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Divisibility properties of Smith matrices

by

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1. Introduction. For any integers x and y, we denote by (x, y) (resp. $[x, y]$) the greatest common divisor (resp. least common multiple) of x and y. Let $e \geq 1$ be an integer and $S = \{x_1, \ldots, x_n\}$ be a set of n distinct positive integers. The $n \times n$ matrix

$$(S^e) = ((x_i, x_j)^e),$$

having the eth power $(x_i, x_j)^e$ as its (i, j)-entry, is called the *eth power GCD matrix* on S. The $n \times n$ matrix

$$[S^e] = ([x_i, x_j]^e),$$

having the eth power $[x_i, x_j]^e$ as its (i, j)-entry, is called the *eth power LCM matrix* on S. These are simply called the GCD matrix and LCM matrix respectively if $e = 1$. The set S is said to be *factor closed* (FC) if it contains every divisor of x for any $x \in S$. The set S is said to be *gcd-closed* if for all i and j, (x_i, x_j) is in S. Evidently, an FC set is gcd-closed but not conversely. A famous theorem of Smith [29] states that the determinant of the matrix $[(i, j)^e]$ equals $\prod_{k=1}^{n} J_e(k)$, where J_e is the Jordan totient function (i.e. $J_e(x) = x^e \prod_{p|x}(1 - 1/p^e)$ for any positive integer x). Smith also gave a formula for the determinant of the power LCM matrix $([i, j]^e)$. Since then many generalizations of Smith's results have been published; see, for example, [1–4, 7, 8, 12, 14, 19, 27, 28]. Later on power GCD matrices and power LCM matrices are called *Smith matrices*. It is known that the power GCD matrix on any set is nonsingular, but an LCM matrix may be singular. There are some papers ([6, 13, 17–19, 23, 24]) studying the nonsingularity of power LCM matrices; also, several authors (see [21, 22, 26, 30]) considered the eigenstructure of power GCD matrices and reciprocal power LCM matrices.

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Divisibility is another central topic in the field of Smith matrices. Bourque and Ligh [5] showed that if S is FC, then $(S^e)\,|\,[S^e]$ in the ring $M_n(\mathbb{Z})$ of $n \times n$ matrices over the integers. That is, there is an $M \in M_n(\mathbb{Z})$ such that $[S^e] = (S^e)M$, or equivalently, $(S^e)^{-1}[S^e] \in M_n(\mathbb{Z})$. Hong [16] proved that such factorization holds when S is either a divisor chain or multiple closed (namely, $y \in S$ if $x\,|\,y\,|\,\mathrm{lcm}(S)$ for all $x \in S$, where $\mathrm{lcm}(S)$ means the least common multiple of all the elements of S). But such factorization is no longer true if S is gcd-closed [15]. For $x, y \in S$ and $x < y$, if $x\,|\,y$ and the conditions $x\,|\,z\,|\,y$ and $z \in S$ imply that $z \in \{x, y\}$, then we say that x is a *greatest-type divisor* of y in S, and we also say that y is a *least-type multiple* of x in S. For $x \in S$, we denote by $G_S(x)$ and $L_S(x)$ the set of all greatest-type divisors of x in S and the set of all least-type multiples of x in S respectively. It follows from [15] that there is a gcd-closed set S with $\max_{x\in S}\{|G_S(x)|\} = 2$ such that $(S)^{-1}[S] \notin M_n(\mathbb{Z})$. However, it is not clear whether there is a gcd-closed set S with $\max_{x\in S}\{|G_S(x)|\} = 1$ such that $(S)^{-1}[S] \notin M_n(\mathbb{Z})$. Hong believed that the answer to this question should be negative. Actually, Hong [19] proposed the following conjectures.

Conjecture 1.1 ([19]). Let S be gcd-closed and $\max_{x\in S}\{|G_S(x)|\} = 1$. Then the GCD matrix $((x_i, x_j))$ on S divides the LCM matrix $([x_i, x_j])$ on S in $M_n(\mathbb{Z})$.

Conjecture 1.2 ([19]). Let S be lcm-closed and $\max_{x\in S}\{|L_S(x)|\} = 1$. Then the GCD matrix $((x_i, x_j))$ on S divides the LCM matrix $([x_i, x_j])$ on S in $M_n(\mathbb{Z})$.

By [16] we know that Conjectures 1.1 and 1.2 are true when S is a divisor chain. Feng, Tan and Zheng [10] showed that Conjecture 1.1 holds if S consists of two relatively prime divisor chains. In this paper, we introduce a new method to investigate the above conjectures. We first show several theorems on the structure and properties of gcd-closed sets S with $\max_{x\in S}\{|G_S(x)|\} = 1$. Using these we then construct an integer matrix which equals the product $(S^e)^{-1}[S^e]$; see Theorem 2.5 below. This in particular implies Conjecture 1.1 is true. Next, we establish a result for the lcm-closed case which confirms Conjecture 1.2. Finally, we make some remarks on the finite arithmetic progression case and raise an open problem.

For any permutation σ on $\{1, \ldots, n\}$, define $S_\sigma := \{x_{\sigma(1)}, \ldots, x_{\sigma(n)}\}$. Then one can easily check that $(S^e)^{-1}[S^e] = P^t(S^e_\sigma)^{-1}[S^e_\sigma]P$, where P is the $n \times n$ permutation matrix whose ith row equals

$$(0, \ldots, 0, \underbrace{1}_{\sigma(i)}, 0, \ldots, 0) \qquad (1 \le i \le n).$$

It follows that $(S^e)^{-1}[S^e] \in M_n(\mathbb{Z}) \Leftrightarrow (S^e_\sigma)^{-1}[S^e_\sigma] \in M_n(\mathbb{Z})$. So for divisibility purposes, we can rearrange the elements of S in case of necessity.

Throughout the paper, for any finite sets T and Q of integers, we denote by $|T|$ and $\max(T)$ the cardinality of T and the maximal element of T respectively, and define $(T, Q) := (\max(T), \max(Q))$. Then $(x, T) = (x, \max(T))$ for any integer x.

2. The gcd-closed case. First we prove three results on the structure of certain gcd-closed sets.

LEMMA 2.1. *Let* $n \geq 2$ *and* $x_1 < \cdots < x_n$. *If* $\max_{x \in S}\{|G_S(x)|\} = 1$ *and* $x_i \mid x_n$ *for all* $1 \leq i \leq n$, *then* $x_1 \mid \cdots \mid x_{n-1} \mid x_n$.

Proof. We use induction on n. If $n = 2$, then the result is obvious, so let $n \geq 3$.

Assume that the assertion is true for $n-1$. Now consider the case of n. Let $S' = \{x_1, \ldots, x_{n-1}\}$. Then $x_{n-1} \mid x_n$ by assumption. Hence x_{n-1} is a greatest-type divisor of x_n in S.

We claim that $x_j \mid x_{n-1}$ for all $1 \leq j \leq n-2$. Indeed, otherwise there exists a j, $1 \leq j \leq n-2$, such that $x_j \nmid x_{n-1}$. Let J be the set of all such j and put $j_0 := \max\{j : j \in J\}$. Then x_{j_0} is another greatest-type divisor of x_n in S. This means that $|G_S(x_n)| \geq 2$, a contradiction.

By the claim we know that $\max_{x \in S'}\{|G_{S'}(x)|\} \geq 1$. Since, on the other hand, $\max_{x \in S}\{|G_S(x)|\} = 1$, we deduce that $\max_{x \in S'}\{|G_{S'}(x)|\} = 1$. It follows from the claim and induction hypothesis that for the set S', we have $x_1 \mid \cdots \mid x_{n-1}$. So $x_1 \mid \cdots \mid x_n$ as required. ∎

Let S be gcd-closed and $\max_{x \in S}\{|G_S(x)|\} = 1$. Then by Lemma 2.1, we can rearrange S into "composite divisor chains" using the following iterative rule:

STEP 1. Pick the biggest element of S and consider the set of all its divisors in S, denoted by $X_1 = \{x_{11}, \ldots, x_{1,a_1}\}$, where $a_1 = |X_1|$. By Lemma 2.1, these numbers form a divisor chain.

STEP 2. If $X_1 = S$, we are done. If $X_1 \neq S$, then Step 1 applied to $S \setminus X_1$ gives us another divisor chain, denoted by $X_2 = \{x_{21}, \ldots, x_{2,a_2}\}$, where $a_2 = |X_2|$. If $X_1 \cup X_2 = S$, we are done. If $X_1 \cup X_2 \neq S$, then by Step 1 applied to $S \setminus (X_1 \cup X_2)$, we get a new divisor chain X_3. Since S is finite, by repeating Step 1 a finite number of times, we can classify S into k disjoint divisor chains $X_1, \ldots, X_k$, i.e., $S = \coprod_{1 \leq i \leq k} X_i$, where $X_i = \{x_{i1}, \ldots, x_{i,a_i}\}$, $a_i = |X_i|$, $x_{i1} < \cdots < x_{i,a_i}$.

Note that $x_{i1} \mid \cdots \mid x_{i,a_i}$ for $1 \leq i \leq k$. Let $A := \{a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_k\}$ and $a_0 = 0$. For $1 \leq i \leq k$ and $1 \leq j \leq a_i$, define $y_{a_0 + a_1 + \cdots + a_{i-1} + j} := x_{ij}$. Now we rearrange the elements of S, and in Lemmas 2.2–2.4 and Theorem

2.5(i) below, we always let

$$S = \coprod_{1\le i\le k} X_i = \{y_1, \ldots, y_n\} \tag{1}$$

with $y_1 = 1$. Obviously $y_a \nmid y_b$ if $1 \le b < a \le n$. If $k = 1$, then $S = X_1$ is a divisor chain. By [16], we have $(S^e) \mid [S^e]$. In what follows we let $k \ge 2$. Define $T_i := \{(X_i, X_j) : 1 \le j < i\}$ for $2 \le i \le k$. For any $y_s \in S$ $(1 \le s \le n)$, define $n_s := |\{2 \le i \le k : \max(T_i) = y_s\}|$. Clearly $n_s = 0$ if $y_s \ne \max(T_i)$ for all $2 \le i \le k$. In particular, $n_s = 0$ if $s \in A$ and thus $\sum_{s=1}^{n} n_s = \sum_{s\notin A} n_s = k - 1$. We have the following results.

Lemma 2.2.

(i) *For any integer* $2 \le i \le k$, T_i *is a divisor chain.*
(ii) *For* $1 < a \le k$, *if* $\max(T_a) \in X_b$, *then* $(X_a, X_b) = \max(T_a)$.

Proof. Since $(X_i, X_j) \mid \max(X_i)$ for all integers $j \ge 1$, by Lemma 2.1 we can easily see that T_i is a divisor chain, proving (i). Clearly $\max(T_a) \mid (X_a, X_b)$ since $\max(T_a) \in X_b$. But T_a is a divisor chain. So $(X_a, X_b) \mid \max(T_a)$ and hence $(X_a, X_b) = \max(T_a)$. This proves (ii). ∎

Lemma 2.3. *Let* $1 \le l \ne m \le k$, $y_t \in X_m$, $y_\alpha \in X_l$ *and* $y_\beta \in S$.

(i) *If* $y_\alpha \nmid y_t$ *and* $y_\alpha \mid y_\beta$, *then* $(y_\alpha, y_t) = (y_\beta, y_t)$.
(ii) *If* $(X_l, y_t) \notin X_l$, *then* $(X_l, y_t) = (T_l, y_t)$.
(iii) *If* $y_t \nmid y_\alpha$, $y_\alpha \nmid y_t$ *and* $(y_\alpha, y_t) = y_\omega$, *then* $n_\omega \ne 0$.

Proof. (i) Let $(y_\alpha, y_t) = a \in S$ and $(y_\beta, y_t) = b \in S$. Clearly $y_\alpha \mid y_\beta$ and $a \mid b \mid y_\beta$. Since $\max_{x\in S}\{|G_S(x)|\} = 1$, Lemma 2.1 applied to $\{b, y_\alpha, y_\beta\}$ tells us that either $b \mid y_\alpha \mid y_\beta$ or $y_\alpha \mid b \mid y_\beta$. If $y_\alpha \mid b = (y_\beta, y_t)$, then $y_\alpha \mid y_t$, contrary to assumption. So we must have $b \mid y_\alpha$ and $b \mid a$. Hence $(y_\alpha, y_t) = (y_\beta, y_t)$ as required.

(ii) Let $(X_l, y_t) = y_{t'} \notin X_l$ and $\max(T_l) \in X_{l'}$ for some positive integers $t' \le n$ and $l' < l$. Then $y_{t'} \in X_{l''}$ for some positive integer $l'' < l$. We then derive that $y_{t'} \mid (X_{l''}, X_l) \mid \max(T_l) \mid \max(X_{l'})$. By Lemma 2.2 we have

$$(T_l, y_t) = ((X_l, X_{l'}), y_t) = ((X_l, y_t), X_{l'}) = (y_{t'}, X_{l'}) = y_{t'} = (X_l, y_t).$$

(iii) Without loss of generality, we may let $l < m$. It suffices to show that $y_\omega = \max(T_i)$ for some $2 \le i \le k$. Let $y_\omega \in X_r$. Then $r \le l < m$. Obviously $y_\omega \nmid \max(X_i)$ for $1 \le i \le r - 1$ and $y_\omega \mid \max(X_r)$ as well as $y_\omega \mid \max(X_m)$. Thus we can define a nonempty index set $\{q_1, \ldots, q_h\} := \{r + 1 \le q \le k : y_\omega \mid \max(X_q)\}$. Clearly $m \in \{q_1, \ldots, q_h\}$.

We claim that there exists some $1 \le j \le h$ such that $(X_{q_j}, X_r) = y_\omega$. Since $y_\alpha \nmid y_t$ and $y_t \nmid \max(X_l)$, we have $y_\omega = (y_\alpha, y_t) = (X_l, y_t) = (X_l, X_m)$ by (i). So if $r = l$, the claim is true. If $r < l$, then $l \in \{q_1, \ldots, q_h\}$. Evidently $y_\omega \mid (X_{q_j}, X_r) \mid \max(X_r)$ for all $1 \le j \le h$. By Lemma 2.1 we know that $\{(X_{q_1}, X_r), \dots, (X_{q_h}, X_r)\}$ is a divisor chain. Assume that the claim is not true. Then $(X_{q_j}, X_r) > y_\omega$ for all $1 \le j \le h$. So $(X_l, X_m) \ge ((X_l, X_r), (X_m, X_r)) = \min((X_l, X_r), (X_m, X_r)) > y_\omega$. This is absurd. The claim is proved.

Now let i be the smallest $r + 1 \le q_j \le k$ such that $(X_{q_j}, X_r) = y_\omega$. It remains to show that $y_\omega = \max(T_i)$. Since $y_\omega = (X_i, X_r)$, we have $y_\omega \mid \max(T_i)$. Let $\max(T_i) \in X_v$ for some $1 \le v \le i - 1$. Then $y_\omega \mid \max(X_v)$ and so $v \in \{r, q_1, \dots, q_h\}$. By Lemma 2.2 we have $(X_i, X_v) = \max(T_i)$. Let $(X_v, X_r) = y_{\omega'}$. Suppose that $\max(T_i) > y_\omega$. Then $X_v \ne X_r$ and so $y_{\omega'} > y_\omega$ by the minimality of i. Since $y_{\omega'} \mid \max(X_v)$ and $\max(T_i) \mid \max(X_v)$, by Lemma 2.1 we have either $y_{\omega'} \mid \max(T_i) \mid \max(X_v)$ or $\max(T_i) \mid y_{\omega'} \mid \max(X_v)$. From this we deduce that $y_\omega = (X_i, X_r) \ge ((X_i, X_v), (X_r, X_v)) = (T_i, y_{\omega'}) = \min(\max(T_i), y_{\omega'}) > y_\omega$, which is impossible. Thus $\max(T_i) = y_\omega$ as desired. ∎

For any $s \in A$, we can define a unique integer $1 \le l(s) \le k$ such that $y_s = y_{a_1 + \dots + a_{l(s)}} = \max(X_{l(s)})$. In the rest of this section, for any given $1 \le t \le n$, let $y_t \in X_{l(t')}$ and $y_{t'} = \max(X_{l(t')})$ for $1 \le l(t') \le k$. Then $t' \in A$.

LEMMA 2.4. *Let* $A_1 := \{s : (y_s, y_t) \notin X_{l(s)},\ s \in A \backslash \{t'\}\}$ *and* $A_2 := \{s : (y_s, y_t) \in X_{l(s)},\ s \in A \backslash \{t'\}\}$. *Then*

(i) $f_1(t) := \displaystyle\sum_{s \in A_1} ((y_s, y_t)^e - (T_{l(s)}, y_t)^e) = 0.$

(ii) $f_2(t) := \displaystyle\sum_{s \in A_2} (y_s, y_t)^e - \sum_{s \in A_2 \cup \{t'\} \backslash \{a_1\}} (T_{l(s)}, y_t)^e = 0.$

Proof. (i) If $s \in A_1$, then $(y_s, y_t) \notin X_{l(s)}$. By Lemma 2.3(ii) we have $(y_s, y_t) = (T_{l(s)}, y_t)$, and so $f_1 = 0$.

(ii) If $t' = a_1$, then clearly $A_2 = \emptyset$. Hence $f_2 = 0$ as required. Let now $t' \ne a_1$. Consider the following two cases:

CASE 1: $t \in A$. Then $t = t'$. Since $(X_1, y_t) \in X_1$, we can define a nonempty index set $\{t_1, \dots, t_r\} := \{1 \le i \le k : (X_i, y_t) \in X_i\}$, where $1 = t_1 < \dots < t_r = l(t)$.

We assert that $(y_t, X_{t_{j-1}}) = \max(T_{t_j}) \in X_{t_{j-1}}$ for all $1 < j \le r$. Let $(y_t, X_{t_j}) = y_{t'_j} \in X_{t_j}$ for $1 < j \le r$. Clearly there exists a unique $1 \le b(j) < t_j$ such that $\max(T_{t_j}) \in X_{b(j)}$. By Lemmas 2.2 and 2.3(i), we have $(y_{t'_j}, X_{b(j)}) = (X_{t_j}, X_{b(j)}) = \max(T_{t_j}) \in X_{b(j)}$. Then $(y_{t'_j}, X_{b(j)}) \mid (y_t, X_{b(j)}) \mid \max(X_{b(j)})$ and so $(y_t, X_{b(j)}) \in X_{b(j)}$. Therefore $X_{b(j)} \in \{X_{t_1}, \dots, X_{t_{j-1}}\}$. Since $y_{t'_j} \mid y_t$ and $(y_t, X_{t_{j-1}}) = y_{t'_{j-1}} \mid y_t$, by Lemma 2.1 we have $y_{t'_{j-1}} \mid y_{t'_j} \mid y_t$. Then $y_{t'_{j-1}} = (y_{t'_j}, y_{t'_{j-1}}) \mid (X_{t_j}, X_{t_{j-1}}) \mid \max(T_{t_j}) \mid \max(X_{t_j})$. Since $y_{t'_{j-1}} \in X_{t_{j-1}}$, we have $\max(T_{t_j}) \in X_{t_{j-1}}$. Clearly $y_{t'_j} \nmid X_{t_{j-1}}$ for all $1 < j \le r$. Then

$(y_t, X_{t_{j-1}}) = (y_{t'_j}, X_{t_{j-1}}) = (X_{t_j}, X_{t_{j-1}}) = \max(T_{t_j})$ by Lemmas 2.3(i) and 2.2. The assertion is proved.

For each $s \in A_2$, we can find a unique $1 \le j < r$ such that $y_s \in X_{t_j}$. Note that $\max(T_{t_j}) \mid y_t$ for all $1 \le j \le r$. Therefore

$$\begin{aligned} f_2 &= \sum_{1\le j<r} (X_{t_j}, y_t)^e - \sum_{1<j\le r} (T_{t_j}, y_t)^e \\ &= \sum_{1\le j<r} \max(T_{t_{j+1}})^e - \sum_{1<j\le r} \max(T_{t_j})^e = 0. \end{aligned}$$

Case 2: $t \notin A$. If $l(s) \in A_2$, then $l(s) < l(t')$ and $y_t \nmid y_s$. Then $(y_s, y_t) = (y_s, y_{t'})$ by Lemma 2.3(i). Since $y_t \nmid y_s$ and $y_t \nmid \max(T_{l(s)})$, by Lemma 2.3(i) we have $(T_{l(s)}, y_t) = (T_{l(s)}, y_{t'})$ for all $l(s) < l(t')$. Then

$$f_2 = \sum_{s\in A_2} (y_s, y_{t'})^e - \sum_{s\in A_2\cup\{t'\}\setminus\{a_1\}} (T_{l(s)}, y_{t'})^e = 0.$$

Let $A'_1 = \{s : (y_s, y_{t'}) \notin X_{l(s)},\ s \in A\setminus\{t'\}\}$ and $A'_2 = \{s : (y_s, y_{t'}) \in X_{l(s)},\ s \in A\setminus\{t'\}\}$. It is easy to see that $A'_1 = A_1$ and $A'_2 = A_2$. If we replace A_2 by A'_2 and t by t', Case 1 gives $f_2 = 0$. ■

Definition. Define a matrix $C := (c_{st}) \in M_n(\mathbb{Z})$, where

$$c_{st} = \frac{y_t^e}{(y_s,\, y_t)^e}\,\delta_{st}$$

and δ_{st} is defined by: $\delta_{s1} = 1$ if $s \in A$; $-n_s$ if $s \notin A$, $\delta_{1t} = 1 - n_1$ if $t > 1$, and for $s, t > 1$,

$$\underbrace{\delta_{st}}_{\text{if } t\in A} = \begin{cases} 1, & s \in A\setminus\{t\}, \\ -n_s, & \text{otherwise}, \end{cases} \qquad \underbrace{\delta_{st}}_{\text{if } t\notin A} = \begin{cases} -1-n_s, & s = t, \\ 1, & s \in A, \\ -n_s, & \text{otherwise}. \end{cases}$$

Now we state the first main result of this paper as follows.

Theorem 2.5.

(i) *Let S be a gcd-closed set such that* $\max_{x\in S}\{|G_S(x)|\} = 1$. *Then* $(S^e)^{-1}[S^e] = C$, *where $C \in M_n(\mathbb{Z})$ is defined as above. In particular, Conjecture* 1.1 *holds.*

(ii) *For each integer $r \ge 2$, there exists a gcd-closed set S such that* $\max_{x\in S}\{|G_S(x)|\} = r$ *and the power GCD matrix (S^e) on S does not divide the power LCM matrix $[S^e]$ on S in $M_n(\mathbb{Z})$.*

Proof. (i) First note that S is as in (1). Then $S = \{y_1, \dots, y_n\}$ with $y_1 = 1$. In what follows we show $[S^e] = (S^e)C$, i.e. $[y_m, y_t]^e = \sum_{s=1}^n (y_m, y_s)^e c_{st}$ for all $1 \le m, t \le n$. Let $y_m \in X_{l(m')}$ and $y_{m'} = \max(X_{l(m')})$ for $1 \le l(m') \le k$ and $m' \in A$. Let $(y_m, y_t) = y_u \in X_{l(u')}$ and $y_{u'} = \max(X_{l(u')})$. Consider the following three cases:

CASE 1: $t = 1$. We have

$$\sum_{s=1}^{n}(y_m, y_s)^e c_{s1} = \sum_{s\in A}(y_m, y_s)^e - \sum_{s\notin A} n_s(y_m, y_s)^e = y_m^e + \Delta,$$

where

$$\Delta = \sum_{s\in A\setminus\{m'\}}(y_m, y_s)^e - \sum_{s\in A\setminus\{a_1\}}(y_m, T_{l(s)})^e.$$

Clearly $\Delta = f_1(m) + f_2(m) = 0$ by Lemma 2.4. Thus $\sum_{s=1}^{n}(y_m, y_s)^e c_{s1} = y_m^e = [y_m, y_1]^e$.

CASE 2: $t \in A$. We have

$$\begin{aligned}\sum_{s=1}^{n}(y_m, y_s)^e c_{st} &= (y_m, y_1)^e y_t^e(1 - n_1) + (y_m, y_t)^e(-n_t)\\ &\quad + \sum_{s\in A\setminus\{t\}}\frac{(y_m, y_s)^e y_t^e}{(y_s, y_t)^e} - \sum_{s\notin A,\, s\neq 1}\frac{n_s(y_m, y_s)^e y_t^e}{(y_s, y_t)^e}\\ &= y_t^e + \sum_{s\in A\setminus\{t\}}\frac{(y_m, y_s)^e y_t^e}{(y_s, y_t)^e} - \sum_{s\notin A}\frac{n_s(y_m, y_s)^e y_t^e}{(y_s, y_t)^e} = y_t^e + y_t^e g_1,\end{aligned}$$

since $n_t = 0$ for $t \in A$, where

$$g_1 = \sum_{s\in A\setminus\{t\}}\frac{(y_m, y_s)^e}{(y_s, y_t)^e} - \sum_{s\in A\setminus\{a_1\}}\frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_t)^e}.$$

Let A_1 and A_2 be as in Lemma 2.4, and let $\{t_1, \dots, t_r\}$ be as in the proof of Lemma 2.4. Consider the following two subcases.

SUBCASE 2-1: $y_m \mid y_t$. Since $[y_m, y_t] = y_t$, it suffices to show $g_1 = 0$. We have

$$\begin{aligned}g_1 &= \left(\sum_{s\in A_1}\frac{(y_s, y_m)^e}{(y_s, y_t)^e} - \sum_{s\in A_1}\frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_t)^e}\right)\\ &\quad + \left(\sum_{s\in A_2}\frac{(y_s, y_m)^e}{(y_s, y_t)^e} - \sum_{s\in A_2\cup\{t\}\setminus\{a_1\}}\frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_t)^e}\right)\\ &= \sum_{s\in A_1}\left(\frac{(y_s, y_m)^e}{(y_s, y_t)^e} - \frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_t)^e}\right) + \left(\sum_{1\le j<r}\frac{(X_{t_j}, y_m)^e}{(X_{t_j}, y_t)^e} - \sum_{1<j\le r}\frac{(T_{t_j}, y_m)^e}{(T_{t_j}, y_t)^e}\right)\\ &= \sum_{s\in A_1}\left(\frac{(y_s, y_m)^e}{(y_s, y_t)^e} - \frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_t)^e}\right) + \sum_{1<j\le r}\left(\frac{(X_{t_{j-1}}, y_m)^e}{(X_{t_{j-1}}, y_t)^e} - \frac{(T_{t_j}, y_m)^e}{(T_{t_j}, y_t)^e}\right)\\ &=: h_1 + h_2.\end{aligned}$$

If $s \in A_1$, then $(y_t, y_s) \notin X_{l(s)}$. Since $(y_m, y_s) \mid (y_t, y_s)$ we have $(y_m, y_s) \notin X_{l(s)}$. From Lemma 2.3(ii), we deduce $(y_s, y_m) = (T_{l(s)}, y_m)$ and $(y_s, y_t) = (T_{l(s)}, y_t)$. So $h_1 = 0$.

We now show $h_2 = 0$. We have proved in Lemma 2.4 that $(y_t, X_{t_{j-1}}) = \max(T_{t_j}) \in X_{t_{j-1}}$ for all $1 < j \le r$. Let first $m' = a_1$, i.e. $y_m \in X_1$. For $2 < j \le r$, we have $\max(T_{t_j}) \nmid y_m$. Then by Lemma 2.3(i), $(T_{t_j}, y_m) = (X_{t_{j-1}}, y_m)$. Let $j = 2$. Then $y_m \,|\, (y_t, y_{m'}) = \max(T_{t_2}) \in X_1$ and so $(T_{t_2}, y_m) = y_m = (X_1, y_m)$. Thus $h_2 = 0$. Let now $m' \ne a_1$. Since $y_m \,|\, (y_t, y_{m'}) \,|\, y_{m'}$ and $y_m, y_{m'} \in X_{l(m')}$, we have $(y_{m'}, y_t) \in X_{l(m')}$, which implies that $l(m') \in \{t_1, \ldots, t_r\}$. Write $l(m') = t_{j_0}$ for some $1 \le j_0 \le r$. Since $(y_t, X_{t_{j-1}}) = \max(T_{t_j}) \in X_{t_{j-1}}$ for all $1 < j \le r$, we have $\max(T_{t_j}) \nmid y_m$ and $(X_{t_{j-1}}, y_m) = (T_{t_j}, y_m)$ for all $j_0 + 2 \le j \le r$ by Lemma 2.3(i). For all $1 < j \le j_0$ we have $y_m \nmid \max(T_{t_j})$ and so $(X_{t_{j-1}}, y_m) = (X_{t_{j-1}}, y_t) = \max(T_{t_j})$ by Lemma 2.3(i). Hence $(X_{t_{j-1}}, y_m) = (T_{t_j}, y_m)$ for all $1 < j \le j_0$. For $j = j_0 + 1$, we have $(T_{t_{j_0+1}}, y_m) = ((y_t, X_{t_{j_0}}), y_m) = (X_{t_{j_0}}, y_m)$. This implies that $h_2 = 0$, which means $g_1 = 0$. (Note that

$$g_1' := \sum_{s \in A \setminus \{t\}} \frac{(y_t, y_s)^e}{(y_s, y_m)^e} - \sum_{s \in A \setminus \{a_1\}} \frac{(T_{l(s)}, y_t)^e}{(T_{l(s)}, y_m)^e} = 0,$$

which will be used in Subcase 3-2).

Subcase 2-2: $y_m \nmid y_t$. Clearly $m' \ne t$. Since $(y_m, y_t) = y_u$ and $y_t \nmid y_m$ for all $y_m \in S$, we have $n_u \ne 0$ by Lemma 2.3(iii). So we can find a $v \in A$ such that $y_u = \max(T_{l(v)})$. Then $(T_{l(v)}, y_m)/(T_{l(v)}, y_t) = y_u/y_u = 1$. Since $y_u \,|\, (y_t, y_{u'}) \,|\, y_{u'}$, we have $(y_t, y_{u'}) \in X_{l(u')}$ and so $l(u') \in \{t_1, \ldots, t_r\}$. Let $t_{j_1} = l(u')$ for some $1 \le j_1 \le r$. By Lemma 2.3(i) and $y_m \nmid y_t$, we have $(y_m, y_{m'})/(y_{m'}, y_t) = y_m/(y_m, y_t)$. Then

$$\sum_{s=1}^{n} (y_m, y_s)^e c_{st} = [y_m, y_t]^e + y_t^e g_2,$$

where

$$g_2 = \sum_{s \in A \setminus \{t, m'\}} \frac{(y_s, y_m)^e}{(y_s, y_t)^e} - \sum_{s \in A \setminus \{a_1, v\}} \frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_t)^e}.$$

In what follows we show that $g_2 = 0$.

Subcase 2-2-1: $m' \in A_2$, i.e. $(y_{m'}, y_t) \in X_{l(m')}$. Then $l(u') \ne l(t) = t_r$. Since $y_m \nmid y_t$, we have $y_u = (y_m, y_t) = (y_{m'}, y_t) \in X_{l(m')}$ by Lemma 2.3(i). Then $l(m') = l(u') = t_{j_1}$. Since $(y_{m'}, y_t) = \max(T_{t_{j_1+1}})$, we may let $l(v) = t_{j_1+1}$. Note that $\{l(s) : s \in A_2\} = \{t_1, \ldots, t_{r-1}\}$. Then

$$g_2 = \sum_{s \in A_1} \left(\frac{(y_s, y_m)^e}{(y_s, y_t)^e} - \frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_t)^e} \right) + \sum_{s \in A_2 \setminus \{m'\}} \frac{(y_s, y_m)^e}{(y_s, y_t)^e}$$

$$- \sum_{s \in A_2 \cup \{t\} \setminus \{a_1, v\}} \frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_t)^e}$$

$$= \sum_{s\in A_1}\left(\frac{(y_s,y_m)^e}{(y_s,y_t)^e}-\frac{(T_{l(s)},y_m)^e}{(T_{l(s)},y_t)^e}\right)+\sum_{1<j\le j_1}\left(\frac{(X_{t_{j-1}},y_m)^e}{(X_{t_{j-1}},y_t)^e}-\frac{(T_{t_j},y_m)^e}{(T_{t_j},y_t)^e}\right)$$

$$+\sum_{j_1+2\le j\le r}\left(\frac{(X_{t_{j-1}},y_m)^e}{(X_{t_{j-1}},y_t)^e}-\frac{(T_{t_j},y_m)^e}{(T_{t_j},y_t)^e}\right)=:h_1+h_3+h_4.$$

For $1<j\le j_1-1$, since $\max(T_{t_j})\in X_{t_{j-1}}$, $\max(T_{t_{j_1+1}})\nmid\max(X_{t_{j-1}})$ and $(y_m,y_t)=\max(T_{t_{j_1+1}})$, we have

$$(X_{t_{j-1}},y_m)=(X_{t_{j-1}},T_{t_{j_1+1}})=(X_{t_{j-1}},y_t)=\max(T_{t_j})=(T_{t_j},y_m)$$

by Lemma 2.3(i). But by Lemmas 2.2 and 2.3(i), $(X_{t_{j_1-1}},y_m)=(X_{t_{j_1-1}},y_{m'})$ $=\max(T_{t_{j_1}})=(T_{t_{j_1}},y_m)$. Since $\max(T_{t_j})\in X_{t_{j-1}}$ and $\max(T_{t_j})\nmid y_m$ for $j_1+2\le j\le r$, we have $(X_{t_{j-1}},y_m)=(T_{t_j},y_m)$ by Lemma 2.3(i). Since $(y_t,X_{t_{j-1}})=\max(T_{t_j})$ for all $1<j\le r$, we have $h_3=h_4=0$.

Now we treat h_1. Let first $s>m'\ge m$ and $s\in A_1$. Then $(y_s,y_m)\notin X_{l(s)}$ for all $s\in A_1$ and so $(y_s,y_m)=(T_{l(s)},y_m)$ by Lemma 2.3(ii). Let $s<m\le m'$ and $s\in A_1$. Clearly $\max(T_{t_{j_1+1}})\nmid\max(X_{l(s)})=y_s$. Then $(y_s,y_m)=(y_s,T_{t_{j_1+1}})=(y_s,y_t)\notin X_{l(s)}$ by Lemma 2.3(i). Hence $(y_s,y_m)=(T_{l(s)},y_m)$ by Lemma 2.3(ii). Since $(y_s,y_t)=(T_{l(s)},y_t)$ for all $s\in A_1$, we have $h_1=0$. Thus $g_2=h_1+h_3+h_4=0$.

SUBCASE 2-2-2: $m'\in A_1$, i.e. $y_u=(y_{m'},y_t)\notin X_{l(m')}$. Define an index set $B:=\{s\in A_1:(y_m,y_s)\notin X_{l(s)}\}$. If $(y_m,y_t)\in X_{l(t)}$, i.e. $l(u')=l(t)$, then let $\{m_r,\dots,m_w\}:=\{i:l(t)\le i\le k,(X_i,X_{l(m')})\in X_i\}$, where $l(t)=t_r=m_r<\dots<m_w=l(m')$. As in the proof of $(y_t,X_{t_{j-1}})=\max(T_{t_j})\in X_{t_{j-1}}$ for all $1<j\le r$, we can show $(y_{m'},X_{m_{j-1}})=\max(T_{m_j})\in X_{m_{j-1}}$ for all $r<j\le w$. Then $y_u=(y_{m'},y_t)=\max(T_{m_{r+1}})$. Hence we may let $l(v)=m_{r+1}$. Since $\{l(s):s\in A_1\backslash B\}=\{m_{r+1},\dots,m_w\}$ and $\{l(s):s\in A_2\}=\{t_1,\dots,t_{r-1}\}$, we have

$$g_2=\sum_{s\in B}\left(\frac{(y_s,y_m)^e}{(y_s,y_t)^e}-\frac{(T_{l(s)},y_m)^e}{(T_{l(s)},y_t)^e}\right)+\sum_{1<j\le r}\left(\frac{(X_{t_{j-1}},y_m)^e}{(X_{t_{j-1}},y_t)^e}-\frac{(T_{t_j},y_m)^e}{(T_{t_j},y_t)^e}\right)$$

$$+\sum_{r+2\le j\le w}\left(\frac{(X_{m_{j-1}},y_m)^e}{(X_{m_{j-1}},y_t)^e}-\frac{(T_{m_j},y_m)^e}{(T_{m_j},y_t)^e}\right)=:h_5+h_6+h_7.$$

If $s\in B$, we have $(y_s,y_t)=(T_{l(s)},y_t)$ and $(y_s,y_m)=(T_{l(s)},y_m)$ by Lemma 2.3(ii). Then $h_5=0$. Since for all $1<j\le r$, $y_u\nmid\max(T_{t_j})$ and $y_u\nmid\max(X_{t_{j-1}})$, by Lemma 2.3(i) we have $(X_{t_{j-1}},y_t)=(X_{t_{j-1}},y_u)=(X_{t_{j-1}},y_m)$ and $(T_{t_j},y_t)=(T_{t_j},y_u)=(T_{t_j},y_m)$. Hence $h_6=0$. As in the proof of $h_4=0$, we can show $h_7=0$. Thus $g_2=0$.

If $(y_m,y_t)\notin X_{l(t)}$, then let $\{m_{j_1},m_{j_1+1},\dots,m_{w'}\}:=\{i:l(u')=m_{j_1}\le i\le k,(X_i,X_{l(m')})\in X_i\}$, where $l(u')=m_{j_1}<m_{j_1+1}<\dots<m_{w'}=l(m')$. Clearly $l(u')\ne t_r$. Since $m',t\in A$, we have $y_u\mid(y_{m'},X_{m_{j_1}})=\max(T_{m_{j_1+1}})\mid y_{u'}$

and $y_u \,|\, (y_{m'}, X_{t_{j_1}}) = \max(T_{t_{j_1+1}}) \,|\, y_{u'}$. Then either $y_u \mid \max(T_{m_{j_1+1}}) \mid \max(T_{t_{j_1+1}}) \,|\, y_{u'}$ or $y_u \mid \max(T_{t_{j_1+1}}) \mid \max(T_{m_{j_1+1}}) \,|\, y_{u'}$ by Lemma 2.1. But

$$y_u = (y_{m'}, y_t) \geq ((y_{m'}, y'_u), (y_t, y_{u'})) = (\max(T_{m_{j_1+1}}), \max(T_{t_{j_1+1}})).$$

Then $y_u = \min(\max(T_{m_{j_1+1}}), \max(T_{t_{j_1+1}}))$. So either $y_u = \max(T_{m_{j_1+1}})$ or $y_u = \max(T_{t_{j_1+1}})$. Then we may let $l(v) = t_{j_1+1}$ or $l(v) = m_{j_1+1}$. Note that $\{l(s) : s \in A_1 \backslash B\} = \{m_{j_1+1}, \ldots, m_{w'}\}$ and $\{l(s) : s \in A_2\} = \{t_1, \ldots, t_{r-1}\}$. If $l(v) = t_{j_1+1}$, then

$$g_2 = \sum_{1<j\leq j_1} \left(\frac{(X_{t_{j-1}}, y_m)^e}{(X_{t_{j-1}}, y_t)^e} - \frac{(T_{t_j}, y_m)^e}{(T_{t_j}, y_t)^e} \right) + \sum_{j_1+2\leq j\leq r} \left(\frac{(X_{t_{j-1}}, y_m)^e}{(X_{t_{j-1}}, y_t)^e} - \frac{(T_{t_j}, y_m)^e}{(T_{t_j}, y_t)^e} \right)$$
$$+ \sum_{j_1+1\leq j\leq w'} \left(\frac{(X_{m_{j-1}}, y_m)^e}{(X_{m_{j-1}}, y_t)^e} - \frac{(T_{m_j}, y_m)^e}{(T_{m_j}, y_t)^e} \right) + h_5 := h_3 + h_4 + h_8 + h_5.$$

As in Subcase 2-2-1, we can prove $h_3 = 0$. As in the proof of $h_6 = 0$, we can show $h_4 = h_8 = 0$. Notice that $h_5 = 0$. Thus $g_2 = 0$. Similarly, we can show that if $l(v) = m_{j_1+1}$, then $g_2 = 0$. Therefore Case 2 is proved.

Case 3: $t \notin A$. We have

$$\sum_{s=1}^{n} (y_m, y_s)^e c_{st} = y_t^e(1 - n_1) - (1 + n_t)(y_t, y_m)^e$$
$$+ \sum_{s\in A} \frac{(y_m, y_s)^e y_t^e}{(y_s, y_t)^e} - \sum_{s\notin A,\, s\neq 1,t} \frac{n_s (y_m, y_s)^e y_t^e}{(y_s, y_t)^e}$$
$$= (y_t^e - (y_t, y_m)^e) + \sum_{s\in A} \frac{(y_m, y_s)^e y_t^e}{(y_s, y_t)^e} - \sum_{s\notin A} \frac{n_s (y_m, y_s)^e y_t^e}{(y_s, y_t)^e}.$$

Consider the following three subcases.

Subcase 3-1: $y_m \,|\, y_t$. We have

$$\sum_{s=1}^{n} (y_m, y_s)^e c_{st} = y_t^e + y_t^e \left(\sum_{s\in A\backslash\{t'\}} \frac{(y_s, y_m)^e}{(y_s, y_t)^e} - \sum_{s\in A\backslash\{a_1\}} \frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_t)^e} \right)$$
$$= y_t^e + y_t^e (h_1 + h_9),$$

where

$$h_9 = \sum_{s\in A_2} \frac{(y_s, y_m)^e}{(y_s, y_t)^e} - \sum_{s\in A_2\cup\{t'\}\backslash\{a_1\}} \frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_t)^e}.$$

As in Subcase 2-1, we have $h_1 = 0$ since t is independent of $t \in A$. If $s \in A_2$, then $y_t \nmid y_s$ and $y_t \nmid \max(T_{l(s)})$. By Lemma 2.3(i), $(y_s, y_t) = (y_s, y_{t'})$ and

$(T_{l(s)}, y_t) = (T_{l(s)}, y_{t'})$. Thus

$$h_9 = \sum_{s\in A_2} \frac{(y_s, y_m)^e}{(y_s, y_{t'})^e} - \sum_{s\in A_2\cup\{t'\}\setminus\{a_1\}} \frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_{t'})^e}.$$

Since $t' \in A$, $h_2 = 0$ gives $h_9 = 0$.

SUBCASE 3-2: $y_t \mid y_m$. We have

$$\begin{aligned}\sum_{s=1}^{n}(y_m, y_s)^e c_{st} = {}& y_m^e + y_t^e\Bigg(\sum_{s\in A\setminus\{m'\}} \frac{(y_s, y_m)^e}{(y_s, y_t)^e} - \sum_{s\in A\setminus\{a_1\}} \frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_t)^e}\Bigg)\\ =:{}& y_m^e + y_t^e g_3.\end{aligned}$$

Let

$$g_3' = \sum_{s\in A\setminus\{m'\}} \frac{(y_s, y_t)^e}{(y_s, y_m)^e} - \sum_{s\in A\setminus\{a_1\}} \frac{(T_{l(s)}, y_t)^e}{(T_{l(s)}, y_m)^e}.$$

Since $y_t \mid y_m$, $h_1 + h_9 = 0$ in Subcase 3-1 gives $g_3' = 0$. Then as in Subcase 2-1, $g_1 = 0$ implying $g_1' = 0$ tells us that $g_3' = 0$ implies that $g_3 = 0$.

SUBCASE 3-3: $y_m \nmid y_t$ and $y_t \nmid y_m$. Since $(y_t, y_m) = y_u$, we have $n_u \neq 0$ by Lemma 2.3(iii). So we also can find a $v \in A$ such that $y_u = \max(T_{l(v)})$ as in Subcase 2-2. Then

$$\sum_{s=1}^{n}(y_m, y_s)^e c_{st} = [y_m, y_t]^e + y_t^e g_4,$$

where

$$g_4 = \sum_{s\in A\setminus\{t',m'\}} \frac{(y_s, y_m)^e}{(y_s, y_t)^e} - \sum_{s\in A\setminus\{a_1,v\}} \frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_t)^e}.$$

In what follows we show that $g_4 = 0$.

SUBCASE 3-3-1: $m' \in A_2$. Since $y_t \nmid \max(T_{l(s)})$ and $y_t \mid y_{t'}$, by Lemma 2.3(i) we have $(T_{l(s)}, y_t) = (T_{l(s)}, y_{t'})$. Then

$$\begin{aligned}g_4 &= h_1 + \Bigg(\sum_{s\in A_2\setminus\{m'\}} \frac{(y_s, y_m)^e}{(y_s, y_t)^e} - \sum_{s\in A_2\cup\{t'\}\setminus\{a_1,v\}} \frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_t)^e}\Bigg)\\ &= h_1 + \Bigg(\sum_{s\in A_2\setminus\{m'\}} \frac{(y_s, y_m)^e}{(y_s, y_{t'})^e} - \sum_{s\in A_2\cup\{t'\}\setminus\{a_1,v\}} \frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_{t'})^e}\Bigg) =: h_1 + h_{10}.\end{aligned}$$

As in Subcase 2-2-1, we have $h_1 = 0$ since t is independent of $t \in A$. Since $t' \in A$, $h_3 + h_4 = 0$ gives $h_{10} = 0$. Therefore $g_4 = 0$.

SUBCASE 3-3-2: $m' \in A_1$. Let B be as in Subcase 2-2-2. Then

$$g_4 = h_{10} + h_5 + \sum_{s\in A_1\setminus B}\Bigg(\frac{(y_s, y_m)^e}{(y_s, y_t)^e} - \frac{(T_{l(s)}, y_m)^e}{(T_{l(s)}, y_t)^e}\Bigg).$$

As in the proof of Subcase 2-2-2, we can show $g_4 = 0$. Part (i) is proved.

(ii) By [31] we know that there is a gcd-closed set S with $\max_{x\in S}\{|G_S(x)|\} = 2$ such that $(S^e)^{-1}[S^e] \notin M_{|S|}(\mathbb{Z})$. Now let $r \geq 3$ and $p_1 < \cdots < p_r$ be prime numbers. Define $x_1 = 1$, $x_i = p_{i-1}$ $(2 \leq i \leq r+1)$ and $x_{r+2} = p_1 \cdots p_r$. Obviously $S := \{x_1, \ldots, x_{r+2}\}$ is gcd-closed and $\max_{x\in S}\{|G_S(x)|\} = r$. By [4], we have

$$((S^e)^{-1})_{ij} = \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{c_{ik} c_{jk}}{b_k},$$

where

$$b_i = \sum_{\substack{d | x_i \\ d \nmid x_t,\, x_t < x_i}} J_e(d) \quad \text{and} \quad c_{ij} = \sum_{\substack{d x_i | x_j \\ d x_i \nmid x_t,\, x_t < x_j}} \mu(d).$$

From [19] we derive $b_{r+2} = \prod_{i=1}^r p_i^e - \sum_{i=1}^r p_i^e + r - 1$. Using these and by some computations, we get $((S^e)^{-1}[S^e])_{22} = -\sigma/b_{r+2}$, where

$$\sigma = \prod_{i=1}^r p_i^e - p_1^e \sum_{i=2}^r p_i^e + (r-2) p_1^e.$$

Since $\sum_{i=2}^r p_i^e > r-1$, we have $b_{r+2} - \sigma > 0$. Clearly $\sigma > 0$. So $0 < \sigma/b_{r+2} < 1$ and $((S^e)^{-1}[S^e])_{22} \notin \mathbb{Z}$. Hence $(S^e) \nmid [S^e]$ as required. Part (ii) is proved. ∎

Remark. By Theorem 2.5(i) we know immediately that the sum of the elements of the tth column of the matrix C equals y_t^e. On the other hand, let S_0 be the gcd-closed set as in (1). Then by Theorem 2.5(i) we have $(S_0^e)^{-1}[S_0^e] = C$. For any general gcd-closed set S, let τ be the permutation such that $S_\tau = S_0$. It follows that $(S^e)^{-1}[S^e] = P^t C P$, where P is the $n \times n$ permutation matrix formed by τ. Now let $e \geq 1$ be a given integer and S be a gcd-closed set with $\max_{x\in S}\{|G_S(x)|\} \geq 2$. It is of interest to have necessary and sufficient conditions on S such that $(S^e) \mid [S^e]$ in $M_{|S|}(\mathbb{Z})$. This is an open problem.

3. The lcm-closed case. The reciprocal set of $S = \{x_1, \ldots, x_n\}$, denoted by mS^{-1}, is defined by $mS^{-1} := \{m/x_1, \ldots, m/x_n\}$. By [19] we know that mS^{-1} is gcd-closed if and only if S is lcm-closed. We can now state the second main result of this paper.

Theorem 3.1.

(i) *Let S be an lcm-closed set such that* $\max_{x\in S}\{|L_S(x)|\} = 1$ *and* mS^{-1} *be the same set as in* (1). *Then* $(S^e)^{-1}[S^e] = \mathrm{diag}(x_1^{-e}, \ldots, x_n^{-e}) \cdot C \cdot \mathrm{diag}(x_1^e, \ldots, x_n^e) \in M_n(\mathbb{Z})$. *In particular, Conjecture* 1.2 *is true.*

(ii) *For each integer $r \geq 2$, there exists an lcm-closed set S with* $\max_{x\in S}\{|L_S(x)|\} = r$ *such that the power GCD matrix (S^e) on S does not divide the power LCM matrix $[S^e]$ on S in $M_n(\mathbb{Z})$.*

Proof. (i) Let $x_i y_i = m$ for all $1 \le i \le n$. Then $S = \{x_1, \dots, x_n\}$ since $mS^{-1} = \{y_1, \dots, y_n\}$. Since

$$(x_i, x_j) = \frac{m}{[m/x_i, m/x_j]} = \frac{x_i x_j}{m} \cdot \left(\frac{m}{x_i}, \frac{m}{x_j}\right),$$

we get

$$(S^e) = 1/m^e \cdot D \cdot ((mS^{-1})^e) \cdot D, \qquad [S^e] = 1/m^e \cdot D \cdot [(mS^{-1})^e] \cdot D,$$

where $D = \mathrm{diag}(x_1^e, \dots, x_n^e)$. We deduce that

$$(S^e)^{-1}[S^e] = D^{-1}((mS^{-1})^e)^{-1}[(mS^{-1})^e]D. \tag{2}$$

Clearly it follows from $\max_{y\in S}\{|L_S(y)|\} = 1$ that $\max_{y \in mS^{-1}}\{|G_{mS^{-1}}(y)|\} = 1$. So Theorem 2.5(i) applied to the set mS^{-1} gives $(S^e)^{-1}[S^e] = D^{-1}CD$, where $C = (c_{ij})$ is defined after Lemma 2.4. Hence the (i,j) entry of the matrix $D^{-1}CD$ is

$$\frac{x_j^e}{x_i^e} c_{ij} = \frac{x_j^e}{x_i^e} \frac{y_j^e}{(y_i, y_j)^e} \delta_{ij} = \frac{x_j^e y_j^e}{x_i^e y_i^e} \frac{y_i^e}{(y_i, y_j)^e} \delta_{ij} = \frac{y_i^e}{(y_i, y_j)^e} \delta_{ij} \in \mathbb{Z}$$

since δ_{ij} is an integer. This implies that $(S^e)^{-1}[S^e] \in M_n(\mathbb{Z})$ as required.

(ii) It is known [31] that there is an lcm-closed set S with $\max_{x\in S}\{|L_S(x)|\} = 2$ such that $(S^e)^{-1}[S^e] \notin M_{|S|}(\mathbb{Z})$. For $r \ge 3$, let $p_1 < \dots < p_r$ be primes and $m = p_1 \cdots p_r$. Set $S = \{m, m/p_1, \dots, m/p_r, 1\}$. Then S is lcm-closed and $mS^{-1} = \{1, p_1, \dots, p_r, p_1 \cdots p_r\}$ is gcd-closed. By (2) and replacing S by mS^{-1} in the proof of Theorem 2.5(ii), we get

$$-1 < ((S^e)^{-1}[S^e])_{22} = (((mS^{-1})^e)^{-1}[(mS^{-1})^e])_{22} = -\frac{\sigma}{b_{r+2}} < 0,$$

where σ and b_{r+2} are as above. So $(S^e)^{-1}[S^e] \notin M_{r+2}(\mathbb{Z})$ as desired. ∎

Remark. Given any integer $e \ge 1$. It is an open question to find necessary and sufficient conditions on an lcm-closed set S with $\max_{x\in S}\{|L_S(x)|\} \ge 2$ so that the power GCD matrix (S^e) divides the power LCM matrix $[S^e]$ in $M_{|S|}(\mathbb{Z})$.

4. Remarks on the finite arithmetic progression case. The renowned Dirichlet theorem states that the arithmetic progression contains infinitely many primes if the first term and the common difference are coprime, while the Green–Tao theorem [11] says that the set of primes contains arbitrarily long arithmetic progressions. Farhi [9] and Hong–Feng [20] investigated the non-trivial lower bounds for the least common multiple of finite arithmetic progressions. Ligh [25] raised the problem of computing the determinants of Smith matrices on a finite arithmetic progression which is still open. We are interested in the divisibility of Smith matrices on a finite arithmetic progression. We can easily check that if $S = \{2, 2+q, 2+2q\}$ and $(2, q) = 1$, then for all integer $e \ge 1$, we have $(S^e)^{-1}[S^e] \in M_3(\mathbb{Z})$. But the set

$S = \{4, 7, 10\}$ gives $(S)^{-1}[S] \notin M_3(\mathbb{Z})$. Now fix an integer $e \geq 1$. We do not know how to characterize the arithmetic progression $S = \{a+b, \ldots, a+nb\}$, where $(a, b) = 1$, such that $(S^e)^{-1}[S^e] \in M_n(\mathbb{Z})$. This problem remains open.

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