On the Beckmann–Black problem

by

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1. Presentation

1.1. The Beckmann–Black problem. This paper is about a question in inverse Galois theory known as the Beckmann–Black problem and which we denote by $BB$. More precisely, if $K$ is a field and $G$ is a finite group, then the $BB$ problem asks whether each Galois extension $E/K$ of the group $G$ is the specialization of some regular Galois extension $F/K(T)$ of $G$ at some unramified point $t_0 \in \mathbb{P}^1(K)$. Recall that “regular” means that $F \cap \overline{K} = K$. Briefly, we will say “a $G$-extension of a group $G$” for a regular Galois extension of $G$. And if $F/K(T)$ is a $G$-extension $F/K(T)$ of $G$ defined over $K$, then we define the specialization of $F/K(T)$ at $t_0$, denoted by $F_{t_0}$, to be the residue field of $F$ at some point over $t_0$ (see §1.3.3).

The $BB$ problem is known to have a positive answer in the following situations:

- $G$ is a symmetric group (Beckmann [Be] if $K$ is a number field, Black [Bl2] for an arbitrary field).
- $G$ is an abelian group (Beckmann [Be] and Black [Bl1] if $K$ is a number field, Dèbes [De1] for an arbitrary field).
- $G$ is the dihedral group $D_n$ of order $2n$ when $n$ is odd (Black [Bl1]).
- $G$ is a finite group and $K$ is P(seudo) A(legebraically) C(losed) [De1]. Recall that a field $K$ is PAC if and only if each geometrically irreducible variety $V$ defined over $K$ has infinitely many $K$-rational points. Moreover P. Dèbes proves in [De1] this stronger version of $BB$: any Galois extension $E/K$ is the specialization of any $G$-extension of $K(T)$ of $G$ at infinitely many unramified $K$-rational points.
- $K$ is an ample field, i.e. each geometrically irreducible smooth curve $C$ defined over $K$ has infinitely many $K$-rational points provided that

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$C(K)$ (the set of all $K$-rational points on $C$) is not empty (Colliot-Thélène [CT] in characteristic 0, Moret-Bailly [MB2] and Haran–Jar- den [HJ] in general).

1.2. Main result. The goal of this paper is to prove the following statement:

**Theorem 1.1.** Let $G$ be a finite group, $H$ be a subgroup of $G$, $K$ be a number field, $E/K$ be a Galois extension of the group $H$, $t_0 \in \mathbb{P}^1(K)$ be a fixed point and $S$ be a finite set of finite places of $K$. Then there exists a finite Galois extension $L/K$ totally split in $K_v$ for all $v \in S$ and a $G$-extension $F/L(T)$ of $G$ such that the specialization $F_{t_0}/L$ at $T = t_0$ satisfies the following:

1. $F_{t_0}/L$ is a Galois extension of $H$ isomorphic to $EL/L$.
2. The $v$-completion $F_{t_0}K_v/K_v$ is isomorphic to $EK_v/K_v$ for each $v \in S$.

The special case when $S = \emptyset$ and $G = H$ asserts that the BB problem has a positive answer over some finite extension $L$ of $K$, while conclusion (2) in case $S \neq \emptyset$ shows that the problem can be solved locally, i.e. after scalar extension to any given completion of $K$. Theorem 1.1 shows in fact that these two conclusions, local and global, can be combined.

The following statement provides two further conclusions: the first one is a uniformizing moduli space version (to be compared with the main result of B. Deschamps [Des]): we show that the $G$-extensions $F/L(T)$ from Theorem 1.1 can all be found on a curve on a Hurwitz space (the same for all $S$). The second one considers the more general case where $K$ is a Hilbertian field (1) of characteristic 0.

**Theorem 1.1 (continued).**

3. The $G$-extensions $F/L(T)$ can be constructed in such a way that the branch point number $r$ and the ramification type $C$ (2) are independent of $S$. That is, the corresponding point on the moduli space lies on the same Hurwitz space $H_r(G,C)$ (3). More precisely, these points can all be picked on a curve $C$ contained in $H_r(G,C)$ and for a given $S$, infinitely many points from $C$ provide a $G$-extension $F/L(T)$ as in Theorem 1.1.

4. In the special situation $S = \emptyset$, the global conclusion (1) holds more generally if $K$ is a Hilbertian field of characteristic 0.

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1. A field $K$ is said to be Hilbertian if for each irreducible polynomial $f(T,Y) \in K(T)[Y]$, there are infinitely many $t \in K$ such that the specialized polynomial $f(t,Y)$ is irreducible in $K[Y]$ (see [V]).

2. For definition, see §1.3.1.

3. For more details, see §1.3.4.
1.3. Preliminary reminders

1.3.1. Ramification type of $G$-extensions. Let $G$ be a finite group, $K$ be an algebraically closed field of characteristic 0, and $F/K(T)$ be a $G$-extension of group $G$. Denote the unordered set of all branch points of this extension by $t = \{t_1, \ldots, t_r\}$, which can be viewed as a $K$-rational divisor of $\mathbb{P}^1$.

With each $j = 1, \ldots, r$, we can associate a conjugacy class $C_j$ of the group $G$ as follows: The point $t_j$ is a branch point of our $G$-extension, so its inertia groups are conjugate and cyclic of order equal to the ramification index. Given one of these inertia groups $I_j$, an element $\sigma \in I_j$ is said to be a distinguished generator if $\sigma(\pi)/\pi = e^{2\pi i/e_j}$, where $\pi$ is equal to $(T - t_j)^{1/e_j}$. Then all the distinguished generators can be shown to be in the same conjugacy class of $G$: this is the conjugacy class $C_j$.

Denote by $C = (C_1, \ldots, C_r)$ the ramification type of $F/K(T)$. We refer to [De2, Chapter 3] for more details.

By branch points and ramification type of some $G$-extension $E/K(T)$ over some non-algebraically closed field $K$ of characteristic 0 we mean those of the $G$-extension $E\mathbb{K}(T)/\mathbb{K}(T)$ (in a given algebraic closure of $\mathbb{K}(T)$).

1.3.2. The fundamental group. Let $r \geq 1$ be an integer and $K$ be a field of characteristic 0. Denote by $\overline{K}$ an algebraic closure of $K$ and by $G_K$ the absolute Galois group of $K$. Let $U_r$ be the variety of all unordered $r$-uples $t = \{t_1, \ldots, t_r\}$ of $(\mathbb{P}^1)^r$ such that $t_i \neq t_j$ for all $1 \leq i \neq j \leq r$. We fix an algebraic closure $\mathbb{K}(T)$ of $\mathbb{K}(T)$. Take $t \in U_r(K)$. The fundamental group of $\mathbb{P}^1 \setminus t$ is defined as follows:

Denote by $\Omega_t$ the maximal Galois extension of $K(T)$ unramified above $\mathbb{P}^1 \setminus t$. Then the geometric fundamental group, $\pi_1(\mathbb{P}^1 \setminus t)_\overline{K}$, of $\mathbb{P}^1 \setminus t$ is the Galois group of $\Omega_t/\overline{K}(T)$. Moreover, as $t \in U_r(K)$, it follows that $\Omega_t/K(T)$ is a Galois extension. By definition, its Galois group is the $K$-fundamental group of $\mathbb{P}^1 \setminus t$ and it is denoted by $\pi_1(\mathbb{P}^1 \setminus t)_K$.

\[
\begin{array}{ccc}
\Omega_t & \xrightarrow{\pi_1(\mathbb{P}^1 \setminus t)_K} & \overline{K}(T) \\
\downarrow \pi_1(\mathbb{P}^1 \setminus t)_K & & \downarrow G_K \\
K(T) & & K(T)
\end{array}
\]

Moreover, the following exact sequence (given by Galois theory) is split as each $K$-rational point $t_0 \in \mathbb{P}^1(K) \setminus t$ provides a section $s_{t_0}$ of the canonical surjection of this sequence:
1.3.3. Specialization. Let $G$ be a finite group, $K$ be a field of characteristic 0, and $F/K(T)$ be a $G$-extension of group $G$ with a $K$-rational branch divisor $t$. This $G$-extension corresponds to some epimorphism $\phi : \pi_1(\mathbb{P}^1 \setminus t)_K \to G$ and the extension $FK/K(T)$ corresponds to the restriction $\bar{\phi} : \pi_1(\mathbb{P}^1 \setminus t)_K \to G$ of $\phi$ to $\pi_1(\mathbb{P}^1 \setminus t)_K$; it is still surjective as $F/K$ is a regular extension. Let $t_0 \in \mathbb{P}^1(K) \setminus t$ be a $K$-rational point and consider the section $s_{t_0}$ corresponding to this point.

The specialization $F_{t_0}$ of $F/K(T)$ at $T = t_0$ (i.e. the residue field of $F$ at some prime above $t_0$ in the extension $F/K(T)$) corresponds to the homomorphism $\phi \circ s_{t_0}$ ([De2, Proposition (2.1)]). More precisely, $F_{t_0}$ is the fixed field in $\overline{K}$ of $\ker(\phi \circ s_{t_0})$. In particular, the specialization $F_{t_0}/K$ is a Galois field extension of the group $\text{Im}(\phi \circ s_{t_0})$. For more details, we refer to [De2, Chapter 3] and to [De1].

1.3.4. Hurwitz spaces. Assume that $G$ is a finite group and $r > 2$ is an integer. Denote by $H_r(G)$ the moduli space of all $G$-extensions of the group $G$ with $r$ branch points. This moduli space is a smooth (not necessarily connected) variety defined over $\mathbb{Q}$ and, for all algebraically closed fields $k$ of characteristic 0, the $k$-rational points in $H_r(G)$ correspond to the isomorphism classes of $G$-extensions $F/k(T)$ of $G$ defined over $k$ with $r$ branch points (see [FrV]).

We fix an $r$-tuple $C = (C_1, \ldots, C_r)$ of conjugacy classes of $G$. We denote by $H_r(G, C) \subseteq H_r(G)$ the subset of all $G$-extensions $F/k(T)$ of $G$ with $r$ branch points and of ramification type $C$. We refer to [V] for more details.

2. Proof of the main result. Let $G$ be a finite group, $K$ be a field of characteristic 0, $t_0 \in \mathbb{P}^1(K)$ be a fixed point and $E/K$ be a Galois extension of a group $H \subset G$.

2.1. BB over a curve. The starting ingredient of the proof is the following:
Under the above hypotheses, there exist:

1. a smooth projective geometrically irreducible curve $C$ defined over $K$ with a $K$-rational point,
2. a $G$-extension $\mathcal{F}/K(C)(T)$ of the group $G$ defined over the function field $K(C)$,

with the following property: For all $x \in C(K)$ off a proper Zariski closed subset $Z$, the extension $\mathcal{F}/K(C)(T)$ specializes to some $G$-extension $\mathcal{F}_x/K(x)(T)$ of $G$ which is unramified above $T = t_0$ and whose specialization $\mathcal{F}_{x,t_0}/K(x)$ at $T = t_0$ is a Galois extension isomorphic to $E(x)/K(x)$.

The following diagram illustrates this double specialization process:

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{x \in C(K) \setminus Z} & \mathcal{F}_x \\
K(C)(T) & \xrightarrow{T = t_0} & K(x)(T) \\
\end{array}
$$

This starting ingredient is essentially an arithmetic translation of Theorem 2.7 of [MB2] which was expressed in geometric terms. In §3, we show in a pure field arithmetic language how to deduce this result from the fact, mentioned in the introduction, that the BB problem has a positive answer if the base field is the complete (thus ample) field $K((X))$ of formal Laurent series with coefficients in $K$.

Denote the branch point number of $\mathcal{F}/K(C)(T)$ by $r$, the branch point set by $t = \{t_1, \ldots, t_r\} \in \mathbb{P}^1(K(C))$ and its ramification type by $C = (C_1, \ldots, C_r)$.

2.2. Proof of Theorem 1.1. Assume further $K$ is a number field. The next step uses the Chebotarev density theorem [EL, Theorem (5.6)]: we can find, for each $g \in H$, a finite valuation $v_g$ of $K$, not in $S$, unramified in $E/K$ and such that its decomposition group, $\text{Gal}(E_v/K_v)$, is conjugate to $\langle g \rangle$ in $H$; and the $v_g$ can further be chosen pairwise distinct. Let $S'$ denote the set of all places $v_g$ with $g \in H$, and $S''$ the union of the set of places $S$ from the statement of Theorem 1.1 and the set $S'$.

Let $K^{\text{tot},S''}$ be the field of all totally $S''$-adic algebraic numbers, i.e. of all $x \in \overline{K}$ such that all $K$-conjugates of $x$ lie in the completion $K_v$, for all $v \in S''$. As $C$ contains a $K$-rational point and $K \subseteq K^{\text{tot},S''}$, the set $C(K^{\text{tot},S''})$ is not empty. The field $K^{\text{tot},S''}$ is known to be ample [MB1]. Using this fact, we deduce that $C(K^{\text{tot},S''})$ is an infinite set.

Fix $x \in C(K^{\text{tot},S''}) \setminus Z$, let $L$ be the Galois closure of $K(x)$, and denote the specialization $\mathcal{F}_x$ by $F$. From statement $(*)$, $F/L(T)$ is a $G$-extension of
G whose specialization at $T = t_0$ is a Galois extension isomorphic to $EL/L$. As $L \subseteq K^{\text{tot} \cdot S''} \subseteq K_v$, the extension $F_{t_0}K_v/K_v$ is isomorphic to $EK_v/K_v$ (for all $v \in S$). This completes the proof of (3).

It remains to prove that $F_{t_0}/L$ is a Galois extension of $H$. In fact, we know that $\text{Gal}(F_{t_0}/L)$ is a subgroup of $H$ and for each $v \in S''$ it contains $\text{Gal}(F_{t_0}K_v/K_v)$ up to conjugation: more precisely, there exists $\sigma_v \in H$ such that
\[
\langle g \rangle^{\sigma_v} \subseteq H.
\]

Via a classical lemma due to Jordan, which states that there is no proper subgroup of $H$ that meets all conjugacy classes of $H$, we conclude that $H = \text{Gal}(F_{t_0}/L)$. This ends the proof of (1)–(2) of Theorem 1.1.

2.3. Proof of Theorem 1.1 (continued). Up to enlarging the Zariski closed subset $Z$, we can claim that $F/L(T)$ has the same number of branch points $r$ and the same ramification type $C$ as $F/K(C)(T)$. Thus if the field $L$ depends on $S$, the number of branch points and the ramification type of $F/L(T)$ do not. Furthermore, the constructed extensions $F/L(T)$ correspond to points on the curve $C$ which is contained in the Hurwitz space $H_r(G, C)$. This completes the proof of (3).

Finally we prove property (4). Assume that $K$ is more generally a Hilbertian field of characteristic 0. From $(*)$, we have the following property: for all $x \in C(K) \setminus Z$, there exists a $G$-extension $F/K(x)(T)$ of $G$ defined over $K(x)$ such that its specialization at $T = t_0$ is a Galois extension isomorphic to $E(x)/K(x)$. To prove our last claim, it suffices to find infinitely many points $x \in C(K) \setminus Z$ such that the extensions $K(x)/K$ and $E/K$ are linearly disjoint (then $E(x)/K(x)$ is a Galois extension of the group $H$).

In fact, let $h(w, z) = 0$ be an affine equation of $C$ with $h(W, Z) \subseteq K[W, Z]$ irreducible in $\overline{K}[W, Z]$. As $E$ is a finite extension of the Hilbertian field $K$, we can find infinitely many points $w_0$ in $K$ (and not only in $E$) such that $h(w_0, Z) \subseteq K[Z]$ is irreducible in $E[Z]$ ([11 Corollary (1.8)]).

Pick $z_0 \in \overline{K}$ such that $h(w_0, z_0) = 0$. As $Z$ is a finite set and infinitely many $w_0 \in K$ satisfy this equality, there exist infinitely many points $x := (w_0, z_0) \in C(\overline{K}) \setminus Z$ (with $w_0 \in K$) such that $h(w_0, Z) \subseteq K[Z]$ is irreducible in $E[Z]$. We deduce that $[K(x) : K] = \deg_Z(h) = [E(x) : E]$. So the extensions $K(x)/K$ and $E/K$ are linearly disjoint.

3. Proof of BB over a curve

3.1. The starting result. The BB problem is known to have a positive answer over any complete valued field. This has been proved by Colliot-Thélène [CT] in characteristic 0, and, more generally, by Moret-Bailly [MB2] and Haran–Jarden [HJ], thanks to some deformation techniques. The com-
complete valued field we will be using is the field $K((X))$ of formal Laurent series with coefficients in $K$.

As $E/K$ is a Galois extension of $H$, the extension $E((X))/K((X))$ is also a Galois extension of $H$. From the paper \[HJ\] which uses as we do field-theoretic language, there exists a $G$-extension $F_{K((X))}/K((X))(T)$ of $G$ unramified above $T = t_0$ whose specialization at $T = t_0$ is a Galois extension of $H$ isomorphic to $E((X))/K((X))$. The construction of $F_{K((X))}$ makes it possible to assume further that all branch points of $F_{K((X))}/K((X))(T)$ lie in $\mathbb{P}^1(N)$ where $N = K((X)) \cap \overline{K(X)}$.

Furthermore, by Section 1.3.3 there exists a split exact sequence of fundamental groups

\[
1 \rightarrow \pi_1(\mathbb{P}^1 \setminus t)_{K((X))} \rightarrow \pi_1(\mathbb{P}^1 \setminus t)_{K((X))} \rightarrow G_{K((X))} \rightarrow 1
\]

As the specialization of $F_{K((X))}/K((X))(T)$ at $T = t_0$ is a Galois extension of $K((X))$ with the group $H$ isomorphic to $E((X))/K((X))$, the image of the homomorphism $\phi \circ s_{t_0}$ is exactly $H$.

3.2. Descent to $N = K((X)) \cap \overline{K(X)}$. We use an argument (see \[DeDes\] Theorem (3.4)) showing that the natural restriction morphism $\pi_1(\mathbb{P}^1 \setminus t)_{K((X))} \rightarrow \pi_1(\mathbb{P}^1 \setminus t)_N$ is an isomorphism.

Indeed, first, the restriction morphism $\pi_1(\mathbb{P}^1 \setminus t)_{K((X))} \rightarrow \pi_1(\mathbb{P}^1 \setminus t)_N$ is an isomorphism (via Riemann’s existence theorem). Second, the restriction morphism $\rho : G_{K((X))} \rightarrow G_N$ is also an isomorphism. We deduce that the natural restriction morphism $\gamma : \pi_1(\mathbb{P}^1 \setminus t)_{K((X))} \rightarrow \pi_1(\mathbb{P}^1 \setminus t)_N$ is an isomorphism.

Furthermore, as $t_0$ is a point in $\mathbb{P}^1(K) \setminus t$, the section $s_{t_0}$ induces a section, still denoted by $s_{t_0}$, of $\pi_1(\mathbb{P}^1 \setminus t)_N \rightarrow G_N$. 
This implies that there exists a unique G-extension $F_N/N(T)$ defined over $N$ of the group $G$ such that $F_N K((X)) = F_K((X))$. Namely this extension, $F_N/N(T)$, corresponds to the epimorphism $\phi_N = \gamma^{-1} \circ \phi : \pi_1(\mathbb{P}^1 \setminus t)_N \to G$. Furthermore, the specialization $F_{N,t_0}/N$ of $F_N/N(T)$ at $T = t_0$ is a Galois extension of $H$ such that $F_{N,t_0} K((X))$ equals the specialization of $F_K((X))$ at $t_0$. By construction of $F_K((X))$, we obtain $F_{N,t_0} K((X)) = E((X))$. On the other hand, as $\rho$ is an isomorphism, we deduce that $F_{N,t_0} = EN$.

To sum up, we now have a G-extension $F_N/N(T)$ defined over $N$ of the group $G$ whose specialization $F_{N,t_0}/N$ at $T = t_0$ is a Galois extension of $H$ isomorphic to $EN/N$.

3.3. Descent to the curve $\mathcal{C}$. We will prove the following claim:

(\textbf{**}) \textit{There exists a finite extension $L/K(X)$ with $L \subset N$ satisfying the following property:}

(i) \textit{There exists a G-extension $F_L/L(T)$ of $G$ such that $F_L N = F_N$.}

(ii) \textit{The specialization $F_{L,t_0}/L$ of $F_L/L(T)$ at $t_0$ is a Galois extension of $H$ such that $F_{L,t_0} = EL$.}

In fact, denote by $y(T)$ a primitive element of the extension $F_N/N(T)$ integral over $N[T]$. All conjugates of $y(T)$ over $N(T)$ can be expressed as rational functions of $T$ and $y(T)$ with coefficients in $N$. Consider the field $L_1$ generated over $K(X)$ by all coefficients of such expressions together with the coefficients in $N$ of the irreducible polynomial of $y(T)$ over $N(T)$. This field, $L_1$, is a finite extension of $K(X)$ contained in $N$. We deduce that $F_{L_1} = L_1(T, y(T))$ satisfies condition (i) of the claim above (with $L_1$ replacing $L$).

To study condition (ii), we view $y(T)$ as a formal power series

$$\sum_{j \geq 0} a_j (T-t_0)^j \in L_1[[T-t_0]].$$
This is possible because $t_0$ is unramified in $F_{L_1}/L_1(T)$. Then we have $F_{L_1,t_0} = L_1(a)$, where $a = \{a_0, a_1, \ldots \}$. As $F_{L_1,t_0}/L_1$ is a finite extension, we can find a finite subset $a_I$ of $a$ such that $L_1(a) = L_1(a_I)$. We also have

$$N(a_I) = F_{N,t_0} = EN.$$  

Consequently, there exists a finite subset $N_0 \subset N$ such that

$$L_1(N_0)(a_I) \subset L_1(N_0)E.$$  

Finally, let $\alpha$ be a primitive element of $E/K$. As $N(a_I) = F_{N,t_0} = EN$, we can find a finite subset $N_1 \subset N$ such that $\alpha \in L_1(N_1)(a_I)$.

Set $L = L_1(N_0 \cup N_1)$ and $F_L = F_{L_1}L$. It is easily checked that $L(a_I) = EL$ and so $F_{L,t_0} = LE$. We conclude that (**) is satisfied for this $L$ and this $F_L$.

The containment $L \subseteq N = K((X)) \cap \overline{K(X)}$ implies that $L$ is regular over $K$ and so is the function field of a smooth projective geometrically irreducible curve $C$ defined over $K$. It follows from $L = K(C) \subseteq K((X))$ that $C(K)$ is not empty.

(**) above shows that there exists a $G$-extension $F/K(C)(T)$ of $G$ such that $E(C)/K(C)$ is the specialization of $F/K(C)(T)$ at $T = t_0$.

$$
\begin{array}{ccc}
F & \longrightarrow & E(C) \\
\downarrow{\scriptstyle T=t_0\in\mathbb{P}^1(K)} & & \downarrow{\scriptstyle K(C)(T) \longrightarrow K(C)} \\
K(C)(T) & & K(C)
\end{array}
$$

3.4. Specialization to points on the curve $C$. As the extension $F/K(C)(T)$ is regular, the Bertini–Noether theorem [FJ, Proposition (8.8)] can be applied. Thus there exists a Zariski closed subset $Z$ of $C$ such that for each $x \in C(K) \setminus Z$, the extension $F/K(C)(T)$ specializes to some $G$-extension $F_x/K(x)(T)$ of $G$, unramified at $T = t_0$. Furthermore, the specialization $F_{x,t_0}/K(x)$ of $F_x/K(x)(T)$ at $T = t_0$ is a Galois extension such that $F_{x,t_0}L = F_{t_0} = EL$.

To finish the proof, we should show that $F_{x,t_0} = EK(x)$. Indeed, we have, first, $F_{x,t_0} = F_{x,t_0}L \cap \overline{K(x)}$ because the extensions $F_{x,t_0}/K(x)$ and $L/K(x)$ are linearly disjoint. As $F_{x,t_0}L = EL$, we deduce that $F_{x,t_0} = EL \cap \overline{K(x)}$. But $EL \cap \overline{K(x)} = EK(x)$ because the extensions $EK(x)/K(x)$ and $L/K$ are linearly disjoint. Thus $F_{x,t_0} = EK(x) = E(x)$.

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