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On disjoint arithmetic progressions

by

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1. Introduction. Let $\{a_i \pmod{m_i}\}_{i=1}^{l}$ be a collection of arithmetic progressions, where $2 \le m_1 < \cdots < m_l \le x$ and

$$a_i \pmod{m_i} \cap a_j \pmod{m_j} = \emptyset \quad \text{if } i \ne j.$$

Let $f(x)$ be the maximum value for such l. In [4], Erdős and Szemerédi proved that

$$\frac{x}{\exp((\log x)^{1/2+\varepsilon})} < f(x) < \frac{x}{(\log x)^c}$$

for some constant $c > 0$ (see also [3]). Let

$$L(c, x) = \exp(c\sqrt{\log x \log\log x}).$$

Recently, E. S. Croot III [2] has proved that

$$\frac{x}{L(\sqrt{2} + o(1), x)} < f(x) < \frac{x}{L(1/6 - o(1), x)}$$

and with the assumption that all m_i are squarefree, the maximum value for such l is less than $x/L(1/2 - o(1), x)$.

In this paper, we improve Croot's methods and prove the following result:

THEOREM.
$$f(x) < \frac{x}{L(1/2 - o(1), x)}.$$

2. Proof of the Theorem. As a direct consequence of [1, Lemma 3.1], we have (see also [2])

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Lemma 1. *Let*

$$\psi(x,y) = \#\{n \le x : p \text{ prime}, p \mid n \Rightarrow p \le y\}.$$

Then, for any constant $c > 0$,

$$\psi(x, L(c,x)) = \frac{x}{L(1/(2c)+o(1), x)}.$$

Lemma 2 (Croot [2]). *There are at most* $x/L(c/2 - o(1), x)$ *positive integers* $n \le x$ *such that*

$$\omega(n) > c\sqrt{\log x/\log\log x},$$

where c *is some positive constant and* $\omega(n) = \sum_{p\mid n,\, p\, prime} 1$.

Definition. Let r and n be positive integers. Write

$$n = ab, \quad (a,b) = 1, \quad a > 0,\ b > 0,$$

such that if p is a prime, $p^\alpha \mid a$ and $p^{\alpha+1} \nmid a$, then either $\alpha = 0$ or $\alpha > r$, and if p is a prime, $p^\beta \mid b$ and $p^{\beta+1} \nmid b$, then $\beta \le r$. Define

$$h_r(n) = a, \quad l_r(n) = b.$$

Here we may take $a = 1$ or $b = 1$.

Lemma 3. *Let* r *be a given positive integer. Then there are at most* $x/L(c/2 - o(1), x)$ *positive integers* $n \le x$ *with* $n = l_r(n)$ *such that*

$$\Omega(n) > c\sqrt{\log x/\log\log x},$$

where c *is some positive constant and* $\Omega(n) = \sum_{p^\alpha \mid n,\, p\, prime} 1$.

Proof. Let

$$T_1 = \{n : n \le x,\ \omega(n) > c\sqrt{\log x/\log\log x}\},$$

$$T_2 = \{n : n \le x, n = l_r(n),\ \omega(n) \le c\sqrt{\log x/\log\log x},\ \Omega(n) > c\sqrt{\log x/\log\log x}\}.$$

By Lemma 2 we have

$$|T_1| \le \frac{x}{L(c/2 - o(1), x)}.$$

Now we estimate $|T_2|$. For a positive integer n, let $n = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$ be the standard factorization of n. Define

$$g(n) = \alpha_1! \cdots \alpha_t!.$$

By a multipolynomial expansion, we have

$$\Big(\sum_{p \le x,\, p \text{ prime}} \frac{1}{p}\Big)^j \ge \sum_{n \le x,\, \Omega(n) = j} \frac{j!}{g(n) n}.$$

Thus

$$\sum_{n\le x,\,\Omega(n)\ge c\sqrt{\log x/\log\log x}} \frac{1}{g(n)n} \le \sum_{j\ge c\sqrt{\log x/\log\log x}} \frac{1}{j!}\Big(\sum_{p\le x,\,p\,\mathrm{prime}} \frac{1}{p}\Big)^j \le \frac{1}{L(c/2-o(1),x)}.$$

For any integer n with $n\in T_2$, we have

$$g(n)\le (r!)^{\omega(n)} \le (r!)^{c\sqrt{\log x/\log\log x}}.$$

Hence

$$|T_2| \le \sum_{n\in T_2}\frac{x}{n} \le x(r!)^{c\sqrt{\log x/\log\log x}}\sum_{n\in T_2}\frac{1}{g(n)n} \le \frac{x}{L(c/2-o(1),x)}. \ \blacksquare$$

Lemma 4. *For any integer $k\ge 3$, we have*

$$\#\{n:1\le n\le x,\ h_{k(k+1)}(n)\ge L(c,x)\} \le \frac{3x}{L(c(1-2/k),x)}.$$

Proof. It is well known that if an integer $\alpha > k(k+1)$, then there exist two positive integers u and v with $\alpha = uk+v(k+1)$. Thus, if $m = h_{k(k+1)}(m)$, then there exist two positive integers a and b such that $m=a^kb^{k+1}$. So

$$S_{k(k+1)}(t) := \#\{m:1\le m\le t,\ m=h_{k(k+1)}(m)\} \le t^{\frac{1}{k}+\frac{1}{k+1}} < t^{2/k}.$$

Hence

$$\#\{n:1\le n\le x,\ h_{k(k+1)}(n)\ge L(c,x)\}$$

$$\le \sum_{m\ge L(c,x),\,m=h_{k(k+1)}(m)} \frac{x}{m} = x\int_{L(c,x)}^{\infty}\frac{1}{t}\,dS_{k(k+1)}(t) \le \frac{3x}{L(c(1-2/k),x)}. \ \blacksquare$$

Lemma 5. *Let r' and m be two integers with $m>0$. Suppose that $\{b_i \pmod{m_i}\}_{i=1}^{l}$ is a collection of disjoint arithmetic progressions, where*

$$\omega(m_i)\le M, \qquad m_i>m,$$
$$m\mid m_i, \qquad b_i\equiv r' \pmod m, \qquad i=1,\dots,l.$$

Then there exists a prime p and an integer r'' such that $pm\mid m_i$ for at least l/M values of i, and

$$pm\mid m_i, \qquad b_i\equiv r'' \pmod{pm}$$

for at least $l/(pM)$ values of i.

Proof. If $l=1$, then we may take $r''=b_1$ and p to be any prime divisor of m_1/m. Now we assume that $l\ge 2$. Let $p_1,\dots,p_t$ be all distinct prime factors of m_1/m. For any $i\ge 2$, since the arithmetic progressions are disjoint, we have $(m_1,m_i)\nmid b_1-b_i$. By $b_1\equiv r'\equiv b_i \pmod m$, we have $(m_1,m_i)\ne m$. That is, $(m_1/m, m_i/m)\ne 1$ $(i=2,\dots,l)$. Hence, each m_i/m must be divisible by at least one of $p_1, \dots, p_t$. Thus, there exists an index j such that p_j divides m_i/m for at least l/t values of i. Let $b_i = r' + ms_i$ $(i = 1, \dots, l)$. Then there exists an integer s such that

$$s_i \equiv s \pmod{p_j}, \qquad p_j \,\Big|\, \frac{m_i}{m}$$

for at least $l/(p_j t)$ values of i. Thus, by $t \le M$, p_j divides m_i/m for at least l/M values of i, and

$$b_i \equiv r' + ms \pmod{p_j m}, \qquad p_j m \mid m_i$$

for at least $l/(p_j M)$ values of i. ■

Lemma 6. *Let r be a given positive integer. If $\{b_i \pmod{m_i}\}_{i=1}^{l}$ is a collection of disjoint arithmetic progressions with $m_i = l_r(m_i)$ $(i = 1, \dots, l)$ and $2 \le m_1 < \dots < m_l \le x$, then*

$$l < \frac{x}{L(1/2 - o(1), x)}.$$

Proof. Let S be the collection of all m_i's such that

(A) $\Omega(m_i) \le \sqrt{\log x / \log\log x}$;

(B) There exists a prime $p > L(1, x)$ such that $p \mid m_i$.

Beginning with $m = 1$ and $r' = 1$, by using Lemma 5 step by step, we obtain a sequence $p_1, \dots, p_v$ of primes (and a sequence $r_1, \dots, r_v$ of integers) such that for each j, $p_1 \cdots p_j \mid m_i$ for at least

$$\frac{|S|}{p_1 \cdots p_{j-1} (\sqrt{\log x / \log\log x})^j}$$

values of i with $m_i \in S$ and $p_1 \cdots p_v$ is some m_{i_0}. Hence, there exists a p_u with $p_u > L(1, x)$, and

$$\begin{aligned} \frac{x}{p_1 \cdots p_{u-1} L(1, x)} &\ge \#\{n \le x : p_1 \cdots p_u \mid n\} \\ &\ge \#\{m_i : m_i \in S, p_1 \cdots p_u \mid m_i\} \\ &\ge \frac{|S|}{p_1 \cdots p_{u-1} (\sqrt{\log x / \log\log x})^u} \\ &\ge \frac{|S|}{p_1 \cdots p_{u-1} L(1/2, x)}. \end{aligned}$$

Thus

$$|S| \le \frac{x}{L(1/2, x)}.$$

By Lemmas 1 and 3, we have

$$\begin{aligned}|S| &\ge l - \#\{m \le x : m = l_r(m), \Omega(m) > \sqrt{\log x/\log\log x}\} - \psi(x, L(1,x))\\ &\ge l - \frac{2x}{L(1/2 - o(1), x)}.\end{aligned}$$

Therefore

$$l \le \frac{x}{L(1/2 - o(1), x)}. \ \blacksquare$$

Proof of the Theorem. Suppose that $\{a_i \pmod{m_i}\}_{i=1}^{f(x)}$ is a collection of disjoint arithmetic progressions, where $2 \le m_1 < \cdots < m_{f(x)} \le x$. Let k be any given integer with $k \ge 4$, and let U be the collection of all m_i with $h_{k(k+1)}(m_i) < L(2, x)$. By the proof of Lemma 4, we have

$$\#\{m : 1 \le m \le L(2,x),\ m = h_{k(k+1)}(m)\} \le L(2,x)^{2/k} = L(4/k, x).$$

So there must exist a positive integer $m < L(2, x)$ for which there are at least $|U|/L(4/k, x)$ of m_i's with $h_{k(k+1)}(m_i) = m$. Thus, there must exist an integer a such that

$$a_i \equiv a \pmod{m}, \qquad h_{k(k+1)}(m_i) = m$$

for at least $|U|/(mL(4/k, x))$ of m_i. In this way, we obtain a collection $\{a_i \pmod{m_i/m}\}$ which has at least $|U|/(mL(4/k, x))$ disjoint arithmetic progressions. Noting that $l_{k(k+1)}(m_i/m) = m_i/m$ for those m_i with $h_{k(k+1)}(m_i) = m$, by Lemma 6, we have

$$\frac{|U|}{mL(4/k, x)} \le \frac{x}{mL(1/2 - o(1), x/m)} \le \frac{x}{mL(1/2 - o(1), x)}.$$

So

$$|U| \le \frac{x}{L(1/2 - 4/k - o(1), x)}.$$

By Lemma 4 we have

$$\#\{n : 1 \le n \le x,\ h_{k(k+1)}(n) \ge L(2, x)\} \le \frac{3x}{L(2(1 - 2/k), x)} \le \frac{3x}{L(1, x)}.$$

Therefore

$$f(x) \le \frac{x}{L(1/2 - 4/k - o(1), x)} + \frac{3x}{L(1, x)}.$$

Since k is arbitrary, the Theorem follows. ■

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