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On the Beckmann–Black problem

by

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1. Presentation

1.1. The Beckmann–Black problem. This paper is about a question in inverse Galois theory known as the **B**eckmann–**B**lack problem and which we denote by **BB**. More precisely, if K is a field and G is a finite group, then the **BB** *problem* asks *whether each Galois extension E/K of the group G is the specialization of some regular Galois extension $F/K(T)$ of G at some unramified point $t_0 \in \mathbb{P}^1(K)$*. Recall that "regular" means that $F \cap \overline{K} = K$. Briefly, we will say "a G-extension of a group G" for a regular Galois extension of G. And if $F/K(T)$ is a G-extension $F/K(T)$ of G defined over K, then we define the *specialization of $F/K(T)$ at t_0*, denoted by F_{t_0}, to be the residue field of F at some point over t_0 (see §1.3.3).

The **BB** problem is known to have a positive answer in the following situations:

- G is a symmetric group (Beckmann [Be] if K is a number field, Black [Bl2] for an arbitrary field).
- G is an abelian group (Beckmann [Be] and Black [Bl1] if K is a number field, Dèbes [De1] for an arbitrary field).
- G is the dihedral group D_n of order $2n$ when n is odd (Black [Bl1]).
- G is a finite group and K is P(seudo) A(lgebraically) C(losed) [De1]. Recall that a field K is PAC if and only if each geometrically irreducible variety V defined over K has infinitely many K-rational points. Moreover P. Dèbes proves in [De1] this stronger version of **BB**: any Galois extension E/K is the specialization of any G-extension of $K(T)$ of G at infinitely many unramified K-rational points.
- K is an *ample* field, i.e. each geometrically irreducible smooth curve $\mathcal{C}$ defined over K has infinitely many K-rational points provided that

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$\mathcal{C}(K)$ (the set of all K-rational points on $\mathcal{C}$) is not empty (Colliot-Thélène [CT] in characteristic 0, Moret-Bailly [MB2] and Haran–Jarden [HJ] in general).

1.2. Main result. The goal of this paper is to prove the following statement:

THEOREM 1.1. *Let G be a finite group, H be a subgroup of G, K be a number field, E/K be a Galois extension of the group H, $t_0 \in \mathbb{P}^1(K)$ be a fixed point and S be a finite set of finite places of K. Then there exists a finite Galois extension L/K totally split in K_v for all $v \in S$ and a G-extension $F/L(T)$ of G such that the specialization F_{t_0}/L at $T = t_0$ satisfies the following:*

(1) *F_{t_0}/L is a Galois extension of H isomorphic to EL/L.*
(2) *The v-completion $F_{t_0}K_v/K_v$ is isomorphic to EK_v/K_v for each $v \in S$.*

The special case when $S = \emptyset$ and $G = H$ asserts that the **BB** problem has a positive answer over some finite extension L of K, while conclusion (2) in case $S \neq \emptyset$ shows that the problem can be solved locally, i.e. after scalar extension to any given completion of K. Theorem 1.1 shows in fact that these two conclusions, local and global, can be combined.

The following statement provides two further conclusions: the first one is a uniformizing moduli space version (to be compared with the main result of B. Deschamps [Des]): we show that the G-extensions $F/L(T)$ from Theorem 1.1 can all be found on a curve on a Hurwitz space (the same for all S). The second one considers the more general case where K is a Hilbertian field ([1]) of characteristic 0.

THEOREM 1.1 (continued).

(3) *The G-extensions $F/L(T)$ can be constructed in such a way that the branch point number r and the ramification type $\mathbf{C}$ ([2]) are independent of S. That is, the corresponding point on the moduli space lies on the same Hurwitz space $H_r(G, \mathbf{C})$ ([3]). More precisely, these points can all be picked on a curve $\mathcal{C}$ contained in $H_r(G, \mathbf{C})$ and for a given S, infinitely many points from $\mathcal{C}$ provide a G-extension $F/L(T)$ as in Theorem* 1.1.
(4) *In the special situation $S = \emptyset$, the global conclusion* (1) *holds more generally if K is a Hilbertian field of characteristic* 0.

([1]) A field K is said to be *Hilbertian* if for each irreducible polynomial $f(T, Y) \in K(T)[Y]$, there are infinitely many $t \in K$ such that the specialized polynomial $f(t, Y)$ is irreducible in $K[Y]$ (see [V]).

([2]) For definition, see §1.3.1.

([3]) For more details, see §1.3.4.

1.3. Preliminary reminders

1.3.1. *Ramification type of G-extensions.* Let G be a finite group, K be an algebraically closed field of characteristic 0, and $F/K(T)$ be a G-extension of group G. Denote the unordered set of all branch points of this extension by $\mathbf{t} = \{t_1, \ldots, t_r\}$, which can be viewed as a K-rational divisor of $\mathbb{P}^1$.

With each $j = 1, \ldots, r$, we can associate a conjugacy class C_j of the group G as follows: The point t_j is a branch point of our G-extension, so its inertia groups are conjugate and cyclic of order equal to the ramification index. Given one of these inertia groups I_j, an element $\sigma \in I_j$ is said to be a *distinguished generator* of I_j if $\sigma(\pi)/\pi = e^{2\pi i/e_j}$, where π is equal to $(T - t_j)^{1/e_j}$. Then all the distinguished generators can be shown to be in the same conjugacy class of G: this is the conjugacy class C_j.

Denote by $\mathbf{C} = (C_1, \ldots, C_r)$ the ramification type of $F/K(T)$. We refer to [De2, Chapter 3] for more details.

By branch points and ramification type of some G-extension $E/K(T)$ over some non-algebraically closed field K of characteristic 0 we mean those of the G-extension $E\overline{K}(T)/\overline{K}(T)$ (in a given algebraic closure of $K(T)$).

1.3.2. *The fundamental group.* Let $r \geq 1$ be an integer and K be a field of characteristic 0. Denote by $\overline{K}$ an algebraic closure of K and by G_K the absolute Galois group of K. Let U_r be the variety of all unordered r-uples $\mathbf{t} = \{t_1, \ldots, t_r\}$ of $(\mathbb{P}^1)^r$ such that $t_i \neq t_j$ for all $1 \leq i \neq j \leq r$. We fix an algebraic closure $\overline{K(T)}$ of $\overline{K}(T)$. Take $\mathbf{t} \in U_r(K)$. The fundamental group of $\mathbb{P}^1 \setminus \mathbf{t}$ is defined as follows:

Denote by $\Omega_{\mathbf{t}}$ the maximal Galois extension of $K(T)$ unramified above $\mathbb{P}^1 \setminus \mathbf{t}$. Then the *geometric fundamental group*, $\pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{\overline{K}}$, of $\mathbb{P}^1 \setminus \mathbf{t}$ is the Galois group of $\Omega_{\mathbf{t}}/\overline{K}(T)$. Moreover, as $\mathbf{t} \in U_r(K)$, it follows that $\Omega_{\mathbf{t}}/K(T)$ is a Galois extension. By definition, its Galois group is the K-fundamental group of $\mathbb{P}^1 \setminus \mathbf{t}$ and it is denoted by $\pi_1(\mathbb{P}^1 \setminus \mathbf{t})_K$.

$$
\begin{tikzcd}
 & \Omega_{\mathbf{t}} \arrow[d, dash, "\pi_1(\mathbb{P}^1\setminus\mathbf{t})_{\overline{K}}"] \arrow[ddl, dash, "\pi_1(\mathbb{P}^1\setminus\mathbf{t})_K"'] \\
 & \overline{K}(T) \arrow[dl, dash, "G_K"] \\
K(T) &
\end{tikzcd}
$$

Moreover, the following exact sequence (given by Galois theory) is split as each K-rational point $t_0 \in \mathbb{P}^1(K) \setminus \mathbf{t}$ provides a section s_{t_0} of the canonical surjection of this sequence:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{\overline{K}} \arrow[r] & \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_K \arrow[r] & G_K \arrow[r] \arrow[l, bend right, "s_{t_0}"'] & 1
\end{tikzcd}$$

For more details, we refer to [De2, Chapter 3].

1.3.3. *Specialization.* Let G be a finite group, K be a field of characteristic 0, and $F/K(T)$ be a G-extension of group G with a K-rational branch divisor $\mathbf{t}$. This G-extension corresponds to some epimorphism $\phi : \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_K \to G$ and the extension $F\overline{K}/\overline{K}(T)$ corresponds to the restriction $\overline{\phi} : \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{\overline{K}} \to G$ of ϕ to $\pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{\overline{K}}$; it is still surjective as F/K is a regular extension. Let $t_0 \in \mathbb{P}^1(K) \setminus \mathbf{t}$ be a K-rational point and consider the section s_{t_0} corresponding to this point.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{\overline{K}} \arrow[r] \arrow[d, "\overline{\phi}"'] & \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_K \arrow[r] \arrow[d, "\phi"] & G_K \arrow[r] \arrow[l, bend right, "s_{t_0}"'] & 1 \\
& G \arrow[r, equal] & G & &
\end{tikzcd}$$

The specialization F_{t_0} of $F/K(T)$ at $T = t_0$ (i.e. the residue field of F at some prime above t_0 in the extension $F/K(T)$) corresponds to the homomorphism $\phi \circ s_{t_0}$ ([De2, Proposition (2.1)]). More precisely, F_{t_0} is the fixed field in $\overline{K}$ of $\ker(\phi \circ s_{t_0})$. In particular, the specialization F_{t_0}/K is a Galois field extension of the group $\mathrm{Im}(\phi \circ s_{t_0})$. For more details, we refer to [De2, Chapter 3] and to [De1].

1.3.4. *Hurwitz spaces.* Assume that G is a finite group and $r > 2$ is an integer. Denote by $H_r(G)$ the moduli space of all G-extensions of the group G with r branch points. This moduli space is a smooth (not necessarily connected) variety defined over $\mathbb{Q}$ and, for all algebraically closed fields k of characteristic 0, the k-rational points in $H_r(G)$ correspond to the isomorphism classes of G-extensions $F/k(T)$ of G defined over k with r branch points (see [FrV]).

We fix an r-tuple $\mathbf{C} = (C_1, \ldots, C_r)$ of conjugacy classes of G. We denote by $H_r(G, \mathbf{C}) \subseteq H_r(G)$ the subset of all G-extensions $F/k(T)$ of G with r branch points and of ramification type $\mathbf{C}$. We refer to [V] for more details.

2. Proof of the main result. Let G be a finite group, K be a field of characteristic 0, $t_0 \in \mathbb{P}^1(K)$ be a fixed point and E/K be a Galois extension of a group $H \subset G$.

2.1. BB over a curve. The starting ingredient of the proof is the following:

(∗) *Under the above hypotheses, there exist:*

(1) *a smooth projective geometrically irreducible curve $\mathcal{C}$ defined over K with a K-rational point,*
(2) *a G-extension $\mathcal{F}/K(\mathcal{C})(T)$ of the group G defined over the function field $K(\mathcal{C})$,*

with the following property: For all $x \in \mathcal{C}(\overline{K})$ off a proper Zariski closed subset Z, the extension $\mathcal{F}/K(\mathcal{C})(T)$ specializes to some G-extension $\mathcal{F}_x/K(x)(T)$ of G which is unramified above $T = t_0$ and whose specialization $\mathcal{F}_{x,t_0}/K(x)$ at $T = t_0$ is a Galois extension isomorphic to $E(x)/K(x)$.

The following diagram illustrates this double specialization process:

$$\begin{array}{ccccc} \mathcal{F} & & \mathcal{F}_x & & \mathcal{F}_{x,t_0} \cong E(x) \\ \Big| & \overset{x\in\mathcal{C}(\overline{K})\setminus Z}{\rightsquigarrow} & \Big| & \overset{T=t_0}{\rightsquigarrow} & \Big| \\ K(\mathcal{C})(T) & & K(x)(T) & & K(x) \end{array}$$

This starting ingredient is essentially an arithmetic translation of Theorem 2.7 of [MB2] which was expressed in geometric terms. In §3, we show in a pure field arithmetic language how to deduce this result from the fact, mentioned in the introduction, that the **BB** problem has a positive answer if the base field is the complete (thus ample) field $K((X))$ of formal Laurent series with coefficients in K.

Denote the branch point number of $\mathcal{F}/K(\mathcal{C})(T)$ by r, the branch point set by $\mathbf{t} = \{t_1, \ldots, t_r\} \in \mathbb{P}^1(\overline{K(\mathcal{C})})$ and its ramification type by $\mathbf{C} = (C_1, \ldots, C_r)$.

2.2. Proof of Theorem 1.1. Assume further K is a number field. The next step uses the Chebotarev density theorem [FJ, Theorem (5.6)]: we can find, for each $g \in H$, a finite valuation v_g of K, not in S, unramified in E/K and such that its decomposition group, $\mathrm{Gal}(E_v/K_v)$, is conjugate to $\langle g\rangle$ in H; and the v_g can further be chosen pairwise distinct. Let S' denote the set of all places v_g with $g \in H$, and S'' the union of the set of places S from the statement of Theorem 1.1 and the set S'.

Let $K^{\mathrm{tot}\, S''}$ be the field of all totally S''-adic algebraic numbers, i.e. of all $x \in \overline{K}$ such that all K-conjugates of x lie in the completion K_v, for all $v \in S''$. As $\mathcal{C}$ contains a K-rational point and $K \subseteq K^{\mathrm{tot}\, S''}$, the set $\mathcal{C}(K^{\mathrm{tot}\, S''})$ is not empty. The field $K^{\mathrm{tot}\, S''}$ is known to be ample [MB1]. Using this fact, we deduce that $\mathcal{C}(K^{\mathrm{tot}\, S''})$ is an infinite set.

Fix $x \in \mathcal{C}(K^{\mathrm{tot}\, S''}) \setminus Z$, let L be the Galois closure of $K(x)$, and denote the specialization $\mathcal{F}_x$ by F. From statement (∗), $F/L(T)$ is a G-extension of

G whose specialization at $T = t_0$ is a Galois extension isomorphic to EL/L. As $L \subseteq K^{\mathrm{tot}\, S''} \subseteq K_v$, the extension $F_{t_0}K_v/K_v$ is isomorphic to EK_v/K_v (for all $v \in S$). This completes the proof of (2).

It remains to prove that F_{t_0}/L is a Galois extension of H. In fact, we know that $\mathrm{Gal}(F_{t_0}/L)$ is a subgroup of H and for each $v \in S''$ it contains $\mathrm{Gal}(F_{t_0}K_v/K_v)$ up to conjugation: more precisely, there exists $\sigma_v \in H$ such that

$$\langle g \rangle^{\sigma_v} \subseteq H.$$

Via a classical lemma due to Jordan, which states that there is no proper subgroup of H that meets all conjugacy classes of H, we conclude that $H = \mathrm{Gal}(F_{t_0}/L)$. This ends the proof of (1)–(2) of Theorem 1.1.

2.3. Proof of Theorem 1.1 (continued). Up to enlarging the Zariski closed subset Z, we can claim that $F/L(T)$ has the same number of branch points r and the same ramification type $\mathbf{C}$ as $\mathcal{F}/K(\mathcal{C})(T)$. Thus if the field L depends on S, the number of branch points and the ramification type of $F/L(T)$ do not. Furthermore, the constructed extensions $F/L(T)$ correspond to points on the curve $\mathcal{C}$ which is contained in the Hurwitz space $H_r(G, \mathbf{C})$. This completes the proof of (3).

Finally we prove property (4). Assume that K is more generally a Hilbertian field of characteristic 0. From $(*)$, we have the following property: for all $x \in \mathcal{C}(\overline{K}) \setminus Z$, there exists a G-extension $F/K(x)(T)$ of G defined over $K(x)$ such that its specialization at $T = t_0$ is a Galois extension isomorphic to $E(x)/K(x)$. To prove our last claim, it suffices to find infinitely many points $x \in \mathcal{C}(\overline{K}) \setminus Z$ such that the extensions $K(x)/K$ and E/K are linearly disjoint (then $E(x)/K(x)$ is a Galois extension of the group H).

In fact, let $h(w, z) = 0$ be an affine equation of $\mathcal{C}$ with $h(W, Z) \in K[W, Z]$ irreducible in $\overline{K}[W, Z]$. As E is a finite extension of the Hilbertian field K, we can find infinitely many points w_0 in K (and not only in E) such that $h(w_0, Z) \in K[Z]$ is irreducible in $E[Z]$ ([V, Corollary (1.8)]).

Pick $z_0 \in \overline{K}$ such that $h(w_0, z_0) = 0$. As Z is a finite set and infinitely many $w_0 \in K$ satisfy this equality, there exist infinitely many points $x := (w_0, z_0) \in \mathcal{C}(\overline{K}) \setminus Z$ (with $w_0 \in K$) such that $h(w_0, Z) \in K[Z]$ is irreducible in $E[Z]$. We deduce that $[K(x) : K] = \deg_Z(h) = [E(x) : E]$. So the extensions $K(x)/K$ and E/K are linearly disjoint.

3. Proof of BB over a curve

3.1. The starting result. The **BB** problem is known to have a positive answer over any complete valued field. This has been proved by Colliot-Thélène [CT] in characteristic 0, and, more generally, by Moret-Bailly [MB2] and Haran–Jarden [HJ], thanks to some deformation techniques. The complete valued field we will be using is the field $K((X))$ of formal Laurent series with coefficients in K.

As E/K is a Galois extension of H, the extension $E((X))/K((X))$ is also a Galois extension of H. From the paper [HJ] which uses as we do field-theoretic language, there exists a G-extension $F_{K((X))}/K((X))(T)$ of G unramified above $T = t_0$ whose specialization at $T = t_0$ is a Galois extension of H isomorphic to $E((X))/K((X))$. The construction of $F_{K((X))}$ makes it possible to assume further that all branch points of $F_{K((X))}/K((X))(T)$ lie in $\mathbb{P}^1(\overline{N})$ where $N = K((X)) \cap \overline{K(X)}$.

Furthermore, by Section 1.3.3, there exists a split exact sequence of fundamental groups

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{\overline{K((X))}} \arrow[r] \arrow[d, "\overline{\phi}"'] & \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{K((X))} \arrow[r] \arrow[d, "\phi"] & G_{K((X))} \arrow[r] \arrow[l, bend right, "s_{t_0}"'] & 1 \\
& G \arrow[r, equal] & G & &
\end{tikzcd}$$

As the specialization of $F_{K((X))}/K((X))(T)$ at $T = t_0$ is a Galois extension of $K((X))$ with the group H isomorphic to $E((X))/K((X))$, the image of the homomorphism $\phi \circ s_{t_0}$ is exactly H.

$$\begin{array}{ccc}
F_{K((X))} & & E((X)) \\
\Big| & \xrightarrow{T=t_0 \in \mathbb{P}^1(K)} & \Big| \\
K((X))(T) & & K((X))
\end{array}$$

3.2. Descent to $N = K((X)) \cap \overline{K(X)}$. We use an argument (see [DeDes, Theorem (3.4)]) showing that the natural restriction morphism $\pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{K((X))} \to \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_N$ is an isomorphism.

Indeed, first, the restriction morphism $\pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{\overline{K((X))}} \to \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{\overline{N}}$ is an isomorphism (via Riemann's existence theorem). Second, the restriction morphism $\rho : G_{K((X))} \to G_N$ is also an isomorphism. We deduce that the natural restriction morphism $\gamma : \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{K((X))} \to \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_N$ is an isomorphism.

Furthermore, as t_0 is a point in $\mathbb{P}^1(K) \setminus \mathbf{t}$, the section s_{t_0} induces a section, still denoted by s_{t_0}, of $\pi_1(\mathbb{P}^1 \setminus \mathbf{t})_N \to G_N$.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{\overline{K((X))}} \arrow[r] \arrow[d, "\cong"] & \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{K((X))} \arrow[r] \arrow[d, "\gamma"] & G_{K((X))} \arrow[r] \arrow[d, "\rho", "\cong"'] \arrow[l, bend right, "s_{t_0}"'] & 1 \\
1 \arrow[r] & \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_{\overline{N}} \arrow[r] & \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_N \arrow[r] & G_N \arrow[r] \arrow[l, bend left, "s_{t_0}"] & 1
\end{tikzcd}$$

This implies that there exists a unique G-extension $F_N/N(T)$ defined over N of the group G such that $F_N K((X)) = F_{K((X))}$. Namely this extension, $F_N/N(T)$, corresponds to the epimorphism $\phi_N = \gamma^{-1} \circ \phi : \pi_1(\mathbb{P}^1 \setminus \mathbf{t})_N \to G$. Furthermore, the specialization $F_{N,t_0}/N$ of $F_N/N(T)$ at $T = t_0$ is a Galois extension of H such that $F_{N,t_0}K((X))$ equals the specialization of $F_{K((X))}$ at t_0. By construction of $F_{K((X))}$, we obtain $F_{N,t_0}K((X)) = E((X))$. On the other hand, as ρ is an isomorphism, we deduce that $F_{N,t_0} = EN$.

To sum up, we now have a G-extension $F_N/N(T)$ defined over N of the group G whose specialization $F_{N,t_0}/N$ at $T = t_0$ is a Galois extension of H isomorphic to EN/N.

$$\begin{tikzcd}
F_N \arrow[d, dash] & & EN \arrow[d, dash] \\
N(T) \arrow[rr, phantom, "\xrightarrow{T=t_0 \in \mathbb{P}^1(K)}"] & & N
\end{tikzcd}$$

3.3. Descent to the curve $\mathcal{C}$. We will prove the following claim:

(∗∗) *There exists a finite extension $L/K(X)$ with $L \subset N$ satisfying the following property:*

(i) *There exists a G-extension $F_L/L(T)$ of G such that $F_L N = F_N$.*

(ii) *The specialization $F_{L,t_0}/L$ of $F_L/L(T)$ at t_0 is a Galois extension of H such that $F_{L,t_0} = EL$.*

In fact, denote by $y(T)$ a primitive element of the extension $F_N/N(T)$ integral over $N[T]$. All conjugates of $y(T)$ over $N(T)$ can be expressed as rational functions of T and $y(T)$ with coefficients in N. Consider the field L_1 generated over $K(X)$ by all coefficients of such expressions together with the coefficients in N of the irreducible polynomial of $y(T)$ over $N(T)$. This field, L_1, is a finite extension of $K(X)$ contained in N. We deduce that $F_{L_1} = L_1(T, y(T))$ satisfies condition (i) of the claim above (with L_1 replacing L).

To study condition (ii), we view $y(T)$ as a formal power series

$$\sum_{j \geq 0} a_j (T - t_0)^j \in \overline{L_1}[[T - t_0]].$$

This is possible because t_0 is unramified in $F_{L_1}/L_1(T)$. Then we have $F_{L_1,t_0} = L_1(\underline{a})$, where $\underline{a} = \{a_0, a_1, \dots\}$. As $F_{L_1,t_0}/L_1$ is a finite extension, we can find a finite subset a_I of $\underline{a}$ such that $L_1(\underline{a}) = L_1(a_I)$. We also have

$$N(a_I) = F_{N,t_0} = EN.$$

Consequently, there exists a finite subset $N_0 \subset N$ such that

$$L_1(N_0)(a_I) \subset L_1(N_0)E.$$

Finally, let α be a primitive element of E/K. As $N(a_I) = F_{N,t_0} = EN$, we can find a finite subset $N_1 \subset N$ such that $\alpha \in L_1(N_1)(a_I)$.

Set $L = L_1(N_0 \cup N_1)$ and $F_L = F_{L_1}L$. It is easily checked that $L(a_I) = EL$ and so $F_{L,t_0} = LE$. We conclude that $(**)$ is satisfied for this L and this F_L.

The containment $L \subseteq N = K((X)) \cap \overline{K(X)}$ implies that L is regular over K and so is the function field of a smooth projective geometrically irreducible curve $\mathcal{C}$ defined over K. It follows from $L = K(\mathcal{C}) \subseteq K((X))$ that $\mathcal{C}(K)$ is not empty.

$(**)$ above shows that there exists a G-extension $F/K(\mathcal{C})(T)$ of G such that $E(\mathcal{C})/K(\mathcal{C})$ is the specialization of $F/K(\mathcal{C})(T)$ at $T = t_0$.

$$\begin{tikzcd}
F \arrow[d, dash] & & E(\mathcal{C}) \arrow[d, dash] \\
K(\mathcal{C})(T) \arrow[rr, phantom, "\xrightarrow{T=t_0\in\mathbb{P}^1(K)}"] & & K(\mathcal{C})
\end{tikzcd}$$

3.4. Specialization to points on the curve $\mathcal{C}$. As the extension $F/K(\mathcal{C})(T)$ is regular, the Bertini–Noether theorem [FJ, Proposition (8.8)] can be applied. Thus there exists a Zariski closed subset Z of $\mathcal{C}$ such that for each $x \in \mathcal{C}(\overline{K})\backslash Z$, the extension $F/K(\mathcal{C})(T)$ specializes to some G-extension $F_x/K(x)(T)$ of G, unramified at $T = t_0$. Furthermore, the specialization $F_{x,t_0}/K(x)$ of $F_x/K(x)(T)$ at $T = t_0$ is a Galois extension such that $F_{x,t_0}L = F_{t_0} = EL$.

To finish the proof, we should show that $F_{x,t_0} = EK(x)$. Indeed, we have, first, $F_{x,t_0} = F_{x,t_0}L \cap \overline{K(x)}$ because the extensions $F_{x,t_0}/K(x)$ and $L/K(x)$ are linearly disjoint. As $F_{x,t_0}L = EL$, we deduce that $F_{x,t_0} = EL \cap \overline{K(x)}$. But $EL \cap \overline{K(x)} = EK(x)$ because the extensions $EK(x)/K(x)$ and L/K are linearly disjoint. Thus $F_{x,t_0} = EK(x) = E(x)$.

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