

Proof. For $\Re(s_3) > 1$, $\Re(s_4) > 1$, we have

$$\lim_{K\to\infty} \int_0^1 \sum_{k=1}^{K} \frac{e^{2\pi ikx}}{k^a} \sum_{l=1}^{K} \frac{e^{2\pi ilx}}{l^b} \sum_{m=1}^{K} \frac{e^{2\pi imx}}{m^{s_3}} \sum_{n=1}^{K} \frac{e^{2\pi inx}}{n^{s_4}}\,dx = 0,$$

$$\lim_{K\to\infty} \int_0^1 \sum_{k=-K}^{-1} \frac{e^{2\pi ikx}}{k^a} \sum_{l=1}^{K} \frac{e^{2\pi ilx}}{l^b} \sum_{m=1}^{K} \frac{e^{2\pi imx}}{m^{s_3}} \sum_{n=1}^{K} \frac{e^{2\pi inx}}{n^{s_4}}\,dx$$
$$= (-1)^a \zeta_{MT,3}(b, s_3, s_4; a),$$

$$\lim_{K\to\infty} \int_0^1 \sum_{k=1}^{K} \frac{e^{2\pi ikx}}{k^a} \sum_{l=-K}^{-1} \frac{e^{2\pi ilx}}{l^b} \sum_{m=1}^{K} \frac{e^{2\pi imx}}{m^{s_3}} \sum_{n=1}^{K} \frac{e^{2\pi inx}}{n^{s_4}}\,dx$$
$$= (-1)^b \zeta_{MT,3}(s_3, s_4, a; b),$$

$$\lim_{K\to\infty} \int_0^1 \sum_{k=-K}^{-1} \frac{e^{2\pi ikx}}{k^a} \sum_{l=-K}^{-1} \frac{e^{2\pi ilx}}{l^b} \sum_{m=1}^{K} \frac{e^{2\pi imx}}{m^{s_3}} \sum_{n=1}^{K} \frac{e^{2\pi inx}}{n^{s_4}}\,dx$$
$$= (-1)^{a+b} G(a, b, s_3, s_4).$$

Therefore

$$(-1)^a \zeta_{MT,3}(b, s_3, s_4; a) + (-1)^b \zeta_{MT,3}(s_3, s_4, a; b) + (-1)^{a+b} G(a, b, s_3, s_4)$$
$$= \int_0^1 \lim_{K\to\infty} \sum_{\substack{k=-K\\ k\neq 0}}^{K} \frac{e^{2\pi ikx}}{k^a} \sum_{\substack{l=-K\\ l\neq 0}}^{K} \frac{e^{2\pi ilx}}{l^b} \sum_{m=1}^{K} \frac{e^{2\pi imx}}{m^{s_3}} \sum_{n=1}^{K} \frac{e^{2\pi inx}}{n^{s_4}}\,dx.$$

Interchanging the limit and integral is justified by bounded convergence. In fact, we need to treat the case $a = 1$ or $b = 1$ carefully. For this case, we know that $\sum_{m=1}^{\infty} \sin(2\pi mx)/m$ is boundedly convergent for $x > 0$ (see [15, p. 15]).

By using (4.8)–(4.11), we obtain (4.12) in this region. By Theorems 4.1 and 4.2, we see that (4.12) holds for all $a, b \in \mathbb{N}$, and all $s_3, s_4 \in \mathbb{C}$ except the singularities of both sides of (4.12).

THEOREM 4.4. *For $a, b \in \mathbb{N}$,*

$$\begin{aligned}
(4.13)\qquad & \zeta_{MT,3}(s_4, a, b; s_3) + (-1)^a G(a, s_3, b, s_4) \\
&\qquad + (-1)^b G(s_3, b, a, s_4) + (-1)^{a+b} \zeta_{MT,3}(a, b, s_3; s_4) \\
&= \frac{2}{a!b!} \sum_{k=0}^{\max([a/2],[b/2])} \left\{ a\binom{b}{2k} + b\binom{a}{2k} \right\} (a+b-2k-1)!(2k)! \\
&\qquad \times \zeta(2k)\zeta_{MT,2}(a+b-2k, s_4; s_3) \\
&\quad + \frac{2(-1)^{a+b}}{a!b!} \sum_{k=0}^{\max([a/2],[b/2])} \left\{ a\binom{b}{2k} + b\binom{a}{2k} \right\} (a+b-2k-1)!(2k)! \\
&\qquad \times \zeta(2k)\zeta_{MT,2}(a+b-2k, s_3; s_4) \\
&\quad + (-1)^{a+1} \frac{(2\pi i)^{a+b} B_{a+b}}{(a+b)!} \zeta(s_3+s_4)
\end{aligned}$$

for all $s_3, s_4 \in \mathbb{C}$ except the singularities of both sides.

Proof. Assume $\Re(s_3) > 1$ and $\Re(s_4) > 1$. Then we have

$$\lim_{K\to\infty} \int_0^1 \sum_{k=1}^{K} \frac{e^{2\pi ikx}}{k^a} \sum_{l=1}^{K} \frac{e^{2\pi ilx}}{l^b} \sum_{m=1}^{K} \frac{e^{-2\pi imx}}{m^{s_3}} \sum_{n=1}^{K} \frac{e^{2\pi inx}}{n^{s_4}}\, dx = \zeta_{MT,3}(s_4, a, b; s_3),$$

$$\lim_{K\to\infty} \int_0^1 \sum_{k=-K}^{-1} \frac{e^{2\pi ikx}}{k^a} \sum_{l=1}^{K} \frac{e^{2\pi ilx}}{l^b} \sum_{m=1}^{K} \frac{e^{-2\pi imx}}{m^{s_3}} \sum_{n=1}^{K} \frac{e^{2\pi inx}}{n^{s_4}}\, dx = (-1)^a G(a, s_3, b, s_4),$$

$$\lim_{K\to\infty} \int_0^1 \sum_{k=1}^{K} \frac{e^{2\pi ikx}}{k^a} \sum_{l=-K}^{-1} \frac{e^{2\pi ilx}}{l^b} \sum_{m=1}^{K} \frac{e^{-2\pi imx}}{m^{s_3}} \sum_{n=1}^{K} \frac{e^{2\pi inx}}{n^{s_4}}\, dx = (-1)^b G(s_3, b, a, s_4),$$

$$\lim_{K\to\infty} \int_0^1 \sum_{k=-K}^{-1} \frac{e^{2\pi ikx}}{k^a} \sum_{l=-K}^{-1} \frac{e^{2\pi ilx}}{l^b} \sum_{m=1}^{K} \frac{e^{-2\pi imx}}{m^{s_3}} \sum_{n=1}^{K} \frac{e^{2\pi inx}}{n^{s_4}}\, dx = (-1)^{a+b} \zeta_{MT,3}(a, b, s_3; s_4).$$

Therefore we can prove Theorem 4.4 in the same way as Theorem 4.3.

We define $K_1(a, b, s_3, s_4)$ and $K_2(a, b, s_3, s_4)$ by the right-hand sides of (4.12) and (4.13) respectively. By the preceding theorems, we obtain the following theorem which essentially includes not only Theorem 3.1 but also the assertion in the triple case given in [20].

THEOREM 4.5. *For $a, b, c \in \mathbb{N}$,*

$$\begin{aligned}(4.14)\qquad & \zeta_{MT,3}(a,b,c;s) - (-1)^{b+c}\zeta_{MT,3}(b,c,s;a)\\ &\quad - (-1)^{c+a}\zeta_{MT,3}(c,s,a;b) - (-1)^{a+b}\zeta_{MT,3}(s,a,b;c)\\ &= (-1)^{a+b}K_2(a,b,c,s) - (-1)^{b}K_1(a,c,b,s) - (-1)^{a}K_1(c,b,a,s)\end{aligned}$$

for all $s \in \mathbb{C}$ except the singularities of both sides.

Proof. By (4.12), we have

$$G(a,b,c,s) = K_1(a,b,c,s) - (-1)^{b}\zeta_{MT,3}(b,c,s;a) - (-1)^{a}\zeta_{MT,3}(c,s,a;b).$$

By changing the order of variables, we have

$$\begin{aligned}G(a,c,b,s) &= K_1(a,c,b,s) - (-1)^{c}\zeta_{MT,3}(b,c,s;a) - (-1)^{a}\zeta_{MT,3}(s,a,b;c),\\ G(c,b,a,s) &= K_1(c,b,a,s) - (-1)^{b}\zeta_{MT,3}(s,a,b;c) - (-1)^{c}\zeta_{MT,3}(c,s,a;b).\end{aligned}$$

Substituting these relations into (4.13), we obtain (4.14).

We denote by $M(a,b,c,s)$ the right-hand side of (4.14). We prove the following explicit formulas for $\zeta_{MT,3}(a,b,c;d)$.

THEOREM 4.6. *For $a, b, c, d \in \mathbb{N}$ with $a+b+c+d \in 2\mathbb{N}$,*

$$\begin{aligned}(4.15)\qquad \zeta_{MT,3}(a,b,c;d) = \frac{1}{4}\{&M(a,b,c,d) - (-1)^{b+c}M(b,c,d,a)\\ &- (-1)^{a+c}M(c,d,a,b) - (-1)^{a+b}M(d,a,b,c)\}.\end{aligned}$$

Proof. By changing variables in (4.14), we have

$$\begin{aligned}M(b,c,d,a) &= \zeta_{MT,3}(b,c,d;a) - (-1)^{c+d}\zeta_{MT,3}(c,d,a;b)\\ &\quad - (-1)^{d+b}\zeta_{MT,3}(d,a,b;c) - (-1)^{b+c}\zeta_{MT,3}(a,b,c;d),\\ M(c,d,a,b) &= \zeta_{MT,3}(c,d,a;b) - (-1)^{d+a}\zeta_{MT,3}(d,a,b;c)\\ &\quad - (-1)^{a+c}\zeta_{MT,3}(a,b,c;d) - (-1)^{c+d}\zeta_{MT,3}(b,c,d;a),\\ M(d,a,b,c) &= \zeta_{MT,3}(d,a,b;c) - (-1)^{a+b}\zeta_{MT,3}(a,b,c;d)\\ &\quad - (-1)^{b+d}\zeta_{MT,3}(b,c,d;a) - (-1)^{d+a}\zeta_{MT,3}(c,d,a;b).\end{aligned}$$

Multiply the above three equations by $(-1)^{b+c}$, $(-1)^{a+c}$, $(-1)^{a+b}$, respectively, and sum them up. Then, by using (4.14) in the case $s = d$, we obtain (4.15).

EXAMPLE 4.7. Put $(a,b,c) = (1,1,1)$ in (4.14). Then we obtain

$$\begin{aligned}&\zeta_{MT,3}(1,1,1;s) - 3\zeta_{MT,3}(s,1,1;1) + 6\zeta_{MT,2}(1,2;s)\\ &\qquad + 6\zeta_{MT,2}(s,2;1) - 6\zeta(2)\zeta(s+1) + 12\zeta(s+3) = 0,\end{aligned}$$

which was essentially given by the first-named and fourth-named authors (see [10, Example 6.1]). Similarly, putting $(a,b,c)=(2,2,2)$ in (4.14), we obtain

$$\begin{aligned}\zeta_{MT,3}&(2,2,2;s)-3\zeta_{MT,3}(2,2,s;2)\\ &=6\{2\zeta_{MT,2}(2,s;4)-\zeta_{MT,2}(4,s;2)-\zeta_{MT,2}(2,4;s)\}\\ &\quad+4\zeta(2)\{\zeta_{MT,2}(2,2;s)-\zeta_{MT,2}(2,s;2)\}+2\zeta(4)\zeta(s+2).\end{aligned}$$

In particular when $s=2$, we obtain the formula for $\zeta_{MT,3}(2,2,2;2)$ given in Example 3.2. Put $(a,b,c,d)=(1,1,1,3)$ in (4.15). Then we obtain

$$\zeta_{MT,3}(1,1,1;3)=-6\zeta(3)^2+\frac{23}{2520}\pi^6,$$

which can also be obtained from Hoffman's [2, Corollary 4.2] and Markett's [4, Corollary 4.3].

Remark 4.8. In [20], the fourth-named author proved, in a different way, that for $k_1,\dots,k_{r+1}\in\mathbb{N}$, $\zeta_{MT,r}(k_1,\dots,k_r;k_{r+1})$ can be expressed as a rational linear combination of products of values of $\zeta_{MT,j}$ $(j<r)$ at positive integers if r and $\sum_{j=1}^{r+1}k_j$ are of different parity. This fact is sometimes called the "parity result" for the Mordell–Tornheim zeta values. From this fact, we know that $\zeta_{MT,3}(a,b,c;d)$ $(a,b,c,d\in\mathbb{N})$ can be expressed as a rational linear combination of products of $\zeta_{MT,2}(p,q;r)$ and $\zeta(s)$ when $a+b+c+d\in 2\mathbb{N}$. Therefore we can interpret the results in Theorem 4.6 as concrete formulas which represent the parity result for $\zeta_{MT,3}$.

The method in this section can be applied to a more general situation, which will be discussed elsewhere.

5. Analytic properties of certain triple zeta-functions. In this section, we prove Theorem 4.2. We mainly use the method established by the first-named author in [5–8].

Let

$$H(s_1,s_2,s_3,s_4):=\sum_{k,l,m=1}^{\infty}\frac{1}{k^{s_1}m^{s_2}(k+l)^{s_3}(l+m)^{s_4}}. \tag{5.1}$$

By (4.1), we have

$$\begin{aligned}G(s_1,s_2,s_3,s_4)&=\sum_{N=1}^{\infty}\sum_{\substack{k,l=1\\k+l=N}}^{\infty}\frac{1}{k^{s_1}l^{s_2}}\sum_{\substack{m,n=1\\m+n=N}}^{\infty}\frac{1}{m^{s_3}n^{s_4}}\\ &=\sum_{k,m=1}^{\infty}\frac{1}{k^{s_1}m^{s_3}}\sum_{N>\max(k,m)}\frac{1}{(N-k)^{s_2}(N-m)^{s_4}}.\end{aligned} \tag{5.2}$$

We decompose the right-hand side of (5.2) as $\sum_{k<m} + \sum_{k=m} + \sum_{k>m}$. Then the first and third terms are equal to $H(s_1, s_4, s_3, s_2)$ and $H(s_3, s_2, s_1, s_4)$, respectively. The second term is $\zeta(s_1 + s_3)\zeta(s_2 + s_4)$. Hence

$$
\begin{aligned}
(5.3) \qquad G(s_1, s_2, s_3, s_4) &= \zeta(s_1 + s_3)\zeta(s_2 + s_4) \\
&\quad + H(s_1, s_4, s_3, s_2) + H(s_3, s_2, s_1, s_4).
\end{aligned}
$$

Therefore we need to consider $H(s_1, s_2, s_3, s_4)$. Actually, $H(s_1, s_2, s_3, s_4)$ is equal to $\zeta_{\mathfrak{sl}(4)}(s_1, 0, s_2, s_3, s_4, 0)$, where $\zeta_{\mathfrak{sl}(4)}(s_1, s_2, s_3, s_4, s_5, s_6)$ is the Witten zeta-function associated with $\mathfrak{sl}(4)$ (see [9]) defined by

$$
\begin{aligned}
(5.4) \qquad & \zeta_{\mathfrak{sl}(4)}(s_1, s_2, s_3, s_4, s_5, s_6) \\
&= \sum_{l,m,n=1}^{\infty} \frac{1}{l^{s_1} m^{s_2} n^{s_3} (l+m)^{s_4} (m+n)^{s_5} (l+m+n)^{s_6}}.
\end{aligned}
$$

First we prove the following lemma. Though it may be regarded as a special case of [9, Theorem 3.5], we can prove it more simply by considering a simple integral representation of $H(s_1, s_2, s_3, s_4)$ (see (5.15) below).

LEMMA 5.1. *The function $H(s_1, s_2, s_3, s_4)$ can be continued meromorphically to $\mathbb{C}^4$, and all of its singularities lie on the subsets of $\mathbb{C}^4$ defined by one of the equations*:

$$
\begin{aligned}
&(5.5) && s_1 + s_3 = 1 - l && (l \in \mathbb{N}_0), \\
&(5.6) && s_2 + s_4 = 1 - l && (l \in \mathbb{N}_0), \\
&(5.7) && s_3 + s_4 = 1 - l && (l \in \mathbb{N}_0), \\
&(5.8) && s_1 + s_3 + s_4 = 2 - l && (l \in \mathbb{N}_0), \\
&(5.9) && s_2 + s_3 + s_4 = 2 - l && (l \in \mathbb{N}_0), \\
&(5.10) && s_1 + s_2 + s_3 + s_4 = 3, &&
\end{aligned}
$$

all points of which are true singularities.

Proof. We use the same notation as in the proof of [8, Theorem 1]. We recall the Mellin–Barnes formula

$$
(5.11) \qquad (1+\lambda)^{-s} = \frac{1}{2\pi i} \int_{(c)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)} \lambda^z \, dz,
$$

where $\Re s > 0$, $|\arg \lambda| < \pi$, $\lambda \neq 0$, $c \in \mathbb{R}$ with $-\Re s < c < 0$, $i = \sqrt{-1}$ and the path (c) of integration is the vertical line $\Re z = c$.

Assume $s_j \in \mathbb{C}$ with $\Re s_j > 1$ ($j = 1, 2, 3, 4$). Then $H(s_1, s_2, s_3, s_4)$ is absolutely convergent. Let $(k + l)^{-s_3} = l^{-s_3}(1 + k/l)^{-s_3}$ in (5.1), and substitute (5.11) with $\lambda = k/l$ into (5.1). Assume $-\Re s_3 < c < 0$. Then

$$\begin{aligned}(5.12)\qquad & H(s_1,s_2,s_3,s_4)\\ &= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\sum_{k=1}^{\infty}\frac{1}{k^{s_1}}\sum_{l,m=1}^{\infty}\frac{1}{l^{s_3}m^{s_2}(l+m)^{s_4}}\left(\frac{k}{l}\right)^{z} dz\\ &= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\zeta(s_1-z)\zeta_{MT,2}(s_2,s_3+z;s_4)\,dz.\end{aligned}$$

Note that, by the assumptions $\Re s_j > 1$ $(1 \le j \le 4)$ and $-\Re s_3 < c < 0$, each series above is absolutely convergent. By [5, Theorem 1], the singularities of $\zeta_{MT,2}(s_1,s_2;s_3)$ lie on the subsets of $\mathbb{C}^3$ defined by one of the equations:

$$s_1+s_3 = 1-l, \qquad s_2+s_3 = 1-l \quad (l\in\mathbb{N}_0), \qquad s_1+s_2+s_3=2.$$

Hence, by considering the singularities of $\Gamma(s)$, $\zeta(s)$ and $\zeta_{MT,2}(s_1,s_2;s_3)$, we see that the singularities of the integrand of (5.12) are determined by $z=-s_3-l$, $z=l$, $z=s_1-1$, $s_2+s_4=1-l$, $z=1-s_3-s_4-l$ $(l\in\mathbb{N}_0)$ and $z=2-s_2-s_3-s_4$.

Now we shift the path $\Re z = c$ to $\Re z = M-\varepsilon$ for sufficiently large $M\in\mathbb{N}$ and sufficiently small positive $\varepsilon\in\mathbb{R}$. Then all the relevant singularities are $z=l$ $(0\le l\le M-1)$ and $z=s_1-1$. Counting their residues, and using the relations

$$(5.13)\qquad \frac{(-1)^l}{l!}\,\frac{\Gamma(s+l)}{\Gamma(s)} = (-1)^l\binom{s+l-1}{l} = \binom{-s}{l},$$

and

$$(5.14)\qquad \Gamma(-l-\delta) = \frac{\Gamma(1-\delta)}{(-\delta)\cdots(-l-\delta)} = -\frac{(-1)^l}{l!}\left(\frac{1}{\delta}+O(1)\right) \qquad (\delta\to 0)$$

for $l\in\mathbb{N}_0$, we find that

$$\begin{aligned}(5.15)\qquad H(s_1,s_2,s_3,s_4) &= \frac{\Gamma(s_1+s_3-1)\Gamma(1-s_1)}{\Gamma(s_3)}\zeta_{MT,2}(s_2,s_1+s_3-1;s_4)\\ &+\sum_{k=0}^{M-1}\binom{-s_3}{k}\zeta(s_1-k)\zeta_{MT,2}(s_2,s_3+k;s_4)\\ &+\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\zeta(s_1-z)\zeta_{MT,2}(s_2,s_3+z;s_4)\,dz,\end{aligned}$$

because $\mathrm{Res}_{z=s_1-1}\,\zeta(s_1-z) = -1$ and

$$\frac{\Gamma(s_3+k)}{\Gamma(s_3)}\operatorname*{Res}_{z=k}\Gamma(-z) = -\binom{-s_3}{k}.$$

Since M can be taken arbitrarily large, (5.15) implies the meromorphic continuation of $H(s_1,s_2,s_3,s_4)$ to $\mathbb{C}^4$.

Fix $(s_1, s_2, s_3, s_4) \in \mathbb{C}^4$. In the above list of singularities of the integrand of (5.12), only the family $s_2 + s_4 = 1 - l$ $(l \in \mathbb{N}_0)$ is independent of z. Therefore, by choosing M sufficiently large, we may assume that the integral term on the right-hand side of (5.15) is holomorphic, except on $s_2+s_4 = 1-l$ $(l \in \mathbb{N}_0)$, around the fixed (s_1, s_2, s_3, s_4). Also we see that the singularities of the first term on the right-hand side of (5.15) are determined by $s_1+s_3 = 1-l$, $s_1 = 1+l$, $s_2+s_4 = 1-l$, $s_1+s_3+s_4 = 2-l$, $s_1+s_2+s_3+s_4 = 3$ $(l \in \mathbb{N}_0)$, and those of the second term are determined by $s_1 = 1 + k$, $s_2 + s_4 = 1 - l$, $s_3 + s_4 = 1 - (k + l)$, $s_2 + s_3 + s_4 = 2 - k$ $(0 \le k \le M - 1;\ l \in \mathbb{N}_0)$. Using the symmetry

$$H(s_2, s_1, s_4, s_3) = H(s_1, s_2, s_3, s_4), \tag{5.16}$$

we see that $s_1 = 1+l$ is not a singularity of $H(s_1, s_2, s_3, s_4)$ because $s_2 = 1+l$ is not. On the other hand,

$$\zeta_{MT,2}(s_1, -l; s_2) \not\equiv 0 \quad (l \in \mathbb{N}_0), \tag{5.17}$$

because

$$\zeta_{MT,2}(l + 2, -l; l + 2) = \sum_{m,n=1}^{\infty} \frac{n^l}{m^{l+2}(m + n)^{l+2}} > 0, \tag{5.18}$$

which is absolutely convergent. Hence, from (5.17), we see that $s_1+s_3 = 1-l$ is not cancelled by the factor of $\zeta_{MT,2}$ in the first term on the right-hand side of (5.15). Hence (5.5) and (5.8)–(5.10) determine true singularities because these equations come from only one term on the right-hand side of (5.15). The singularities $s_3 + s_4 = 1 - l$ come from the terms corresponding to $0 \le k \le l$ in the sum part on the right-hand side of (5.15); but these are not cancelled, because the residues coming from different terms have different order with respect to s_3. Hence (5.7) also gives true singularities. Furthermore, combining (5.16) with the fact that (5.5) determines true singularities as mentioned above, we conclude that (5.6) also determines true singularities. This completes the proof of Lemma 5.1.

Remark 5.2. From [9, Theorem 3.5], the singularities of $H(s_1, s_2, s_3, s_4) = \zeta_{\mathfrak{sl}(4)}(s_1, 0, s_2, s_3, s_4, 0)$ are given by (5.5)–(5.10) and

$$s_1 + s_2 + s_3 + s_4 = 2 - l \quad (l \in \mathbb{N}_0), \tag{5.19}$$

though (5.19) does not appear in Lemma 5.1. In fact, we can check that (5.19) does not determine singularities of $H(s_1, s_2, s_3, s_4)$ as follows. The possible singularity (5.19) comes from [9, (3.43)], the singularities of $\zeta_{\mathfrak{sl}(4)}$, which come from

$$S_1 = \frac{\Gamma(s_3 + s_5 + s_6 + n - 1)\Gamma(1 - s_3 - s_5 - n)}{\Gamma(s_6)} \times \zeta_{MT,2}(s_1, s_2 - n; s_3 + s_4 + s_5 + s_6 + n - 1)$$

corresponding to $s_1+(s_3+s_4+s_5+s_6+n-1)=1-l$ ($l\in\mathbb{N}_0$). These are indeed true singularities of $\zeta_{\mathfrak{sl}(4)}(s_1,s_2,s_3,s_4,s_5,s_6)$. However, in the above argument, we consider the case $(s_2,s_6)=(0,0)$. Hence these singularities are cancelled by $\Gamma(s_6)$ as $s_6\to 0$. Thus (5.19) does not determine singularities of $H(s_1,s_2,s_3,s_4)$.

From Lemma 5.1, we now deduce Theorem 4.2, that is, determine the true singularities of $G(s_1,s_2,s_3,s_4)$.

Proof of Theorem 4.2. The meromorphic continuation of G comes from that of H and of $\zeta(s)$. From Lemma 5.1, the true singularities of $H(s_1,s_4,s_3,s_2)$ are determined by $s_1+s_3=1-l$, $s_2+s_4=1-l$, $s_2+s_3=1-l$, $s_1+s_2+s_3=2-l$, $s_2+s_3+s_4=2-l$ and $s_1+s_2+s_3+s_4=3$ ($l\in\mathbb{N}_0$), and those of $H(s_3,s_2,s_1,s_4)$ are determined by $s_1+s_3=1-l$, $s_2+s_4=1-l$, $s_1+s_4=1-l$, $s_1+s_3+s_4=2-l$, $s_1+s_2+s_4=2-l$ and $s_1+s_2+s_3+s_4=3$. Furthermore, those of $\zeta(s_1+s_3)\zeta(s_2+s_4)$ are determined by $s_1+s_3=1$ and $s_2+s_4=1$. Hence we only have to check that $s_1+s_3=1-l$, $s_2+s_4=1-l$ and $s_1+s_2+s_3+s_4=3$ determine true singularities.

Using the relation $G(s_1,s_2,s_3,s_4)=G(s_2,s_1,s_3,s_4)=G(s_1,s_2,s_4,s_3)$ and the fact that $s_1+s_4=1-l$ determines true singularities as mentioned above, we conclude that $s_1+s_3=1-l$ and $s_2+s_4=1-l$ also determine true singularities.

On the other hand, this kind of argument using symmetry is not enough to prove that $s_1+s_2+s_3+s_4=3$ determines true singularities. Hence we have to give a more detailed justification. From [5, (5.3)], we have

$$
\begin{aligned}
(5.20)\qquad & \zeta_{MT,2}(s_1,s_2;s_3)\\
&= \frac{\Gamma(s_2+s_3-1)\Gamma(1-s_2)}{\Gamma(s_3)}\zeta(s_1+s_2+s_3-1)\\
&\quad+\sum_{k=0}^{M-1}\binom{-s_3}{k}\zeta(s_1+s_3+k)\zeta(s_2-k)\\
&\quad+\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_3+z)\Gamma(-z)}{\Gamma(s_3)}\zeta(s_1+s_3+z)\zeta(s_2-z)\,dz.
\end{aligned}
$$

Therefore the singular part of $\zeta_{MT,2}(s_2,s_1+s_3-1;s_4)$ corresponding to $s_1+s_2+s_3+s_4=3$ comes from $\zeta(s_1+s_2+s_3+s_4-2)$. Hence the corresponding singular part of $H(s_1,s_4,s_3,s_2)$ is

$$
(5.21)\qquad \frac{\Gamma(s_1+s_3-1)\Gamma(1-s_1)}{\Gamma(s_3)}\,\frac{\Gamma(s_1+s_2+s_3-2)\Gamma(2-s_1-s_3)}{\Gamma(s_2)}\times\zeta(s_1+s_2+s_3+s_4-2).
$$

Similarly, the corresponding singular part of $H(s_3,s_2,s_1,s_4)$ is

$$\frac{\Gamma(s_1+s_3-1)\Gamma(1-s_3)}{\Gamma(s_1)}\,\frac{\Gamma(s_1+s_3+s_4-2)\Gamma(2-s_1-s_3)}{\Gamma(s_4)} \times\zeta(s_1+s_2+s_3+s_4-2). \tag{5.22}$$

Therefore the corresponding singular part of $G(s_1,s_2,s_3,s_4)$ is

$$\Gamma(s_1+s_3-1)\Gamma(2-s_1-s_3)\zeta(s_1+s_2+s_3+s_4-2) \times\left\{\frac{\Gamma(1-s_1)\Gamma(s_1+s_2+s_3-2)}{\Gamma(s_2)\Gamma(s_3)}+\frac{\Gamma(1-s_3)\Gamma(s_1+s_3+s_4-2)}{\Gamma(s_1)\Gamma(s_4)}\right\}. \tag{5.23}$$

If we substitute $s_4=3-s_1-s_2-s_3$ into the part in the curly parentheses in (5.23), we find that it is equal to

$$\frac{\Gamma(1-s_1)\Gamma(s_1+s_2+s_3-2)}{\Gamma(s_2)\Gamma(s_3)}+\frac{\Gamma(1-s_3)\Gamma(1-s_2)}{\Gamma(s_1)\Gamma(3-s_1-s_2-s_3)}.$$

We can check that this quantity is $\not\equiv 0$, by letting $\Im s_1\to\infty$, or by observing the value at (s_1,s_2,s_3) with $s_2=s_3=1/2$ and $s_1\to 1$. Thus $s_1+s_2+s_3+s_4=3$ determines true singularities. This completes the proof of Theorem 4.2. (We will give another expression for (5.23) in Remark 6.3.)

Remark 5.3. In the proof of Theorem 4.2, we concluded that $s_1+s_3=1-l$ ($l\in\mathbb{N}_0$) gives true singularities by the argument using symmetry of indices and the fact that $s_1+s_4=1-l$ determines a true singularity. On the other hand, we can prove this fact directly as follows.

Singularities determined by $s_1+s_3=1-l$ ($l\in\mathbb{N}_0$) come from $\Gamma(s_1+s_3-1)$ in the second term on the right-hand side of (5.12). Suppose $l\in\mathbb{N}$. Then the corresponding singular part of $G(s_1,s_2,s_3,s_4)$ is

$$\Gamma(s_1+s_3-1)\left\{\frac{\Gamma(1-s_1)}{\Gamma(s_3)}\,\zeta_{MT,2}(s_4,s_1+s_3-1;s_2) +\frac{\Gamma(1-s_3)}{\Gamma(s_1)}\,\zeta_{MT,2}(s_2,s_1+s_3-1;s_4)\right\}. \tag{5.24}$$

If we substitute $s_3=1-s_1-l$ into the part in the curly parentheses, we obtain

$$\begin{aligned}\frac{\Gamma(1-s_1)}{\Gamma(1-s_1-l)}\,\zeta_{MT,2}(s_4,-l;s_2)+\frac{\Gamma(s_1+l)}{\Gamma(s_1)}\,\zeta_{MT,2}(s_2,-l;s_4) &= s_1(s_1+1)\cdots(s_1+l-1)\\ &\quad\times\{\zeta_{MT,2}(s_4,-l;s_2)+(-1)^l\zeta_{MT,2}(s_2,-l;s_4)\}\not\equiv 0.\end{aligned} \tag{5.25}$$

In fact, if l is even, then by putting $s_2=s_4=l+2$ and using (5.18), we see that (5.25) holds. If l is odd, then by putting $s_2=l+2$ and $s_4=l+3$ we have

$$\zeta_{MT,2}(l+3,-l;l+2)-\zeta_{MT,2}(l+2,-l;l+3)=\zeta_{MT,2}(l+3,-l-1;l+3)>0,$$

hence (5.25) holds. This implies that $s_1+s_3=1-l$ ($l\in\mathbb{N}$) determine true singularities.

Suppose $l=0$. Then the corresponding singular part of $G(s_1,s_2,s_3,s_4)$ is (5.20) plus $\zeta(s_1+s_3)\zeta(s_2+s_4)$, which can be written as

$$\frac{1}{s_1+s_3-1}\left\{\zeta(s_2+s_4)+\frac{\Gamma(1-s_1)}{\Gamma(s_3)}\zeta_{MT,2}(s_4,s_1+s_3-1;s_2)\right.$$
$$\left.+\frac{\Gamma(1-s_3)}{\Gamma(s_1)}\zeta_{MT,2}(s_2,s_1+s_3-1;s_4)\right\}+O(1). \tag{5.26}$$

If we substitute $s_3=1-s_1$ into the part in the curly parentheses, we obtain

$$\zeta(s_2+s_4)+\zeta_{MT,2}(s_2,0;s_4)+\zeta_{MT,2}(s_4,0;s_2)=\zeta(s_2)\zeta(s_4)\not\equiv 0.$$

This implies that $s_1+s_3=1$ determines true singularities.

6. True singularities of $\zeta_{MT,r}$ and some remarks. In this section, we consider further applications of the method used in Section 5.

First we determine the true singularities of $\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})$. Actually, in [8, Theorem 1], the first-named author showed that $\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})$ can be continued meromorphically to $\mathbb{C}^{r+1}$ and gave the list ((6.1) below) of the *possible* singularities. By combining this method with our present method, we can determine the true singularities of $\zeta_{MT,r}$ as follows. Note that the case $r=3$ of this theorem coincides with Theorem 4.1.

THEOREM 6.1. *The function $\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})$ can be continued meromorphically to $\mathbb{C}^{r+1}$ and its singularities lie on the subsets of $\mathbb{C}^{r+1}$ given by one of the following equations*:

$$\begin{cases}(s_j-1)+s_{r+1}=-l \quad (1\le j\le r,\ l\in\mathbb{N}_0),\\ (s_{j_1}-1)+(s_{j_2}-1)+s_{r+1}=-l \quad (1\le j_1<j_2\le r,\ l\in\mathbb{N}_0),\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \displaystyle\sum_{\nu=1}^{r-1}(s_{j_\nu}-1)+s_{r+1}=-l \quad (1\le j_1<\cdots<j_{r-1}\le r, l\in\mathbb{N}_0),\\ s_1+\cdots+s_{r+1}=r,\end{cases} \tag{6.1}$$

all points of which are true singularities.

Proof. We will prove this theorem by induction on $r\ge 1$.

In the case $r=1$, we see that $\zeta_{MT,1}(s_1;s_2)=\zeta(s_1+s_2)$. Hence only $s_1+s_2=1$ determines singularities of $\zeta_{MT,1}(s_1;s_2)$. Thus we have the assertion. Actually the case $r=2$ has also been proved in [5, Theorem 1].

Assume that the assertion holds for $r-1$ ($r>1$), and consider the case of r. From [8, (3.2)], we have

$$\begin{aligned}
(6.2)\qquad &\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})\\
&\quad=\frac{\Gamma(s_r+s_{r+1}-1)\Gamma(1-s_r)}{\Gamma(s_{r+1})}\,\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_r+s_{r+1}-1)\\
&\quad+\sum_{k=0}^{M-1}\binom{-s_{r+1}}{k}\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_{r+1}+k)\zeta(s_r-k)\\
&+\frac{1}{2\pi i}\int_{(M-\varepsilon)}\frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\,\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_{r+1}+z)\zeta(s_r-z)\,dz,
\end{aligned}$$

where M $(\in\mathbb{N})$ is sufficiently large and ε $(\in\mathbb{R}_+)$ is sufficiently small. Since M can be taken arbitrarily large, (6.2) implies the meromorphic continuation of $\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})$ to $\mathbb{C}^{r+1}$, by the assumption of induction.

Now, by induction, we take M so large that the right-hand side of (6.2) is holomorphic on a certain neighbourhood of $(s_1,\ldots,s_{r+1})$. Then, by the assumption again, the singularities of the first term on the right-hand side of (6.2) are determined by

$$(6.3)\qquad \sum_{j\in J}(s_j-1)+(s_r+s_{r+1}-1)=-l\quad(l\in\mathbb{N}_0),$$

$$(6.4)\qquad \sum_{j=1}^{r-1}(s_j-1)+(s_r+s_{r+1}-1)=0,$$

$$(6.5)\qquad s_r+s_{r+1}=1-l\quad(l\in\mathbb{N}_0),$$

$$(6.6)\qquad s_r=1+l\quad(l\in\mathbb{N}_0),$$

where J $(\neq\emptyset)$ runs over all proper subsets of $\{1,\ldots,r-1\}$. Similarly, the singularities of the second term on the right-hand side of (6.2) are determined by

$$(6.7)\qquad \sum_{j\in J}(s_j-1)+(s_{r+1}+k)=-l\quad(k,l\in\mathbb{N}_0),$$

$$(6.8)\qquad \sum_{j=1}^{r-1}(s_j-1)+(s_{r+1}+k)=0\quad(k\in\mathbb{N}_0),$$

$$(6.9)\qquad s_r-k=1\quad(k\in\mathbb{N}_0)$$

for J as above.

First we claim that (6.6), that is, (6.9), is not a singularity of $\zeta_{MT,r}$. In fact, since $s_1=1+l$ $(l\in\mathbb{N}_0)$ is not singular because $r>1$, we see from the symmetry of indices

$$\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})=\zeta_{MT,r}(s_r,s_1,\ldots,s_{r-1};s_{r+1})$$

that $s_r=1+l$ $(l\in\mathbb{N}_0)$ is not singular either. Note that this fact has already been obtained from [8, (3.3)], by checking the cancellation directly. However, the above argument is much simpler than that in [8].

Next we claim that (6.3)–(6.5) determine true singularities of $\zeta_{MT,r}$. In fact, these come from only the first term on the right-hand side of (6.2). Furthermore, (6.3) and (6.4) are not cancelled by the Gamma factors, hence determine true singularities. On the other hand, we need to check whether (6.5) is cancelled by the factor of $\zeta_{MT,r-1}$ or not. For this, we claim that

$$\zeta_{MT,r}(s_1,\dots,s_r;-l)\not\equiv 0 \tag{6.10}$$

for $l\in\mathbb{N}_0$. Actually, this comes from

$$\begin{aligned}\zeta_{MT,r}(l+2,\dots,l+2;-l)&=\sum_{m_1,\dots,m_r=1}^{\infty}\frac{(m_1+\cdots+m_r)^l}{m_1^{l+2}\cdots m_r^{l+2}}\\&=\sum_{\substack{k_1,\dots,k_r\in\mathbb{N}_0\\k_1+\cdots+k_r=l}}\frac{l!}{k_1!\cdots k_r!}\zeta(l+2-k_1)\cdots\zeta(l+2-k_r)>0\end{aligned}$$

by an argument similar to the deduction of (5.17) from (5.18). From these facts, we see that (6.5) is not cancelled with the factor of $\zeta_{MT,r-1}$, hence (6.5) determines true singularities.

Lastly we consider (6.7) and (6.8). In fact, from the symmetry of indices and by the fact that (6.3) determines true singularities, we see that (6.7) and (6.8) also determine true singularities. Thus, from the above considerations, the assertion holds for r, completing the proof of Theorem 6.1.

Remark 6.2. Here we give an alternative proof of (6.10) by induction on $r\in\mathbb{N}$. In the case $r=1$, it is obvious. Hence we assume that (6.10) holds for $r-1$ $(r>1)$, and prove it for r. Put $s_r=-l$ in (6.2). Then the first and third terms on the right-hand side of (6.2) vanish because of the Gamma factor. Therefore

$$\begin{aligned}&\zeta_{MT,r}(s_1,\dots,s_r;-l)\\&\qquad=\sum_{k=0}^{M-1}\binom{l}{k}\zeta_{MT,r-1}(s_1,\dots,s_{r-1};k-l)\zeta(s_r-k).\end{aligned}\tag{6.11}$$

We see that as a set of meromorphic functions, $\{\zeta(s-k)\mid k\in\mathbb{N}_0\}$ is linearly independent over $\mathbb{C}$. In fact, we only have to consider each pole of $\zeta(s-k)$ $(k\in\mathbb{N}_0)$. From the assumption of induction, we have

$$\zeta_{MT,r-1}(s_1,\dots,s_{r-1};-l)\neq 0 \tag{6.12}$$

for some $(s_1,\dots,s_{r-1})\in\mathbb{C}^{r-1}$. If we regard (6.11) as a linear relation for functions of s_r, then (6.12) implies that the coefficient of $\zeta(s_r)$ does not vanish. Hence $\zeta_{MT,r}(s_1,\dots,s_r;-l)\not\equiv 0$, proving the assertion.

REMARK 6.3. Since (6.4) is not cancelled by the Gamma factor, we can prove that the singular part of $\zeta_{MT,r}(s_1,\ldots,s_r;s_{r+1})$ corresponding to (6.4) can be written as

$$\frac{\Gamma(1-s_1)\cdots\Gamma(1-s_r)}{(s_1+\cdots+s_{r+1}-r)\Gamma(s_{r+1})}+O(1) \tag{6.13}$$

as $s_1+\cdots+s_{r+1}\to r$, by induction on r. In fact, for $r=1$, (6.13) is

$$\frac{\Gamma(1-s_1)}{(s_1+s_2-1)\Gamma(s_2)}+O(1)=\frac{1}{s_1+s_2-1}+O(1), \tag{6.14}$$

which coincides with the singular part of $\zeta_{MT,1}(s_1;s_2)$ $(=\zeta(s_1+s_2))$ corresponding to s_1+s_2-1. Hence we have the assertion for $r=1$. Assume that the case of $r-1$ holds. Then the singular part of $\zeta_{MT,r-1}(s_1,\ldots,s_{r-1};s_r+s_{r+1}-1)$ corresponding to (6.4) is

$$\frac{\Gamma(1-s_1)\cdots\Gamma(1-s_{r-1})}{(s_1+\cdots+s_{r+1}-r)\Gamma(s_r+s_{r+1}-1)}+O(1). \tag{6.15}$$

By substituting (6.15) into (6.2), we immediately obtain the assertion in the case of r.

Applying (6.13) with $r=2$ to (5.15) shows that (5.21) can be written as

$$\begin{aligned}\frac{\Gamma(s_1+s_3-1)\Gamma(1-s_1)}{\Gamma(s_3)}&\frac{\Gamma(1-s_4)\Gamma(2-s_1-s_3)}{(s_1+s_2+s_3+s_4-3)\Gamma(s_2)}+O(1)\\ &=-\frac{\Gamma(1-s_1)\Gamma(1-s_2)\Gamma(1-s_3)\Gamma(1-s_4)}{s_1+s_2+s_3+s_4-3}\frac{\sin(\pi s_2)\sin(\pi s_3)}{\pi\sin(\pi(s_1+s_3))}+O(1),\end{aligned} \tag{6.16}$$

because $\Gamma(z)\Gamma(1-z)=\pi/\sin(\pi z)$. Similarly, (5.22) can be written as

$$-\frac{\Gamma(1-s_1)\Gamma(1-s_2)\Gamma(1-s_3)\Gamma(1-s_4)}{s_1+s_2+s_3+s_4-3}\frac{\sin(\pi s_4)\sin(\pi s_1)}{\pi\sin(\pi(s_1+s_3))}+O(1). \tag{6.17}$$

Since it can be elementarily shown that

$$\begin{aligned}\sin(\pi s_1)\sin(\pi s_4)+\sin(\pi s_2)\sin(\pi s_3)-\sin(\pi(s_1+s_2))\sin(\pi(s_1+s_3))\\ =\{\sin(\pi s_4)-\sin(\pi(s_1+s_2+s_3))\}\sin(\pi s_1),\end{aligned}$$

we have

$$\begin{aligned}&\sin(\pi s_1)\sin(\pi s_4)+\sin(\pi s_2)\sin(\pi s_3)\\ &\qquad=\sin(\pi(s_1+s_2))\sin(\pi(s_1+s_3))+O(s_1+s_2+s_3+s_4-3).\end{aligned} \tag{6.18}$$

Using (6.18), we see that (5.23), that is, the singular part of $G(s_1,s_2,s_3,s_4)$

corresponding to (5.10), can be obtained as (6.16) plus (6.17), i.e.

$$
\begin{aligned}
(6.19) \qquad & -\frac{\Gamma(1-s_1)\Gamma(1-s_2)\Gamma(1-s_3)\Gamma(1-s_4)}{s_1+s_2+s_3+s_4-3} \\
&\qquad\qquad \times \frac{\sin(\pi s_2)\sin(\pi s_3)+\sin(\pi s_4)\sin(\pi s_1)}{\pi\sin(\pi(s_1+s_3))} \\
&= -\frac{\Gamma(1-s_1)\Gamma(1-s_2)\Gamma(1-s_3)\Gamma(1-s_4)}{s_1+s_2+s_3+s_4-3}\,\frac{\sin(\pi(s_1+s_2))}{\pi}+O(1).
\end{aligned}
$$

From this expression it is obvious that (6.19), that is, (5.23), is indeed singular at $s_1+s_2+s_3+s_4=3$.

We conclude this paper with a comment on the Witten multiple zeta-function (5.4) associated with $\mathfrak{sl}(4)$. From (5.4), we see that

$$(6.20) \qquad \zeta_{\mathfrak{sl}(4)}(s_1,s_2,s_3,s_4,s_5,s_6)=\zeta_{\mathfrak{sl}(4)}(s_3,s_2,s_1,s_5,s_4,s_6).$$

In [9, Section 4], it was shown that true singularities of $\zeta_{\mathfrak{sl}(4)}(s_1,s_2,s_3,s_4,s_5,s_6)$ satisfy one of the equations

$$
\begin{aligned}
(6.21) \qquad & s_1+s_4+s_6=1-l && (l\in\mathbb{N}_0),\\
(6.22) \qquad & s_2+s_4+s_5+s_6=1-l && (l\in\mathbb{N}_0),\\
(6.23) \qquad & s_3+s_5+s_6=1-l && (l\in\mathbb{N}_0),\\
(6.24) \qquad & s_1+s_2+s_4+s_5+s_6=2-l && (l\in\mathbb{N}_0),\\
(6.25) \qquad & s_1+s_3+s_4+s_5+s_6=2-l && (l\in\mathbb{N}_0),\\
(6.26) \qquad & s_2+s_3+s_4+s_5+s_6=2-l && (l\in\mathbb{N}_0),\\
(6.27) \qquad & s_1+s_2+s_3+s_4+s_5+s_6=3.
\end{aligned}
$$

Using (6.20), we see that (6.23) and (6.26) determine true singularities, because (6.21) and (6.24) do. This argument for (6.23) and (6.26) is much simpler than the original method in [9]. Hence we can see that this kind of argument using symmetry is convenient for checking whether singularities are true or not. On the other hand, the method used in the latter part of the proof of Theorem 4.2 and in Remarks 5.3 and 6.3 is convenient for getting explicit information about singularities. Therefore it seems that we should use these two methods, depending on the case by case.

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