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A spectral sequence for de Rham cohomology

by

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1. Introduction. Let R be a complete discrete valuation ring with mixed characteristic, π a uniformizer of R, $\mathcal{S}$ the π-adic formal scheme $\mathrm{Spf}(R)$, k the residue field of R, and K the field of fractions of R. Let X be an algebraic scheme proper and strictly semi-stable over $\mathrm{Spec}(R)$ so that the generic fiber X_K of X is smooth. Let X_s be the special fiber of X, $\mathcal{X}$ the π-adic formal scheme associated to X, and $\mathcal{X}_K$ the Raynaud generic fiber of $\mathcal{X}$, which is a rigid space.

It is well known that the algebraic de Rham cohomology of X_K is naturally isomorphic to the analytic de Rham cohomology of $\mathcal{X}_K$. The purpose of this paper is to compare the (analytic) de Rham cohomology of $\mathcal{X}_K$ and the rigid cohomology of X_s.

Let Y_i ($1 \le i \le n$) be all irreducible components of X_s. For any nonempty subset I of $\{1, \ldots, n\}$, put $Y_I = \bigcap_{i\in I} Y_i$ and $U_I = Y_I \backslash \bigcup_{I' \supsetneq I} Y_{I'}$. For an integer $i \ge 0$, put $Y^{(i)} = \bigcup_{|I|=i} Y_I$. Let H^*_{rig} and $H^*_{c,\mathrm{rig}}$ denote the rigid cohomology and the rigid cohomology with proper support respectively. Then

$$H^*_{c,\mathrm{rig}}(Y^{(i)} \backslash Y^{(i+1)}) = \bigoplus_{|I|=i} H^*_{c,\mathrm{rig}}(U_I)$$

and we have the long exact sequence

$$\cdots \to H^m_{c,\mathrm{rig}}(Y^{(i)} \backslash Y^{(i+1)}) \to H^m_{\mathrm{rig}}(Y^{(i)}) \to H^m_{\mathrm{rig}}(Y^{(i+1)}) \to \cdots.$$

Let γ_I (resp. γ_i) denote the inclusion map

$$]Y_I \backslash U_I[_{\mathcal{X}} \hookrightarrow]Y_I[_{\mathcal{X}} \quad (\text{resp. }]Y^{(i+1)}[_{\mathcal{X}} \hookrightarrow]Y^{(i)}[_{\mathcal{X}}),$$

where $]\cdot[_{\mathcal{X}}$'s denote the tubes in $\mathcal{X}_K$. Let $\Omega^\cdot_{c,I;\mathcal{X}}$ and $\Omega^\cdot_{c,i;\mathcal{X}}$ be the total complexes of the bicomplexes

$$\Omega^\cdot_{]Y_I[_{\mathcal{X}}/K} \to \gamma_{I*}\Omega^\cdot_{]Y_I \backslash U_I[_{\mathcal{X}}/K} \quad \text{and} \quad \Omega^\cdot_{]Y^{(i)}[_{\mathcal{X}}/K} \to \gamma_{i*}\Omega^\cdot_{]Y^{(i+1)}[_{\mathcal{X}}/K}$$

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respectively. Then we have

$$H^*(]Y^{(i)}[_{\mathcal{X}}, \Omega^{\cdot}_{c,i;\mathcal{X}}) = \bigoplus_{|I|=i} H^*(]Y_I[_{\mathcal{X}}, \Omega^{\cdot}_{c,I;\mathcal{X}}).$$

(See Lemma 1.) The triangle

$$\Omega^{\cdot}_{c,i;\mathcal{X}} \to \Omega^{\cdot}_{]Y^{(i)}[_{\mathcal{X}}} \to \gamma_{i*}\Omega^{\cdot}_{]Y^{(i+1)}[_{\mathcal{X}}} \xrightarrow{+1}$$

in $D^+(]Y^{(i)}[_{\mathcal{X}})$ induces the long exact sequence

$$\cdots \to H^m(]Y^{(i)}[_{\mathcal{X}}, \Omega^{\cdot}_{c,i;\mathcal{X}}) \to H^m_{\mathrm{dR}}(]Y^{(i)}[_{\mathcal{X}}) \to H^m_{\mathrm{dR}}(]Y^{(i+1)}[_{\mathcal{X}}) \to \cdots . \tag{1.1}$$

The main result of this paper is the following theorem.

Theorem 1. *If I is a subset of $\{1,\dots,n\}$ such that $|I| \geq 2$, then there is a spectral sequence converging to $H^*(]Y_I[_{\mathcal{X}}, \Omega^{\cdot}_{c,I;\mathcal{X}})$ with*

$$E_2^{pq} = H^p_{c,\mathrm{rig}}(U_I/K) \otimes_K \bigwedge^q(V_I'),$$

where V_I' is a K-vector space of dimension $|I|-1$ defined in §3.

If $|I| = 1$, then it is well known that

$$H^*_{c,\mathrm{rig}}(U_I/K) = H^*(]Y_I[_{\mathcal{X}}, \Omega^{\cdot}_{c,I;\mathcal{X}}). \tag{1.2}$$

Put

$$\chi_{\mathrm{dR}}(\mathcal{X}_K) := \sum_{m\geq 0} (-1)^m \dim_K H^m_{\mathrm{dR}}(\mathcal{X}_K/K),$$
$$\chi_{\mathrm{rig}}(X_s) := \sum_{m\geq 0} (-1)^m \dim_K H^m_{\mathrm{rig}}(X_s/K).$$

As an application of Theorem 1, we obtain a description of $\chi_{\mathrm{rig}}(X_s) - \chi_{\mathrm{dR}}(\mathcal{X}_K)$ by the geometry of X.

Proposition 1. *We have*

$$\begin{aligned}\chi_{\mathrm{rig}}(X_s) - \chi_{\mathrm{dR}}(\mathcal{X}_K) &= \sum_{|I|\geq 2} \chi_c(U_I) \\ &= \sum_{|I|\geq 2} (-1)^{|I|}(|I|-1)(\triangle Y_I.\triangle Y_I),\end{aligned} \tag{1.3}$$

where $\chi_c(U_I)$ is the rigid Euler–Poincaré characteristic with proper support of U_I and $(\triangle Y_I.\triangle Y_I)$ is the self-intersection number of Y_I.

This paper is organized as follows. In §2 we recall the theory of rigid cohomology and provide some basic facts on de Rham cohomology. In §3 we present a result on relative de Rham complexes. Then in §4 we prove Theorem 1 by using the result of §3 and a generalization of Grothendieck's spectral sequence given in §4.1. Finally we prove Proposition 1.

Notation. Throughout this paper, a triangle of the form

$$\begin{tikzcd} & C \arrow[dl, "+1"'] & \\ A \arrow[rr] & & B \arrow[ul] \end{tikzcd}$$

is always denoted by

$$A \to B \to C \xrightarrow{+1} .$$

2. Rigid cohomology and de Rham cohomology

2.1. Rigid cohomology. We recall some basic facts about rigid cohomology developed by Berthelot.

Let X be a proper k-variety, U an open subset of X, and $Z = X\backslash U$. Assume that X admits a closed immersion into a smooth π-adic formal scheme $\mathcal{P}$ over R. As in [4, 5], we define tubes $]X[_{\mathcal{P}}$, $]U[_{\mathcal{P}}$ and $]Z[_{\mathcal{P}}$ in $\mathcal{P}_K$, which are also denoted by $]X[$, $]U[$ and $]Z[$ respectively if there is no confusion. We call an admissible open subset $V \subset\,]X[$ a *strict neighborhood* of $]U[$ in $]X[$ if the covering of $]X[$ by V and $]Z[$ is admissible. For any sheaf $\mathscr{E}$ on $]X[$, put

$$j^{\dagger}_{]U[}\mathscr{E} = \varinjlim_{V} j_{V*}j_V^{-1}\mathscr{E}$$

where V runs through all strict neighborhoods of $]U[$ in $]X[$ and j_V is the immersion $V \hookrightarrow\,]X[$. Then $H^*_{\mathrm{rig}}(U/K)$ is defined by

$$H^*_{\mathrm{rig}}(U/K) := H^*(]X[, j^{\dagger}_{]U[}\Omega^{\cdot}_{]X[/K}).$$

E. Grosse-Klönne [6] showed that $H^*_{\mathrm{rig}}(U/K)$ is a finite-dimensional K-vector space.

There also exists rigid cohomology with proper support defined in [3] as follows. Let α denote the inclusion map $]Z[\, \hookrightarrow\,]X[$ and let $\Omega^{\cdot}_{c,]U[/K}$ denote the total complex of the bicomplex

$$\Omega^{\cdot}_{]X[/K} \to \alpha_*\Omega^{\cdot}_{]Z[/K}.$$

The rigid cohomology $H^*_{c,\mathrm{rig}}(U/K)$ with proper support is defined by

$$H^*_{c,\mathrm{rig}}(U/K) := H^*(]X[, \Omega^{\cdot}_{c,]U[/K}) = H^*(X, \mathbb{R}\mathrm{sp}_*\Omega^{\cdot}_{c,]U[/K}),$$

where sp denotes the specialization map $]X[\, \to X$. If U is proper, then the canonical map

$$H^*_{c,\mathrm{rig}}(U/K) \to H^*_{\mathrm{rig}}(U/K)$$

is an isomorphism. One has a long exact sequence

$$\cdots \to H^i_{c,\mathrm{rig}}(U/K) \to H^i_{\mathrm{rig}}(X/K) \to H^i_{\mathrm{rig}}(Z/K) \to \cdots . \tag{2.1}$$

In general, X cannot always be embedded into a smooth formal scheme. In this case one can define the above cohomologies via the technique of "diagrams of topos". We recall the definition of the rigid cohomology with proper support.

We can always find an open covering $\{T_\nu\}$ of X and for each ν a closed imbedding $T_\nu \hookrightarrow \mathcal{P}_\nu$ in a smooth π-adic formal scheme. For a set of indices $\nu_0, \ldots, \nu_n$, there is a closed imbedding

$$T_{\nu_0\cdots\nu_n} := T_{\nu_0} \cap \cdots \cap T_{\nu_n} \hookrightarrow \mathcal{P}_{\nu_0\cdots\nu_n} := \mathcal{P}_{\nu_0} \times_{\mathcal{S}} \cdots \times_{\mathcal{S}} \mathcal{P}_{\nu_n}.$$

From now on, we will always denote $\times_{\mathcal{S}}$ by $\times$ for simplicity.

The $T_{\nu_0\cdots\nu_n}$'s form a diagram of topos $T.$ endowed with Zariski topology. There is a natural map $\epsilon : T. \to X_{\mathrm{Zar}}$. Let sp denote specialization maps, and i denote the closed immersions

$$Z \cap T_{\nu_0\cdots\nu_n} \hookrightarrow T_{\nu_0\cdots\nu_n}.$$

The bicomplexes of sheaves

$$\mathrm{sp}_* \Omega^\cdot_{]T_{\nu_0\cdots\nu_n}[_{\mathcal{P}_{\nu_0\cdots\nu_n}}/K} \to i_*\, \mathrm{sp}_* \Omega^\cdot_{]Z \cap T_{\nu_0\cdots\nu_n}[_{\mathcal{P}_{\nu_0\cdots\nu_n}}/K}$$

form a bicomplex of sheaves on $T.$. The total complex of this bicomplex is denoted by $\mathbb{R}\mathrm{sp}_* \Omega^\cdot_{c,]U[_{\mathcal{P}}/K}$. The rigid cohomology with proper support of U is defined by

$$H^*_{c,\mathrm{rig}}(U/K) := H^*(X, \mathbb{R}\epsilon_* \mathbb{R}\mathrm{sp}_* \Omega^\cdot_{c,]U[_{\mathcal{P}}/K}).$$

2.2. De Rham cohomology. We keep using the notation of §1.

Lemma 1. *We have*

$$H^*(]Y^{(i)}[_{\mathcal{X}}, \Omega^\cdot_{c,i;\mathcal{X}}) = \bigoplus_{|I|=i} H^*(]Y_I[_{\mathcal{X}}, \Omega^\cdot_{c,I;\mathcal{X}}). \tag{2.2}$$

Proof. Note that all of $]Y_I[_{\mathcal{X}}$ with $|I| = i$ form an admissible covering of $]Y^{(i)}[_{\mathcal{X}}$. Since the restriction of the complex $\Omega^\cdot_{c,i;\mathcal{X}}$ to $]Y^{(i+1)}[_{\mathcal{X}}$ is quasi-isomorphic to zero, for any distinct $I_1, \ldots, I_j$, $j \geq 2$, with $|I_1| = \cdots = |I_j| = i$, and any $k \geq 0$, we have

$$H^k(]Y_{I_1}[_{\mathcal{X}} \cap \cdots \cap]Y_{I_j}[_{\mathcal{X}}, \Omega^\cdot_{c,i;\mathcal{X}}) = 0.$$

From this and the theory of Čech cohomology we obtain

$$H^*(]Y^{(i)}[_{\mathcal{X}}, \Omega^\cdot_{c,i;\mathcal{X}}) = \bigoplus_{|I|=i} H^*(]Y_I[_{\mathcal{X}}, \Omega^\cdot_{c,i;\mathcal{X}}) = \bigoplus_{|I|=i} H^*(]Y_I[_{\mathcal{X}}, \Omega^\cdot_{c,I;\mathcal{X}}),$$

as desired. ∎

Assume that Y_I can be embedded into a smooth π-adic formal scheme $\mathcal{P}$. Put $\mathcal{Q} = \mathcal{X} \times \mathcal{P}$. The composition of $\triangle_{Y_I} : Y_I \hookrightarrow Y_I \times Y_I$ and $Y_I \times Y_I \hookrightarrow \mathcal{X} \times \mathcal{P}$ is a closed immersion $Y_I \hookrightarrow \mathcal{Q}$.

THEOREM 2 ([5, Theorem 1.4]). *Let Y be a k-scheme of finite type, $i : Y \to \mathcal{X}$ and $i' : Y \to \mathcal{Q}$ two closed immersions into π-adic formal schemes, and $u : \mathcal{Q} \to \mathcal{X}$ a morphism smooth in a neighborhood of Y such that $i = i' \circ u$. If the Raynaud generic fibers of $\mathcal{X}$ and $\mathcal{Q}$ are smooth, then the canonical homomorphism*

$$\Omega^{\cdot}_{]Y[_{\mathcal{X}}/K} \to \mathbb{R}u_{K*}\Omega^{\cdot}_{]Y[_{\mathcal{Q}}/K} \tag{2.3}$$

is an isomorphism.

Note that the assumption of this theorem is a little different from that of [5], but their proofs are the same.

Theorem 2 tells us that

$$H^*_{\mathrm{dR}}(]Y_I[_{\mathcal{X}}/K) = H^*_{\mathrm{dR}}(]Y_I[_{\mathcal{Q}}/K).$$

Let α_I denote the inclusion map

$$]Y_I\backslash U_I[_{\mathcal{Q}} \hookrightarrow]Y_I[_{\mathcal{Q}},$$

and $\Omega^{\cdot}_{c,I;\mathcal{Q}}$ the total complex of the bicomplex

$$\Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/K} \to \alpha_{I*}\Omega^{\cdot}_{]Y_I\backslash U_I[_{\mathcal{Q}}/K}.$$

PROPOSITION 2. *We have*

$$H^*(]Y_I[_{\mathcal{X}}, \Omega^{\cdot}_{c,I;\mathcal{X}}) = H^*(]Y_I[_{\mathcal{Q}}, \Omega^{\cdot}_{c,I;\mathcal{Q}}). \tag{2.4}$$

Proof. Let $Z = Y_I\backslash U_I$. By Theorem 2,

$$\Omega^{\cdot}_{]Y_I[_{\mathcal{X}}/K} \to \mathbb{R}u_{K*}\Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/K} \quad \text{and} \quad \Omega^{\cdot}_{]Z[_{\mathcal{X}}/K} \to \mathbb{R}u_{K*}\Omega^{\cdot}_{]Z[_{\mathcal{Q}}/K}$$

are isomorphisms. As γ_I and α_I are quasi-Stein, we have

$$\begin{aligned}\mathbb{R}u_{K*}\alpha_{I*}\Omega^{\cdot}_{]Z[_{\mathcal{Q}}/K} &= \mathbb{R}(u_{K*}\circ \alpha_{I*})\Omega^{\cdot}_{]Z[_{\mathcal{Q}}/K} = \mathbb{R}(\gamma_{I*}\circ u_{K*})\Omega^{\cdot}_{]Z[_{\mathcal{Q}}/K}\\ &= \mathbb{R}\gamma_{I*}\mathbb{R}u_{K*}\Omega^{\cdot}_{]Z[_{\mathcal{Q}}/K} = \mathbb{R}\gamma_{I*}\Omega^{\cdot}_{]Z[_{\mathcal{X}}/K} = \gamma_{I*}\Omega^{\cdot}_{]Z[_{\mathcal{X}}/K}.\end{aligned}$$

Hence we get an isomorphism $\Omega^{\cdot}_{c,I;\mathcal{X}} \to \mathbb{R}u_{K*}\Omega^{\cdot}_{c,I;\mathcal{Q}}$, as desired. ∎

We generalize the above proposition to the case that Y_I need not have an embedding in a smooth π-adic formal scheme.

Let $\{T_\nu\}$ be an open covering of Y_I such that for each ν there exists a closed imbedding $T_\nu \hookrightarrow \mathcal{P}_\nu$ in a smooth π-adic formal scheme. For a set of indices $\nu_0, \ldots, \nu_n$, put

$$T_{\nu_0\cdots\nu_n} := T_{\nu_0} \cap \cdots \cap T_{\nu_n}.$$

The $T_{\nu_0\cdots\nu_n}$'s form a diagram of Zariski topos $T.$ and there is a natural map $\epsilon : T. \to Y_I$. Put

$$\mathcal{P}_{\nu_0\cdots\nu_n} := \mathcal{P}_{\nu_0} \times \cdots \times \mathcal{P}_{\nu_n}, \quad \mathcal{Q}_{\nu_0\cdots\nu_n} := \mathcal{X} \times \mathcal{P}_{\nu_0\cdots\nu_n}.$$

Embed $T_{\nu_0\cdots\nu_n}$ into $\mathcal{P}_{\nu_0\cdots\nu_n}$ and $\mathcal{Q}_{\nu_0\cdots\nu_n}$ naturally.

Let α denote the inclusion maps

$$](Y_I\backslash U_I)\cap T_{\nu_0\cdots\nu_n}[_{\mathcal{Q}_{\nu_0\cdots\nu_n}} \hookrightarrow \ ]T_{\nu_0\cdots\nu_n}[_{\mathcal{Q}_{\nu_0\cdots\nu_n}}.$$

The bicomplexes of sheaves

$$\Omega^{\cdot}_{]T_{\nu_0\cdots\nu_n}[_{\mathcal{Q}_{\nu_0\cdots\nu_n}}/K} \to \alpha_* \ \Omega^{\cdot}_{](Y_I\backslash U_I)\cap T_{\nu_0\cdots\nu_n}[_{\mathcal{Q}_{\nu_0\cdots\nu_n}}/K}$$

form a bicomplex of sheaves on the diagram of rigid spaces $]T_{\nu_0\cdots\nu_n}[_{\mathcal{Q}_{\nu_0\cdots\nu_n}}$. The total complex of this bicomplex is denoted by $\Omega^{\cdot}_{c,I;\mathcal{Q}}$.

Lemma 2. *The natural map*

$$\mathbb{R}\mathrm{sp}_*\Omega^{\cdot}_{c,I;\mathcal{X}} \to \mathbb{R}\epsilon_*\mathbb{R}\mathrm{sp}_*\Omega^{\cdot}_{c,I;\mathcal{Q}}$$

is an isomorphism.

Proof. From the proof of Proposition 2 we see that $\mathbb{R}\mathrm{sp}_*\Omega^{\cdot}_{c,I;\mathcal{Q}}$ is isomorphic to $\epsilon^*\mathbb{R}\mathrm{sp}_*\Omega^{\cdot}_{c,I;\mathcal{X}}$. On the other hand, cohomological descent holds for ϵ ([2]), so

$$\mathbb{R}\epsilon_*\mathbb{R}\mathrm{sp}_*\Omega^{\cdot}_{c,I;\mathcal{Q}} = \mathbb{R}\epsilon_*\epsilon^*\mathbb{R}\mathrm{sp}_*\Omega^{\cdot}_{c,I;\mathcal{X}} = \mathbb{R}\mathrm{sp}_*\Omega^{\cdot}_{c,I;\mathcal{X}},$$

as expected. ∎

Corollary 1. *We have*

$$H^*(]Y_I[_{\mathcal{X}}, \Omega^{\cdot}_{c,I;\mathcal{X}}) = H^*(Y_I, \mathbb{R}\epsilon_*\mathbb{R}\mathrm{sp}_*\Omega^{\cdot}_{c,I;\mathcal{Q}}).$$

3. Relative differentials. Let X, X_s, $\mathcal{X}$, Y_j and Y_I be as in §1. Here, we do not assume that X is proper but assume that X_s can be embedded into a smooth π-adic formal scheme $\mathcal{P}$. Put $\mathcal{Q} = \mathcal{X}\times\mathcal{P}$. Let p_1 and p_2 be the projections from $\mathcal{Q}_K$ to $\mathcal{X}_K$ and $\mathcal{P}_K$ respectively.

For every irreducible component Y_j of X_s, we associate with Y_j a section s_{Y_j} of $\mathscr{H}^1(\Omega^{\cdot}_{\mathcal{Q}_K/\mathcal{P}_K})$ in §3.1. For any nonempty subset I of $\{1,\ldots,n\}$, let V_I be the K-vector space of dimension $|I|$ generated by $\{s_{Y_j} : j\in I\}$, and V_I' the quotient space of V_I modulo the subspace $K\sum_{j\in I} s_{Y_j}$.

Let α_I and β_I be the inclusion maps

$$\alpha_I :]Y_I\backslash U_I[_{\mathcal{Q}} \hookrightarrow]Y_I[_{\mathcal{Q}} \quad\text{and}\quad \beta_I :]Y_I\backslash U_I[_{\mathcal{P}} \hookrightarrow]Y_I[_{\mathcal{P}}.$$

Proposition 3. *If $|I|\ge 2$, then for any integer $i\ge 0$ we have*

$$\begin{aligned}(3.1)\qquad (\mathscr{O}_{]Y_I[_{\mathcal{P}}}\otimes_K {\textstyle\bigwedge}^i(V_I') \to \beta_{I*}\beta_I^{-1}\mathscr{O}_{]Y_I[_{\mathcal{P}}}\otimes_K {\textstyle\bigwedge}^i(V_I'))\\ = (R^ip_{2*}\Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}} \to R^ip_{2*}(\alpha_{I*}\alpha_I^{-1}\Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}}))\end{aligned}$$

in the derived category $D^+(]Y_I[_{\mathcal{P}})$.

The proof will be given in §3.3.

3.1. Definition of s_{Y_j}. Let $Y = Y_j$ be an irreducible component of X_s. If f is a local equation defining Y in $\mathcal{X}$, then f divides π. Thus f is invertible in the structure sheaf of $\mathcal{X}_K$ and $\frac{df}{f}$ is a local section of $\Omega^1_{\mathcal{X}_K/K}$. We use d_1 to denote the differential of $\mathcal{Q}_K$ relative to $\mathcal{P}_K$. Then

$$p_1^* \frac{df}{f} = \frac{d_1(p_1^* f)}{p_1^* f},$$

which is denoted as $\frac{d_1 f}{f}$ for simplicity. In general, $\frac{d_1 f}{f}$ depends on the choice of f.

Proposition 4. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be open subsets of $\mathcal{X}$. Let $f \in \Gamma(\mathcal{X}_1, \mathcal{O}_{\mathcal{X}})$ and $g \in \Gamma(\mathcal{X}_2, \mathcal{O}_{\mathcal{X}})$ be regular elements defining $Y \cap \mathcal{X}_1$ and $Y \cap \mathcal{X}_2$ respectively. Then on the tube of $\mathcal{X}_{1s} \cap \mathcal{X}_{2s}$ in $\mathcal{Q}_K$, we have*

$$\frac{d_1 f}{f} \equiv \frac{d_1 g}{g} \ \textit{modulo}\ d_1 \mathcal{O}_{\mathcal{Q}_K}. \tag{3.2}$$

This proposition says that the image of $\frac{d_1 f}{f}$ in $\mathscr{H}^1(\Omega^{\cdot}_{\mathcal{Q}_K/\mathcal{P}_K})$ does not depend on the choice of f, which is denoted by $s_{Y,\mathcal{P}}$.

Let $i_1 : X_s \hookrightarrow \mathcal{P}_1$ and $i_2 : X_s \hookrightarrow \mathcal{P}_2$ be closed immersions into smooth π-adic formal schemes, and u a morphism $\mathcal{P}_2 \to \mathcal{P}_1$ such that $i_1 = u \circ i_2$. Then $u^* s_{Y,\mathcal{P}_1} = s_{Y,\mathcal{P}_2}$. In other words, $\{s_{Y,\mathcal{P}}\}_{\mathcal{P}}$'s form a compatible system. We will use s_Y to denote $s_{Y,\mathcal{P}}$.

Let $\mathcal{Q}'$ be the completion of $\mathcal{Q} = \mathcal{X} \times \mathcal{P}$ along X_s. In general, $\mathcal{Q}'$ is not a π-adic formal scheme, but it can also be associated with a rigid space $\mathcal{Q}'_K$ as its generic fiber. Locally we can write $\mathcal{Q}' = \mathrm{Spf}(A)$ with the ideal of definition generated by $f_1, \ldots, f_r \in A$. Put

$$B_m = A\langle T_1, \ldots, T_r\rangle/(f_1^m - \pi T_1, \ldots, f_r^m - \pi T_r). \tag{3.3}$$

If $m' \geq m$, then there is an inclusion map

$$\mathrm{Spm}(B_m \otimes_R K) \hookrightarrow \mathrm{Spm}(B_{m'} \otimes_R K)$$

defined by the canonical homomorphism $B_{m'} \to B_m$. Berthelot [4] defined $\mathcal{Q}'_K$ to be the union of $\mathrm{Spm}(B_m \otimes_R K)$'s and showed that $\mathcal{Q}'_K$ is just the tube of X_s in $\mathcal{X}_K \times \mathcal{P}_K$.

Proof of Proposition 4. We may assume that $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}$. Since the question is local, it suffices to consider the case of $\mathcal{X}$ and $\mathcal{P}$ being affine, say $\mathcal{X} = \mathrm{Spf}(A_1)$ and $\mathcal{P} = \mathrm{Spf}(A_2)$.

Let $\varphi : A_2 \to A_{1k}$ be the homomorphism defining the embedding $X_s \hookrightarrow \mathcal{P}$. Let $\mathcal{I}$ be the kernel of the homomorphism

$$A_1 \otimes_R A_2 \to A_{1k}.$$

Let $f_1, \ldots, f_r$ be generators of $\mathcal{I}$. If A is the $\mathcal{I}$-adic completion of $A_1 \otimes_R A_2$ and B_m's are the R-algebras defined by (3.3), then $\mathcal{Q}' = \mathrm{Spf}(A)$ and $\mathcal{Q}'_K$ is the union of $\mathrm{Spm}(B_m \otimes_R K)$'s.

It remains to find some $h_m \in B_m \otimes_R K$ for every m such that
$$\frac{d_1 f}{f} - \frac{d_1 g}{g} = d_1 h_m.$$
As φ is surjective, there is some $u \in A_2$ such that $\varphi(u)$ is equal to the reduction of $f^{-1}g$. Let $v := g^{-1}fu \in A$. Then
$$\frac{d_1 f}{f} = \frac{d_1 f}{f} + \frac{d_1 u}{u} = \frac{d_1 g}{g} + \frac{d_1 v}{v}.$$
As $v \in 1 + \mathcal{I}$, the series
$$h_m := \log(v) = \sum_{i=1}^{+\infty} (-1)^{i-1} \frac{(v-1)^i}{i}$$
belongs to $B_m \otimes_R K$. Thus $\frac{d_1 v}{v} = d_1 h_m$, as expected. ■

3.2. A lemma. Let $m \le r$ be positive integers. Let $\mathrm{D}(0,1)^r$ be the affinoid rigid space $\mathrm{Spm}(K\langle T_1, \ldots, T_r\rangle)$, $\mathrm{D}(0,1^-)^r$ the subdomain of $\mathrm{D}(0,1)^r$ defined by
$$|T_1| < 1, \ \ldots, \ |T_r| < 1,$$
and D the subdomain defined by
$$|T_1| < 1, \ \ldots, \ |T_r| < 1, \qquad \pi < |T_1 \cdots T_m|.$$

For a rigid space Z, let $\Omega^{\cdot}_{D\times Z/Z}$ denote the relative de Rham complex of $D \times Z$ over Z, and V the subspace of $\Gamma(D \times Z, \Omega^1_{D\times Z/Z})$ defined as
$$V := K\frac{dT_1}{T_1} \oplus \cdots \oplus K\frac{dT_m}{T_m}.$$

LEMMA 3. *In the above notation, let p_2 denote the projection $D \times Z \to Z$. Then*
$$R^i p_{2*} \Omega^{\cdot}_{D\times Z/Z} = \mathscr{O}_Z \otimes_K \textstyle\bigwedge^i(V). \tag{3.4}$$

Proof. It suffices to show that for any affinoid open subset $W = \mathrm{Spm}(B)$ of Z,
$$H^i(D \times W, \Omega^{\cdot}_{D\times W/W}) = B \otimes_K \textstyle\bigwedge^i(V).$$
As $D \times W$ is quasi-Stein, $H^i(D \times W, \Omega^{\cdot}_{D\times W/W})$ is the ith cohomology of the complex $\Gamma(D \times W, \Omega^{\cdot}_{D\times W/W})$. For any $0 \le i \le r$ put
$$\Gamma^i = \Gamma(D \times W, \Omega^i_{D\times W/W}).$$
Then $\Gamma^i = \Gamma^0 \otimes_K \bigwedge^i(\tilde{V})$, where
$$\tilde{V} = KdT_1 \oplus \cdots \oplus KdT_r.$$
Let $Z^i \subset \Gamma^i$ be the space of closed i-forms.

A formal series

$$\sum_{\substack{t_1,\dots,t_m\\ t_{m+1}\ge 0,\dots,t_r\ge 0}} b_{t_1,\dots,t_r} T_1^{t_1}\cdots T_r^{t_r}$$

with coefficients in B belongs to Γ^0 if and only if for any given $\epsilon > 0$ and $0<\rho<1$ almost all of the following relations hold:

$$\tag{3.5} \begin{aligned} |b_{t_1,\dots,t_r}|_B\cdot\rho^{t_1+\dots+t_r} &< \epsilon \quad \text{if } \min(t_1,\dots,t_r)\ge 0,\\ |b_{t_1,\dots,t_r}|_B\cdot\rho^{t_1+\dots+t_r-(m+1)N}|\pi|^N &< \epsilon \quad \text{if } \min(t_1,\dots,t_r)=N<0, \end{aligned}$$

where $|\cdot|_B$ is a norm on B.

Every element ω in Γ^i can be written as a formal sum of monomials

$$b_{\gamma,I}T^\gamma dT_I = b_{\gamma,I}T_1^{t_1}\cdots T_r^{t_r}dT_{l_1}\wedge\dots\wedge dT_{l_i},$$

where $b_{\gamma,I}\in B$, $\gamma=(t_1,\dots,t_r)$ and $I=\{l_1,\dots,l_i\}\subseteq\{1,\dots,r\}$ with $l_1<\dots<l_i$. We associate with any monomial $b_{\gamma,I}T^\gamma dT_I$ a number

$$n_\delta(b_{\gamma,I}T^\gamma dT_I) := \#(\{l\in I : t_l\ne -1\}\cup\{l\notin I : 1\le l\le r,\, t_l\ne 0\}),$$

which satisfies

$$0\le n_\delta(b_{\gamma,I}T^\gamma dT_I)\le r.$$

We call this number the δ-*number* of $b_{\gamma,I}T^\gamma dT_I$. Let Γ^i_j be the subspace of Γ^i consisting of i-forms which are formal sums of monomials with δ-number j. Then $\Gamma^i=\bigoplus_{j=0}^r\Gamma^i_j$. Put $Z^i_j=Z^i\cap\Gamma^i_j$. Note that $\Gamma^i_0=Z^i_0=B\otimes_K\bigwedge^i(V)$. If $\omega\in\Gamma^i_j$, then $d\omega\in\Gamma^{i+1}_j$. Thus $Z^i=\bigoplus_{j=0}^r Z^i_j$.

Put

$$\begin{aligned}\delta(T^\gamma dT_I) := \sum_{\substack{1\le\mu\le i\\ t_{l_\mu}\ne -1}}(-1)^{\mu-1}\frac{1}{t_{l_\mu}+1}T_{l_\mu}\cdot T^\gamma&\\ \times\, dT_{l_1}\wedge\dots\wedge dT_{l_{\mu-1}}\wedge dT_{l_{\mu+1}}\wedge\dots\wedge dT_{l_i}&.\end{aligned}$$

By (3.5) we can extend δ to a continuous B-linear map $\delta:\Gamma^i\to\Gamma^{i-1}$. It is easy to check that, if $\omega\in\Gamma^i_j$, then

$$(d\delta+\delta d)\omega = j\omega.$$

In other words, we have $\bigoplus_{j=1}^r Z^i_j\subset d\Gamma^{i-1}$. Since $Z^i_0\cap d\Gamma^{i-1}=0$, we have

$$Z^i/d\Gamma^{i-1}\cong Z^i_0 = B\otimes_K\textstyle\bigwedge^i(V). \ \blacksquare$$

3.3. Proof of Proposition 3. Let I be a nonempty subset of $\{1,\dots,n\}$, and I_1 a subset of I such that $|I_1|=|I|-1$. As $R^ip_{2*}\Omega^{\cdot}_{]Y_I[_{\mathcal Q}/]Y_I[_{\mathcal P}}$ is the sheaf associated to the presheaf

$$W\mapsto H^i(p_2^{-1}(W),\Omega^{\cdot}_{]Y_I[_{\mathcal Q}/]Y_I[_{\mathcal P}}),$$

where W's are admissible open subsets of $]Y_I[_{\mathcal{P}}$, there is a canonical map

$$\mathscr{O}_{]Y_I[_{\mathcal{P}}} \otimes_K \bigwedge^i(V_{I_1}) \to R^i p_{2*}\Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}}. \tag{3.6}$$

Proposition 5. *Under the above map, we have*

$$\mathscr{O}_{]U_I[_{\mathcal{P}}} \otimes_K \bigwedge^i(V_{I_1}) = R^i p_{2*}\Omega^{\cdot}_{]U_I[_{\mathcal{Q}}/]U_I[_{\mathcal{P}}}. \tag{3.7}$$

Proposition 6. *If $|I| \geq 2$, then the homomorphism of complexes*

$$\begin{aligned}(\mathscr{O}_{]Y_I[_{\mathcal{P}}} \otimes_K \bigwedge^i(V_{I_1}) \to \beta_{I*}\beta_I^{-1}\mathscr{O}_{]Y_I[_{\mathcal{P}}} \otimes_K \bigwedge^i(V_{I_1}))\\ \to (R^i p_{2*}\Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}} \to R^i p_{2*}(\alpha_{I*}\alpha_I^{-1}\Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}}))\end{aligned} \tag{3.8}$$

is a quasi-isomorphism.

Proposition 3 follows immediately from Proposition 6.

For the proofs of Propositions 5 and 6, we assume that $I = \{1, \ldots, m\}$, where $1 \leq m \leq n$. The questions are local, so we may assume that

- $\mathcal{X}$ and $\mathcal{P}$ are affine, say $\mathcal{X} = \mathrm{Spf}(A_1)$ and $\mathcal{P} = \mathrm{Spf}(A_2)$,
- there is an étale morphism $\theta : \mathcal{X} \to \mathcal{X}_0 = \mathrm{Spf}(A_0)$ of π-adic formal schemes over R, where $A_0 = R\langle T_1, \ldots, T_d\rangle/(T_1 \cdots T_q - \pi)$ with $m \leq q \leq \min(d, n)$,
- Y_i ($1 \leq i \leq q$) is defined by $\varphi(T_i)$, where $\varphi : A_0 \to A_1$ is the R-algebra homomorphism defining θ.

Here a morphism θ of π-adic formal schemes is called *étale* if $\theta \otimes_R R/\pi^i R$ ($i \geq 1$) are all étale (cf. [1]).

Proof of Proposition 5. The composition of $Y_I \hookrightarrow \mathcal{X} \times \mathcal{P}$ and $\theta \times \mathrm{id}_{\mathcal{P}}$ is an inclusion map $Y_I \hookrightarrow \mathcal{X}_0 \times \mathcal{P}$. As $\theta \times \mathrm{id}_{\mathcal{P}}$ is étale, the tube of Y_I in $\mathcal{Q}_K = \mathcal{X}_K \times \mathcal{P}_K$ is isomorphic to the tube of Y_I in $\mathcal{X}_{0K} \times \mathcal{P}_K$, i.e.,

$$]Y_I[_{\mathcal{Q}} \cong]Y_I[_{\mathcal{X}_0 \times \mathcal{P}}.$$

Let $\mathcal{X}_{0s}$ be the special fiber of $\mathcal{X}_0$ and put $Z = \mathcal{X}_{0s} \times Y_I$. Consider the diagram

$$\begin{tikzcd} & Z \arrow[r] \arrow[d] & \mathcal{X}_0 \times \mathcal{P} \arrow[d] \\ Y_I \arrow[ur] \arrow[r, "\mathrm{id}"] & Y_I \arrow[r] & \mathcal{P} \end{tikzcd}$$

where the square is cartesian. Let t_i ($m+1 \leq i \leq d$) be elements of A_2 such that $\phi(t_i)$ is equal to $\varphi(T_i) \bmod \pi$, where $\phi : A_2 \to A_{1k}$ is the algebra homomorphism associated to the embedding $X_s \hookrightarrow \mathcal{P}$. Then the morphism $Y_I \hookrightarrow Z$ in the above diagram is a closed immersion defined by the images of $T_1, \ldots, T_m, T_{m+1} - t_{m+1}, \ldots, T_d - t_d$ in $\Gamma(Z, \mathscr{O})$. Thus $]Y_I[_{\mathcal{X}_0 \times \mathcal{P}}$ is the intersection of $]Z[_{\mathcal{X}_0 \times \mathcal{P}} = \mathcal{X}_{0K} \times]Y_I[_{\mathcal{P}}$ and the subdomain of $\mathcal{X}_{0K} \times \mathcal{P}_K$

defined by

$$|T_1| < 1, \dots, |T_m| < 1, \quad |T_{m+1} - t_{m+1}| < 1, \dots, |T_d - t_d| < 1.$$

Consider the homomorphism

$$R\langle T'_1, \dots, T'_d\rangle \to \Gamma(\mathcal{X}_0 \times \mathcal{P}, \mathscr{O}),$$
$$T'_1, \dots, T'_d \mapsto T_1, \dots, T_m, T_{m+1} - t_{m+1}, \dots, T_d - t_d,$$

which defines a morphism $\mathcal{X}_0 \times \mathcal{P} \to \mathbb{A}^d$, where $\mathbb{A}^d$ is the π-adic formal scheme $\mathrm{Spf}(R\langle T'_1, \dots, T'_d\rangle)$. Combining this morphism with the projection $\mathcal{X}_0 \times \mathcal{P} \to \mathcal{P}$ we obtain a closed immersion

$$\mathcal{X}_0 \times \mathcal{P} \hookrightarrow \mathbb{A}^d \times \mathcal{P}, \tag{3.9}$$

which is defined by

$$T'_1 \cdots T'_m (T'_{m+1} + t_{m+1}) \cdots (T'_q + t_q) - \pi.$$

Let D be the subdomain of $\mathrm{D}(0,1)^{m-1} = \mathrm{Spm}(K\langle T'_1, \dots, T'_{m-1}\rangle)$ defined by

$$|T'_1| < 1, \ \dots, \ |T'_{m-1}| < 1 \quad \text{and} \quad |\pi| < |T'_1 \cdots T'_{m-1}|.$$

Then (3.9) induces an inclusion map

$$\iota :]Y_I[_{\mathcal{X}_0 \times \mathcal{P}} \hookrightarrow D \times \mathrm{D}(0,1^-)^{d-m} \times]Y_I[_{\mathcal{P}}$$

and an isomorphism

$$]U_I[_{\mathcal{X}_0 \times \mathcal{P}} \xrightarrow{\sim} D \times \mathrm{D}(0,1^-)^{d-m} \times]U_I[_{\mathcal{P}}. \tag{3.10}$$

Now the validity of Proposition 5 is ensured by Lemma 3. ∎

The proof of Proposition 6 needs the following lemma.

Lemma 4. *The map* (3.6) *is an injection.*

Proof. Let W be an affinoid open subset of $]Y_I[_{\mathcal{P}}$. By Proposition 5 we see that, if $I \subseteq I' \subseteq \{1, \dots, n\}$, then the map

$$\Gamma(W \cap]U_{I'}[_{\mathcal{P}}, \mathscr{O}_{]Y_I[_{\mathcal{P}}} \otimes_K \textstyle\bigwedge^i (V_{I_1})) \to \Gamma(W \cap]U_{I'}[_{\mathcal{P}}, R^i p_{2*} \Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}})$$

is injective. On the other hand, the map

$$\Gamma(W, \mathscr{O}_{]Y_I[_{\mathcal{P}}}) \to \prod_{I' \supseteq I} \Gamma(W \cap]U_{I'}[_{\mathcal{P}}, \mathscr{O}_{]Y_I[_{\mathcal{P}}})$$

is also injective. Hence (3.6) is an injection. ∎

Proof of Proposition 6. We keep the notation of the proof of Proposition 5.

We identify $]Y_I[_{\mathcal{X}_0 \times \mathcal{P}}$ with a subset of $D \times \mathrm{D}(0,1^-)^{d-m} \times]Y_I[_{\mathcal{P}}$ via ι. Let q_2 be the projection

$$q_2 : D \times \mathrm{D}(0,1^-)^{d-m} \times]Y_I[_{\mathcal{P}} \to]Y_I[_{\mathcal{P}},$$

and α'_I the inclusion map
$$\alpha'_I : q_2^{-1}(]Y_I\backslash U_I[_{\mathcal{P}}) \to q_2^{-1}(]Y_I[_{\mathcal{P}}).$$
Let $\Omega^{\cdot}_{c,]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}}$ and $\Omega^{\cdot}_{c,q_2^{-1}(]Y_I[_{\mathcal{P}})/]Y_I[_{\mathcal{P}}}$ denote the total complexes of the bicomplexes
$$\Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}} \to \alpha_{I*}\alpha_I^{-1}\Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}}$$
and
$$\Omega^{\cdot}_{q_2^{-1}(]Y_I[_{\mathcal{P}})/]Y_I[_{\mathcal{P}}} \to \alpha'_{I*}\alpha'^{-1}_I\Omega^{\cdot}_{q_2^{-1}(]Y_I[_{\mathcal{P}})/]Y_I[_{\mathcal{P}}}$$
respectively.

LEMMA 5. *$]Y_I[_{\mathcal{X}_0\times\mathcal{P}}$ and $q_2^{-1}(]Y_I\backslash U_I[_{\mathcal{P}})$ form an admissible covering of $D\times \mathrm{D}(0,1^-)^{d-m}\times\,]Y_I[_{\mathcal{P}}$.*

The following proof is due to the referee.

Proof. The isomorphism (3.10) ensures that $]Y_I[_{\mathcal{X}_0\times\mathcal{P}}$ and $q_2^{-1}(]Y_I\backslash U_I[_{\mathcal{P}})$ indeed form a covering of $D\times \mathrm{D}(0,1)^{d-m}\times]Y_I[_{\mathcal{P}}$. To prove that the covering is admissible, we may assume that $\mathcal{X}_0$ and $\mathcal{P}$ are affine, since the question is local.

Write $Z_I = Y_I\backslash U_I$ and $M = D\times \mathrm{D}(0,1^-)^{d-m}\times\,]Y_I[_{\mathcal{P}}$. By the definition of an admissible covering, it suffices to prove that, for any affinoid rigid analytic space W and any morphism of rigid spaces $u : W\to M$, the covering $\{u^{-1}(]Y_I[_{\mathcal{X}_0\times\mathcal{P}}), u^{-1}(q_2^{-1}(]Z_I[_{\mathcal{P}}))\}$ can be refined by a finite covering by affinoid open subspaces. Denote by the same letters the pullbacks by u of functions on M. Note that, as a closed subscheme of Y_I, Z_I is defined by the restriction of $t_{m+1}\cdots t_q$ to Y_I. Hence $]Z_I[_{\mathcal{P}}$ is the open subspace of $]Y_I[_{\mathcal{P}}$ defined by the condition $|t_{m+1}\cdots t_q| < 1$. For any $\lambda < 1$, let $V_\lambda\subset M$ be the open subset defined by $|t_{m+1}\cdots t_q|\le\lambda$. For any $\eta<1$, let $[Y_I]_{\mathcal{X}_0\times\mathcal{P},\eta}$ be the closed tube of radius η for Y_I in $\mathcal{X}_0\times\mathcal{P}$, viewed via ι as a subspace of M; $[Y_I]_{\mathcal{X}_0\times\mathcal{P},\eta}$ is the open subset of $D\times \mathrm{D}(0,1^-)^{d-m}\times[Y_I]_{\mathcal{P},\eta}$ described by the inequalities:

$$|T'_i|\le\eta \quad \text{for } i\le m-1 \text{ and } m+1<i\le d, \tag{3.11}$$
$$|T'_1\cdots T'_{m-1}(T'_{m+1}+t_{m+1})\cdots(T'_q+t_q)|\ge|\pi|/\eta. \tag{3.12}$$

If some integral powers of λ and η belong to the multiplicative group of absolute values of $K^\times$, then $u^{-1}([Y_I]_{\mathcal{X}_0\times\mathcal{P},\eta})$ and $u^{-1}(V_\lambda)$ are affinoid open subsets of W. So it suffices to check that their union is equal to W for λ, η close enough to 1.

Since W is affinoid, the maximum modulus principle implies that there exists $\rho<1$ such that the inequalities
$$|T'_i|\le\rho \quad \text{for } i\le m-1 \text{ and } m+1<i\le d$$

and
$$|T'_1 \cdots T'_{m-1}| \geq |\pi|/\rho$$
are satisfied on W. Let λ be such that $\rho < \lambda < 1$. Let $x \in W$ be a point which is not in $u^{-1}(V_\lambda)$. Then $|(t_{m+1} \cdots t_q)(x)| > \lambda$. As $|t_i(x)| \leq 1$ for all i, it follows that $|t_i(x)| > \lambda$ for $m+1 \leq i \leq q$. Therefore $|(T'_i + t_i)(x)| = |t_i(x)| > \lambda$ for $m+1 \leq i \leq q$. We obtain
$$|(T'_1 \cdots T'_{m-1}(T'_{m+1} + t_{m+1}) \cdots (T'_q + t_q))(x)| > \frac{|\pi|}{\rho}\lambda^{q-m}.$$
We can choose λ close enough to 1 such that $\rho < \lambda^{q-m}$ and take $\eta = \rho/\lambda^{q-m} \geq \rho$. Then inequalities (3.11) and (3.12) are satisfied at x, and it follows that $W = u^{-1}([Y_I]_{\mathcal{X}_0 \times \mathcal{P}, \eta}) \cup u^{-1}(V_\lambda)$. ∎

Lemma 6. *We have*
$$R^i p_{2*} \Omega^{\cdot}_{c,]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}} = R^i q_{2*} \Omega^{\cdot}_{c, q_2^{-1}(]Y_I[_{\mathcal{P}})/]Y_I[_{\mathcal{P}}}.$$

Proof. Let W be an admissible open subset of $]Y_I[_{\mathcal{P}}$. By Lemma 5, $p_2^{-1}(W) = q_2^{-1}(W) \cap]Y_I[_{\mathcal{X}_0 \times \mathcal{P}}$ and $q_2^{-1}(W \cap\ ]Y_I \backslash U_I[_{\mathcal{P}})$ form an admissible covering of $q_2^{-1}(W)$. Since the restriction of $\Omega^{\cdot}_{c, q_2^{-1}(]Y_I[_{\mathcal{P}})/]Y_I[_{\mathcal{P}}}$ to $q_2^{-1}(W \cap\ ]Y_I \backslash U_I[_{\mathcal{P}})$ is quasi-isomorphic to zero, we have
$$\begin{aligned} H^i(q_2^{-1}(W), \Omega^{\cdot}_{c, q_2^{-1}(]Y_I[_{\mathcal{P}})/]Y_I[_{\mathcal{P}}}) &= H^i(p_2^{-1}(W), \Omega^{\cdot}_{c, q_2^{-1}(]Y_I[_{\mathcal{P}})/]Y_I[_{\mathcal{P}}}) \\ &= H^i(p_2^{-1}(W), \Omega^{\cdot}_{c,]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}}), \end{aligned}$$
as expected. ∎

Since α'_I and β_I are quasi-Stein, we have
$$\begin{aligned} R^i q_{2*}(\alpha'_{I*}\alpha'^{-1}_I \Omega^{\cdot}_{q_2^{-1}(]Y_I[_{\mathcal{P}})/]Y_I[_{\mathcal{P}}}) &= R^i(q_{2*} \circ \alpha'_{I*}) \Omega^{\cdot}_{q_2^{-1}(]Y_I \backslash U_I[_{\mathcal{P}})/]Y_I \backslash U_I[_{\mathcal{P}}} \\ &= R^i(\beta_{I*} \circ q_{2*}) \Omega^{\cdot}_{q_2^{-1}(]Y_I \backslash U_I[_{\mathcal{P}})/]Y_I \backslash U_I[_{\mathcal{P}}} \\ &= \beta_{I*} \mathcal{O}_{]Y_I \backslash U_I[_{\mathcal{P}}} \otimes_K \textstyle\bigwedge^i(V_{I_1}) \quad \text{(by Lemma 3)} \\ &= \beta_{I*}\beta_I^{-1} \mathcal{O}_{]Y_I[_{\mathcal{P}}} \otimes_K \textstyle\bigwedge^i(V_{I_1}). \end{aligned}$$
Here, the projection $q_2^{-1}(]Y_I \backslash U_I[_{\mathcal{P}}) \to\ ]Y_I \backslash U_I[_{\mathcal{P}}$ is also denoted by q_2. Again by Lemma 3 we have
$$R^i q_{2*} \Omega^{\cdot}_{q_2^{-1}(]Y_I[_{\mathcal{P}})/]Y_I[_{\mathcal{P}}} = \mathcal{O}_{]Y_I[_{\mathcal{P}}} \otimes_K \textstyle\bigwedge^i(V_{I_1}).$$
Thus from the distinguished triangles
$$\Omega^{\cdot}_{c,]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}} \to \Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}} \to \alpha_{I*}\alpha_I^{-1} \Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}} \xrightarrow{+1}$$
and
$$\Omega^{\cdot}_{c, q_2^{-1}(]Y_I[_{\mathcal{P}})/]Y_I[_{\mathcal{P}}} \to \Omega^{\cdot}_{q_2^{-1}(]Y_I[_{\mathcal{P}})/]Y_I[_{\mathcal{P}}} \to \alpha'_{I*}\alpha'^{-1}_I \Omega^{\cdot}_{q_2^{-1}(]Y_I[_{\mathcal{P}})/]Y_I[_{\mathcal{P}}} \xrightarrow{+1}$$

we get a commutative diagram of exact sequences

$$\begin{tikzcd}
R^i q_{2*}\Omega^{\cdot}_{c,q_2^{-1}(]Y_I[_{\mathcal{P}})/]Y_I[_{\mathcal{P}}} \arrow[r] \arrow[d, equal] & \mathcal{O}_{]Y_I[_{\mathcal{P}}} \otimes_K \bigwedge^i(V_{I_1}) \arrow[r] \arrow[d, "u"] & \beta_{I*}\beta_I^{-1}\mathcal{O}_{]Y_I[_{\mathcal{P}}} \otimes_K \bigwedge^i(V_{I_1}) \arrow[d] \\
R^i p_{2*}\Omega^{\cdot}_{c,]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}} \arrow[r] & R^i p_{2*}\Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}} \arrow[r] & R^i p_{2*}(\alpha_{I*}\alpha_I^{-1}\Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}})
\end{tikzcd}$$

The map (3.8) is just given by the right square of this diagram.

Let $\ker_i$ and cok_i be the kernel and cokernel of

$$\mathcal{O}_{]Y_I[_{\mathcal{P}}} \otimes_K \bigwedge^i(V_{I_1}) \to \beta_{I*}\beta_I^{-1}\mathcal{O}_{]Y_I[_{\mathcal{P}}} \otimes_K \bigwedge^i(V_{I_1}),$$

and $\ker'_i$ and cok'_i the kernel and cokernel of

$$R^i p_{2*}\Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}} \to R^i p_{2*}(\alpha_{I*}\alpha_I^{-1}\Omega^{\cdot}_{]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}}).$$

The map (3.8) induces two maps $\ker_i \to \ker'_i$ and $\mathrm{cok}_i \to \mathrm{cok}'_i$. From the above commutative diagram we see that $\ker_i \to \ker'_i$ is surjective. By Lemma 4, u is injective, and so is $\ker_i \to \ker'_i$. Thus $\ker_i \to \ker'_i$ is an isomorphism. From the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{cok}_i \arrow[r] \arrow[d] & R^{i+1}q_{2*}\Omega^{\cdot}_{c,q_2^{-1}(]Y_I[_{\mathcal{P}})/]Y_I[_{\mathcal{P}}} \arrow[r] \arrow[d, equal] & \ker_{i+1} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{cok}'_i \arrow[r] & R^{i+1}p_{2*}\Omega^{\cdot}_{c,]Y_I[_{\mathcal{Q}}/]Y_I[_{\mathcal{P}}} \arrow[r] & \ker'_{i+1} \arrow[r] & 0
\end{tikzcd}$$

we see that $\mathrm{cok}_i \to \mathrm{cok}'_i$ is also an isomorphism. Hence (3.8) is a quasi-isomorphism. ∎

4. The proof of Theorem 1

4.1. A generalization of Grothendieck's spectral sequence. For the proof of Theorem 1, we need the following lemma.

LEMMA 7. *Let C_1, C_2 and C_3 be abelian categories with enough injective objects, $F : C_1 \to C_2$ and $G : C_2 \to C_3$ additive functors, $M^{\cdot\cdot}$ a first quadrant bicomplex in C_1, and $K^{\cdot}$ the total complex of $M^{\cdot\cdot}$. Suppose that F sends injective objects of C_1 to G-acyclic objects. Then we have two spectral sequences*

$$'E_2^{pq} = R^pG(R^q_{II}F(M^{\cdot\cdot})) \;\Rightarrow\; R^{p+q}(G\circ F)K^{\cdot} \tag{4.1}$$

and

$$''E_2^{pq} = R^pG(R^q_{I}F(M^{\cdot\cdot})) \;\Rightarrow\; R^{p+q}(G\circ F)K^{\cdot}. \tag{4.2}$$

If $M^{ij} = 0$ unless $j = 0$, then (4.2) is just Grothendieck's spectral sequence.

Proof. We shall only show (4.1). The proof of (4.2) is similar.

Let $N^{\cdots}$ be a Cartan–Eilenberg resolution of first type of $M^{\cdot\cdot}$. (We mean that $N^{\cdots}$ is a triple complex of injective objects in C_1 such that if $i<0$, $j<0$ or $l<0$ then $N^{ijl}=0$, and for every i the bicomplexes $N^{i\cdot\cdot}$, $B^i_I(N^{\cdots})$, $Z^i_I(N^{\cdots})$ and $H^i_I(N^{\cdots})$ are injective resolutions of $M^{i\cdot}$, $B^i_I(M^{\cdot\cdot})$, $Z^i_I(M^{\cdot\cdot})$ and $H^i_I(M^{\cdot\cdot})$ respectively. Cartan–Eilenberg resolutions of second type are defined similarly.) Put

$$M_1^{ij}=\bigoplus_{r+s=j} FN^{irs},$$

and let $K_1^{\cdot}$ be the total complex of $M_1^{\cdot\cdot}$. It is clear that

$$R^q_{II}F(M^{\cdot\cdot})=H^q_{II}(M_1^{\cdot\cdot}).$$

Let $N_1^{\cdots}$ be a Cartan–Eilenberg resolution of second type of $M_1^{\cdot\cdot}$, and $M_2^{\cdot\cdot}$ the bicomplex defined by

$$M_2^{ij}=\bigoplus_{r+s=i} N_1^{rjs}.$$

Then $H^q_{II}(M_2^{\cdot\cdot})$ is a complex of injective objects, quasi-isomorphic to $H^q_{II}(M_1^{\cdot\cdot})$. Thus

$$R^pG(R^q_{II}F(M^{\cdot\cdot}))=R^pG(H^q_{II}F(M_1^{\cdot\cdot}))=H^p(GH^q_{II}(M_2^{\cdot\cdot})).$$

As $M_2^{p\cdot}$ is a complex such that $Z^q(M_2^{p\cdot})$, $B^q(M_2^{p\cdot})$ and $H^q(M_2^{p\cdot})$ are all injective, we see that

$$GH^q_{II}(M_2^{\cdot\cdot})=H^q_{II}(GM_2^{\cdot\cdot}).$$

Hence

$$R^pG(R^q_{II}F(M^{\cdot\cdot}))=H^p(H^q_{II}(GM_2^{\cdot\cdot})).$$

As F sends injective objects of C_1 to G-acyclic objects, we have

$$R^{p+q}(G\circ F)K^{\cdot}=H^{p+q}G(K_1^{\cdot})=H^{p+q}G(K_2^{\cdot}),$$

where $K_2^{\cdot}$ is the total complex of $M_2^{\cdot\cdot}$. (Notice that $K_1^{\cdot}$ and $K_2^{\cdot}$ are complexes of injective objects, quasi-isomorphic to each other.) As a consequence, the spectral sequence (4.1) comes from the first spectral sequence for the bicomplex $G(M_2^{\cdot\cdot})$. ■

4.2. Proofs of Theorem 1 and Proposition 1. Choose an open covering $\{U_\nu\}$ of X_s such that U_ν admits a closed immersion into a smooth π-adic formal scheme $\mathcal{P}_\nu$. Put $T_\nu=Y_I\cap U_\nu$. In the following, the notation $\{\nu\}$ means a finite set of indices $\nu_0,\ldots,\nu_n$. Put

$$T_{\{\nu\}}=T_{\nu_0}\cap\cdots\cap T_{\nu_n}.$$

As before, we use $T.$ to denote the diagram of Zariski topos formed by $T_{\nu_0\cdots\nu_n}$'s.

Put $\mathcal{P}_{\{\nu\}}=\mathcal{P}_{\nu_0}\times\cdots\times\mathcal{P}_{\nu_n}$ and $\mathcal{Q}_{\{\nu\}}=\mathcal{X}\times\mathcal{P}_{\{\nu\}}$. Then there are closed immersions $T_{\{\nu\}}\hookrightarrow\mathcal{P}_{\{\nu\}}$ and $T_{\{\nu\}}\hookrightarrow\mathcal{Q}_{\{\nu\}}$. We use $]T_{\{\nu\}}[_{\mathcal{P}}$ (resp.

$]T_{\{\nu\}}[_{\mathcal{Q}})$ to denote the tube $]T_{\{\nu\}}[_{\mathcal{P}_{\{\nu\}}}$ (resp. $]T_{\{\nu\}}[_{\mathcal{Q}_{\{\nu\}}})$. Then $]T_{\{\nu\}}[_{\mathcal{X}}$'s (resp. $]T_{\{\nu\}}[_{\mathcal{P}}$'s, $]T_{\{\nu\}}[_{\mathcal{Q}}$'s) form a diagram of rigid spaces, which is denoted as $]T.[_{\mathcal{X}}$ (resp. $]T.[_{\mathcal{P}}$, $]T.[_{\mathcal{Q}}$). Let p_1 and p_2 denote the projections $]T.[_{\mathcal{Q}} \to]T.[_{\mathcal{X}}$ and $]T.[_{\mathcal{Q}} \to]T_\nu[_{\mathcal{P}}$ respectively.

Put

$$\begin{aligned}\Omega^{ij}_{\{\nu\}} &:= \mathscr{O}_{]T_{\{\nu\}}[_{\mathcal{Q}}} \otimes_{(p_1^{-1}\mathscr{O}_{]T_{\{\nu\}}[_{\mathcal{X}}} \otimes p_2^{-1}\mathscr{O}_{]T_{\{\nu\}}[_{\mathcal{P}}})} (p_1^{-1}\Omega^i_{]T_{\{\nu\}}[_{\mathcal{X}}/K} \otimes_K p_2^{-1}\Omega^j_{]T_{\{\nu\}}[_{\mathcal{P}}/K}) \\ &= p_1^{-1}\Omega^i_{]T_{\{\nu\}}[_{\mathcal{X}}/K} \otimes_{p_1^{-1}\mathscr{O}_{]T_{\{\nu\}}[_{\mathcal{X}}}} \Omega^j_{]T_{\{\nu\}}[_{\mathcal{Q}}/]T_{\{\nu\}}[_{\mathcal{X}}} \\ &= \Omega^i_{]T_{\{\nu\}}[_{\mathcal{Q}}/]T_{\{\nu\}}[_{\mathcal{P}}} \otimes_{p_2^{-1}\mathscr{O}_{]T_{\{\nu\}}[_{\mathcal{P}}}} p_2^{-1}\Omega^j_{]T_{\{\nu\}}[_{\mathcal{P}}/K}.\end{aligned}$$

Then $\Omega^{\cdot\cdot}_{\{\nu\}}$ is a bicomplex with the horizontal differentials given by the differentials of $\Omega^{\cdot}_{]T_{\{\nu\}}[_{\mathcal{Q}}/]T_{\{\nu\}}[_{\mathcal{P}}}$ and the vertical differentials given by the differentials of $\Omega^{\cdot}_{]T_{\{\nu\}}[_{\mathcal{Q}}/]T_{\{\nu\}}[_{\mathcal{X}}}$ up to sign. For any fixed j the complex $\Omega^{\cdot j}$ is just

$$\Omega^{\cdot}_{]T_{\{\nu\}}[_{\mathcal{Q}}/]T_{\{\nu\}}[_{\mathcal{P}}} \otimes_{p_2^{-1}\mathscr{O}_{]T_{\{\nu\}}[_{\mathcal{P}}}} p_2^{-1}\Omega^j_{]T_{\{\nu\}}[_{\mathcal{P}}/K}.$$

Let $(\Omega^{ij\,l}_{c,I;\{\nu\}})_{ijl}$ be the tricomplex

$$\Omega^{ij}_{\{\nu\}} \to \alpha_{I*}\alpha_I^{-1}\Omega^{ij}_{\{\nu\}},$$

where α_I is the inclusion map $]T_{\{\nu\}} \cap (Y_I \backslash U_I)[_{\mathcal{Q}} \hookrightarrow]T_{\{\nu\}}[_{\mathcal{Q}}$. Note that $\Omega^{ij\,l}_{c,I;\{\nu\}} = 0$ unless $l = 0, 1$. Let $M^{\cdot\cdot}_{\{\nu\}}$ be the bicomplex defined by

$$M^{ij}_{\{\nu\}} = \bigoplus_{r+s=j} \Omega^{irs}_{c,I;\{\nu\}}.$$

Thus we get a bicomplex $M^{\cdot\cdot}$ on $]T.[_{\mathcal{Q}}$. The total complex of $M^{\cdot\cdot}$ is just $\Omega^{\cdot}_{c,I;\mathcal{Q}}$.

From Lemma 7, Theorem 1 can be deduced as follows.

Proof of Theorem 1. Let C_1, C_2 and C_3 be respectively the category of abelian sheaves on $]T.[_{\mathcal{Q}}$, the category of abelian sheaves on $]T.[_{\mathcal{P}}$ and the category of abelian groups. Put $F = p_{2*}$ and $G = \Gamma \circ \epsilon_* \circ \mathrm{sp}_{\mathcal{P}*}$, where $\mathrm{sp}_{\mathcal{P}}$ is the specialization map $]T.[_{\mathcal{P}} \to T.$, ϵ is the natural map $T. \to Y_I$ and $\Gamma = \Gamma(Y_I, \cdot\,)$. Then $G \circ F = \Gamma \circ \epsilon_* \circ \mathrm{sp}_{\mathcal{Q}*}$, where $\mathrm{sp}_{\mathcal{Q}}$ is the specialization map $]T.[_{\mathcal{Q}} \to T.$. Let $M^{\cdot\cdot}$ be as above. By Proposition 3 we have

$$R^q_I F(M^{\cdot\cdot}) = R^q_I p_{2*}(M^{\cdot\cdot}) = \Omega^{\cdot}_{c,I;\mathcal{P}} \otimes_K \textstyle\bigwedge^q(V_I').$$

Hence

$$\begin{aligned}R^p G R^q_I F(M^{\cdot\cdot}) &= H^p(Y_I, \mathbb{R}\epsilon_*\mathbb{R}\mathrm{sp}_{\mathcal{P}*}\Omega^{\cdot}_{c,I;\mathcal{P}}) \otimes_K \textstyle\bigwedge^q(V_I') \\ &= H^p_{c,\mathrm{rig}}(U_I/K) \otimes_K \textstyle\bigwedge^q(V_I').\end{aligned}$$

On the other hand, Corollary 1 implies that

$$R^{p+q}(G\circ F)\Omega^{\boldsymbol{\cdot}}_{c,I;\mathcal{Q}} = H^{p+q}(Y_I, \mathbb{R}\epsilon_*\mathbb{R}\mathrm{sp}_{\mathcal{Q}*}\Omega^{\boldsymbol{\cdot}}_{c,I;\mathcal{Q}}) = H^{p+q}(]Y_I[_{\mathcal{X}}, \Omega^{\boldsymbol{\cdot}}_{c,I;\mathcal{X}}).$$

Now Theorem 1 follows immediately from Lemma 7. ■

Proof of Proposition 1. By Theorem 1, if $|I|\geq 2$, then

$$\begin{aligned}
\sum_{i\geq 0}&(-1)^i \dim_K H^i(]Y_I[_{\mathcal{X}}, \Omega^{\boldsymbol{\cdot}}_{c,I;\mathcal{X}}) \\
&= \sum_{p\geq 0,\, q\geq 0} (-1)^{p+q} \dim_K (H^p_{c,\mathrm{rig}}(U_I/K)\otimes_K {\textstyle\bigwedge}^q(V'_I)) \\
&= \sum_{p\geq 0,\, q\geq 0} (-1)^{p+q} \dim_K H^p_{c,\mathrm{rig}}(U_I/K) \dim_K {\textstyle\bigwedge}^q(V'_I) \\
&= \sum_{p\geq 0}(-1)^p \dim_K H^p_{c,\mathrm{rig}}(U_I/K) \sum_{q\geq 0}(-1)^q \dim_K {\textstyle\bigwedge}^q(V'_I).
\end{aligned}$$

When $|I|\geq 2$,

$$\sum_{q\geq 0}(-1)^q \dim_K {\textstyle\bigwedge}^q(V'_I) = 0,$$

so

$$\sum_{i\geq 0}(-1)^i \dim_K H^i(]Y_I[_{\mathcal{X}}, \Omega^{\boldsymbol{\cdot}}_{c,I;\mathcal{X}}) = 0.$$

Combining this equality, (1.2) and the equality

$$\begin{aligned}
\chi_{\mathrm{dR}}(\mathcal{X}_K) &= \sum_{i\geq 0}(-1)^i \dim_K H^i_{\mathrm{dR}}(\mathcal{X}_K/K) \\
&= \sum_{|I|\geq 1}\sum_{i\geq 0}(-1)^i \dim_K H^i(]Y_I[_{\mathcal{X}}, \Omega^{\boldsymbol{\cdot}}_{c,I;\mathcal{X}}) \quad \text{(by (1.1) and Lemma 1)},
\end{aligned}$$

we get

$$\begin{aligned}
\chi_{\mathrm{dR}}(\mathcal{X}_K) &= \sum_{|I|=1}\sum_{i\geq 0}(-1)^i \dim_K H^i(]Y_I[_{\mathcal{X}}, \Omega^{\boldsymbol{\cdot}}_{c,I;\mathcal{X}}) \\
&= \sum_{|I|=1}\sum_{i\geq 0}(-1)^i \dim_K H^i_{c,\mathrm{rig}}(U_I/K) = \sum_{|I|=1}\chi_c(U_I).
\end{aligned}$$

On the other hand, we have

$$\chi_{\mathrm{rig}}(X_s) = \sum_{\emptyset\neq I\subseteq\{1,\ldots,n\}} \chi_c(U_I).$$

Thus

$$\chi_{\mathrm{rig}}(X_s) - \chi_{\mathrm{dR}}(\mathcal{X}_K) = \sum_{|I|\geq 2}\chi_c(U_I). \tag{4.3}$$

From the equality

$$\chi_{\mathrm{rig}}(Y_I) = \sum_{J \supseteq I} \chi_c(U_J)$$

we get

$$\chi_c(U_I) = \sum_{J \supseteq I} (-1)^{|I|+|J|} \chi_{\mathrm{rig}}(Y_J).$$

By this equality and (4.3) we see that

$$\begin{aligned}\chi_{\mathrm{rig}}(X_s) - \chi_{\mathrm{dR}}(\mathcal{X}_K) &= \sum_{|I|\geq 2} \sum_{J\supseteq I} (-1)^{|I|+|J|} \chi_{\mathrm{rig}}(Y_J) \\ &= \sum_{|J|\geq 2} (-1)^{|J|} \chi_{\mathrm{rig}}(Y_J) \sum_{I\subseteq J,\, |I|\geq 2} (-1)^{|I|} \\ &= \sum_{|J|\geq 2} (-1)^{|J|} (|J|-1) \chi_{\mathrm{rig}}(Y_J).\end{aligned}$$

As the rigid cohomology is a Weil cohomology in the sense of Kleiman [7], we have

$$\chi_{\mathrm{rig}}(Y_J) = (\triangle Y_J . \triangle Y_J).$$

So,

$$\chi_{\mathrm{rig}}(X_s) - \chi_{\mathrm{dR}}(\mathcal{X}_K) = \sum_{|J|\geq 2} (-1)^{|J|} (|J|-1)(\triangle Y_J . \triangle Y_J),$$

as expected. ∎

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References

[1] V. G. Berkovich, *Vanishing cycles for formal schemes*, Invent. Math. 115 (1994), 539–571.

[2] P. Berthelot, *Cohomologie cristalline des schémas de caractéristique $p > 0$*, Lecture Notes in Math. 407, Springer, Berlin, 1974.

[3] —, *Géométrie rigide et cohomologie des variétés algébriques de caractéristique p*, in: Introduction aux cohomologies p-adiques (Luminy, 1984), Mém. Soc. Math. France 23 (1986), 7–32.

[4] —, *Cohomologie rigide et cohomologie rigide à supports propres, I*, Publ. IRMAR 96-03, Univ. de Rennes, 1996.

[5] —, *Finitude et pureté cohomologique en cohomologie rigide*, Invent. Math. 128 (1997), 329–377.

[6] E. Grosse-Klönne, *Finiteness of de Rham cohomology in rigid analysis*, Duke Math. J. 113 (2002), 57–91.

[7] S. Kleiman, *Algebraic cycles and the Weil conjectures*, in: Dix exposés sur la cohomologie des schémas, North-Holland, Amsterdam, and Masson, Paris, 1968, 359–386.

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