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On the sum of the first n values of the Euler function

by

R. Balasubramanian (Chennai),
Florian Luca (Santiago de Querétaro and Johannesburg)
and Dimbinaina Ralaivaosaona (Muizenberg)

1. Introduction. Let $\phi(n)$ be the Euler function. Put

$$E(x)=\sum_{n\le x}\phi(n)-\frac{3}{\pi^2}x^2.$$

Walfisz [3] proved that

$$E(x)=O(x(\log x)^{2/3}(\log\log x)^{4/3}).$$

We shall denote by $\delta(x)$ all functions which are bounded above by

$$\exp\big(-A(\log x)^{3/5}(\log\log x)^{-1/5}\big)$$

as $x\to\infty$, where $A>0$ is some constant. The mean value of $E(n)$ was considered in [2], where it was shown that

$$\sum_{n\le x}E(n)=\frac{3x^2}{2\pi^2}+O(x^2\delta(x)). \tag{1.1}$$

Here, we give an estimate for the mean value of $E(n)^2$.

Theorem 1.1. *The estimate*

$$\sum_{n\le x}E(n)^2=\left(\frac{1}{6\pi^2}+\frac{2}{\pi^4}\right)x^3+O(x^3\delta(x))$$

holds for $x>10$ with a suitable value of A.

2. The proof of Theorem 1.1. We begin by stating a similar result essentially due to Chowla [1].

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LEMMA 2.1. *We have the estimate*

$$\int_0^x E(u)^2\,du = \frac{x^3}{6\pi^2} + O(x^3\delta(x)) \qquad \textit{as } x \to \infty. \tag{2.1}$$

Note here that the result in [1] is not quite as precise as in the estimate (2.1), because Chowla started his proof with a weaker estimate of the Mertens function:

$$M(x) := \sum_{n\le x} \mu(n) = O\left(\frac{x}{(\log x)^{30}}\right)$$

as $x \to \infty$. Since it is well known that the Mertens function satisfies

$$M(x) = O(x\delta(x)) \qquad \text{as } x \to \infty,$$

the arguments in [1] can be adapted to deduce (2.1).

We will now see that Theorem 1.1 can easily be deduced from Lemma 2.1. Let us assume, without loss of generality, that x is an integer. Then we have

$$\int_0^x E(u)^2\,du = \sum_{k=0}^{x-1} \int_0^1 E(k+u)^2\,du.$$

Furthermore, for any integer k and $u \in (0,1)$ we have

$$\begin{aligned} E(k+u)^2 &= \left(E(k) - \frac{3}{\pi^2}(2uk+u^2)\right)^2 \\ &= E(k)^2 + \frac{9}{\pi^4}(2uk+u^2)^2 - \frac{12}{\pi^2}ukE(k) - \frac{6}{\pi^2}u^2E(k). \end{aligned}$$

Hence,

$$\begin{aligned} &\int_0^x E(u)^2\,du \\ &= \sum_{k=0}^{x-1} E(k)^2 + \sum_{k=0}^{x-1}\int_0^1 \left(\frac{9}{\pi^4}(2uk+u^2)^2 - \frac{12}{\pi^2}ukE(k) - \frac{6}{\pi^2}u^2E(k)\right) du \\ &= \sum_{k=0}^{x-1} E(k)^2 + \frac{4}{\pi^4}x^3 - \frac{6}{\pi^2}\sum_{k=0}^{x-1} kE(k) + \mathcal{O}(x^2). \end{aligned} \tag{2.2}$$

Finally,

$$\begin{aligned}\sum_{k=0}^{x-1} kE(k) &= x\sum_{k=0}^{x-1} E(k) - \sum_{k=0}^{x-1}(x-k)E(k)\\ &= x\sum_{k=0}^{x-1} E(k) - \sum_{l=0}^{x-1}\sum_{k=0}^{l} E(k)\\ &= \frac{1}{\pi^2}x^3 + O(x^3\delta(x)).\end{aligned}$$

Here, we used (1.1) to estimate both terms of the second line and deduce the last line. Combining this with (2.2), we get

$$\sum_{k=0}^{x-1} E(k)^2 = \int_0^x E(u)^2\,du + \frac{2}{\pi^4}x^3 + O(x^3\delta(x)).$$

Thus, Theorem 1.1 follows from the estimate (2.1) of Lemma 2.1.

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R. Balasubramanian
The Institute of Mathematical Sciences
Chennai 600 113, India
E-mail: balu@imsc.res.in

Dimbinaina Ralaivaosaona
African Institute for Mathematical Sciences
6 Melrose Ave.
Muizenberg, South Africa
E-mail: naina@aims.ac.za

Florian Luca
Mathematical Institute
UNAM Juriquilla
76230 Santiago de Querétaro, México
and
School of Mathematics
University of the Witwatersrand
P.O. Box Wits 2050
Johannesburg, South Africa
E-mail: fluca@matmor.unam.mx

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