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Jumps of ternary cyclotomic coefficients

by

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1. Introduction. The nth *cyclotomic polynomial* is

$$\Phi_n(x) = \prod_{1\le m\le n,\,(m,n)=1} (x-\zeta_n^m) = \sum_{k\in\mathbb{Z}} a_n(k)x^k, \quad \text{where} \quad \zeta_n = e^{2\pi i/n}.$$

We say that Φ_n is *binary* if n is a product of two distinct odd primes, *ternary* if n is a product of three distinct odd primes, etc.

The coefficients of cyclotomic polynomials are a popular object of study. One of the intensively studied directions is estimating the maximal absolute value of the coefficients of Φ_n [1, 3–6, 8]. There are also papers on the sum of the absolute values of the coefficients [3, 8] and on the number θ_n of nonzero coefficients [7, 9].

Ternary cyclotomic polynomials have an interesting property discovered by Gallot and Moree [10]: the absolute difference between $a_{pqr}(k)$ and $a_{pqr}(k-1)$ never exceeds 1. In this paper, for a given ternary cyclotomic polynomial Φ_{pqr}, we characterize all k such that $|a_{pqr}(k) - a_{pqr}(k-1)| = 1$. Also we determine the number of k's for which this equality holds.

We say that the coefficient $a_{pqr}(k)$ is *jumping up* if $a_{pqr}(k) = a_{pqr}(k-1) + 1$. Analogously we define jumping down coefficients. Cyclotomic polynomials are known to be palindromic, i.e. $a_n(k) = a_n(\varphi(n)-k)$, where $\varphi(n)$ is the Euler function and the degree of Φ_n. Therefore the number of jumping up coefficients and the number of jumping down ones are equal; we denote this number by J_{pqr}.

One of our main results is the following theorem.

Theorem 1.1. *For a ternary cyclotomic polynomial Φ_n we have*

$$J_n > n^{1/3}.$$

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Half the total number of jumping (up or down) coefficients is a lower bound for the number of odd coefficients of Φ_{pqr} and thus it is a lower bound for the number θ_{pqr} of nonzero coefficients of Φ_{pqr}. So we have

COROLLARY 1.2. *Let Φ_n be a ternary cyclotomic polynomial. Then*

$$\theta_n > n^{1/3}.$$

We do not know if for every $\epsilon > 0$ there exist infinite classes of ternary cyclotomic polynomials Φ_n with $J_n < n^{1/3+\varepsilon}$. However, under some strong assumptions, we can prove that they do exist.

THEOREM 1.3. *Let $0 < \varepsilon < 1/2$. If q is a Germain prime, $q + 1$ has a prime divisor $p > q^{1-\varepsilon}$ and $r = 2q+1$, then $J_n < 10n^{1/(3-\varepsilon)}$, where $n = pqr$.*

If the celebrated Schinzel Hypothesis H is true then there exist infinitely many triples of primes (p, q, r) satisfying the conditions of Theorem 1.3. For example, we can put $(p, q, r) = (m, 6m - 1, 12m - 1)$ and take $m > 6$.

The paper is organized as follows. In Section 2 we recall some results from our earlier work [6]. In Section 3 we give a criterion on k determining the value of $V(k) = a_{pqr}(k) - a_{pqr}(k-1) \in \{-1, 0, 1\}$. In Section 4 we derive a formula for J_{pqr} and prove Theorem 1.1. In Section 5 we prove Theorem 1.3 and discuss the case of inclusion-exclusion polynomials.

We remark that Liu [12] independently obtained a similar criterion on k determining $V(k)$ by a different method.

2. Preliminaries. Throughout the paper we fix distinct odd primes p, q, r. Let us emphasize that every fact we prove for (p, q, r) also has an appropriate symmetric version.

By $a^{-1}(b)$ we denote the inverse of a modulo b for $(a, b) = 1$. We treat this number as an integer from the set $\{1, \dots, b - 1\}$.

For every integer k we define $F_k \in \mathbb{Z}$ and $a_k \in \{0, 1, \dots, p - 1\}$, $b_k \in \{0, 1, \dots, q - 1\}$, $c_k \in \{0, 1, \dots, r - 1\}$ by the equation

$$k + F_k pqr = a_k qr + b_k rp + c_k pq,$$

which clearly has a unique solution (F_k, a_k, b_k, c_k) depending on k. In [6] we proved the following properties of the numbers F_k.

PROPOSITION 2.1 ([6, remark before Lemma 2.1]). *For $-(qr+rp+pq) < k < pqr$ we have $F_k \in \{0, 1, 2\}$.*

PROPOSITION 2.2 ([6, Lemma 2.2]). *We have*

$$F_k - F_{k-q} = \begin{cases} -1 & \text{if } a_k < r^{-1}(p) \text{ and } c_k < p^{-1}(r), \\ 1 & \text{if } a_k \geq r^{-1}(p) \text{ and } c_k \geq p^{-1}(r), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.3 ([6, Lemma 2.3]). *We have*

$$F_k - F_{k-q} - F_{k-r} + F_{k-q-r} = \begin{cases} -1 & \text{if } a_k \in \mathcal{A}_1^p, \\ 1 & \text{if } a_k \in \mathcal{A}_3^p, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\mathcal{A}_1^p = \{0,1,\dots,p-1\} \cap \big[q^{-1}(p) + r^{-1}(p) - p, \min\{q^{-1}(p), r^{-1}(p)\}\big),$$
$$\mathcal{A}_3^p = \{0,1,\dots,p-1\} \cap \big[\max\{q^{-1}(p), r^{-1}(p)\}, q^{-1}(p) + r^{-1}(p)\big).$$

Proposition 2.4 ([6, Lemma 2.4]). *We have*

$$F_k - F_{k-p} - F_{k-q} - F_{k-r} + F_{k-q-r} + F_{k-r-p} + F_{k-p-q} - F_{k-p-q-r} = 0.$$

Proposition 2.5 ([6, Lemma 5.1]). *For $k = 0,1,\dots,pqr-1$ we have*

$$\begin{aligned} &a_{pqr}(k) - a_{pqr}(k-1) \\ &\quad = N_0(F_k, F_{k-q-r}, F_{k-r-p}, F_{k-p-q}) - N_0(F_{k-p}, F_{k-q}, F_{k-r}, F_{k-p-q-r}) \\ &\quad = N_2(F_k, F_{k-q-r}, F_{k-r-p}, F_{k-p-q}) - N_2(F_{k-p}, F_{k-q}, F_{k-r}, F_{k-p-q-r}) \\ &\quad = \tfrac{1}{2}\big(N_1(F_{k-p}, F_{k-q}, F_{k-r}, F_{k-p-q-r}) - N_1(F_k, F_{k-q-r}, F_{k-r-p}, F_{k-p-q})\big), \end{aligned}$$

where $N_t(s)$ denotes the number of t's in the sequence (s).

3. A criterion on jumping coefficients.

We define five sets:

$$\mathcal{A}_0^p = \{0,1,\dots,p-1\} \cap \big[0, q^{-1}(p) + r^{-1}(p) - p\big),$$
$$\mathcal{A}_1^p = \{0,1,\dots,p-1\} \cap \big[q^{-1}(p) + r^{-1}(p) - p, \min\{q^{-1}(p), r^{-1}(p)\}\big),$$
$$\mathcal{A}_2^p = \{0,1,\dots,p-1\} \cap \big[\min\{q^{-1}(p), r^{-1}(p)\}, \max\{q^{-1}(p), r^{-1}(p)\}\big),$$
$$\mathcal{A}_3^p = \{0,1,\dots,p-1\} \cap \big[\max\{q^{-1}(p), r^{-1}(p)\}, q^{-1}(p) + r^{-1}(p)\big),$$
$$\mathcal{A}_4^p = \{0,1,\dots,p-1\} \cap \big[q^{-1}(p) + r^{-1}(p), p\big).$$

Note that if $q^{-1}(p) + r^{-1}(p) = p$ then both $\mathcal{A}_0^p$ and $\mathcal{A}_4^p$ are empty, otherwise precisely one of $\mathcal{A}_0^p$ and $\mathcal{A}_4^p$ is empty. Further, $\mathcal{A}_1^p$ and $\mathcal{A}_3^p$ are not empty, and $\mathcal{A}_2^p$ is empty if and only if $q^{-1}(p) = r^{-1}(p)$.

Similarly we define $\mathcal{A}_j^q$ and $\mathcal{A}_j^r$ for $j = 0,1,2,3,4$.

By Proposition 2.5, we have to consider 8-tuples

$$\mathrm{oct}(k) = (F_k, F_{k-p}, F_{k-q}, F_{k-r}, F_{k-q-r}, F_{k-r-p}, F_{k-p-q}, F_{k-p-q-r}).$$

We write $\mathrm{oct}(k) \sim (t_1,\dots,t_8)$ if $\mathrm{oct}(k) = (t_1+u,\dots,t_8+u)$ for some integer u. Put also

$$V(k) = a_{pqr}(k) - a_{pqr}(k-1)$$

and

$$\delta_{qr} = \begin{cases} 1 & \text{if } q^{-1}(p) < r^{-1}(p), \\ 0 & \text{otherwise.} \end{cases}$$

Analogously we define δ_{rq}, δ_{rp}, δ_{pr}, δ_{pq} and δ_{pq}.

The following theorem gives a criterion for whether the kth coefficient of Φ_{pqr} is jumping up or down or remains constant.

THEOREM 3.1. *The value $V(k)$ depends on which one of the sets $\mathcal{A}_{j_1}^p \times \mathcal{A}_{j_2}^q \times \mathcal{A}_{j_3}^r$ contains (a_k, b_k, c_k), in the way described in Table 1. The notation $(j_1 j_2 j_3)$ in the first column means $(a_k, b_k, c_k) \in \mathcal{A}_{j_1}^p \times \mathcal{A}_{j_2}^q \times \mathcal{A}_{j_3}^r$.*

In order to prove Theorem 3.1 we need the following simple fact.

LEMMA 3.2. *If $(a_k, b_k, c_k) \in \mathcal{A}_{j_1}^p \times \mathcal{A}_{j_2}^q \times \mathcal{A}_3^r$ and $(a_{k'}, b_{k'}, c_{k'}) \in \mathcal{A}_{j_1}^p \times \mathcal{A}_{j_2}^q \times \mathcal{A}_4^r$, then*

$$\mathrm{oct}(k') \sim (F_k, F_{k-p}, F_{k-q}, F_{k-r}, F_{k-q-r}, F_{k-r-p}, F_{k-p-q} - 1, F_{k-p-q-r} - 1).$$

Similarly, if $k \in \mathcal{A}_1^p \times \mathcal{A}_{j_2}^q \times \mathcal{A}_{j_3}^r$ and $k' \in \mathcal{A}_0^p \times \mathcal{A}_{j_2}^q \times \mathcal{A}_{j_3}^r$, then

$$\mathrm{oct}(k') \sim (F_k, F_{k-p}, F_{k-q}, F_{k-r}, F_{k-q-r} + 1, F_{k-r-p}, F_{k-p-q}, F_{k-p-q-r} + 1).$$

Proof. Let us consider the first situation. By Proposition 2.2 and its symmetric versions, it follows that

$$F_k - F_{k-p} = F_{k'} - F_{k'-p}, \quad F_k - F_{k-q} = F_{k'} - F_{k'-q}, \quad F_k - F_{k-r} = F_{k'} - F_{k'-r}.$$

Then, by Proposition 2.3 and its symmetric versions,

$$F_k - F_{k-q-r} = F_{k'} - F_{k'-q-r}, \qquad F_k - F_{k-r-p} = F_{k'} - F_{k'-r-p}.$$

Once more we use symmetric versions of Propositions 2.3 and 2.4 to obtain

$$F_k - F_{k-p} - F_{k-q} + F_{k-p-q} = F_{k-r} - F_{k-r-p} - F_{k-q-r} + F_{k-p-q-r} = 1,$$
$$F_{k'} - F_{k'-p} - F_{k'-q} + F_{k'-p-q} = F_{k'-r} - F_{k'-r-p} - F_{k'-q-r} + F_{k'-p-q-r} = 0.$$

Thus the first claim is true. The proof of the second one is similar. ■

Proof of Theorem 3.1. We determine $\mathrm{oct}(k)$ up to adding an integer, so in each row of Table 1 we can choose F_k arbitrarily. First, consider the cases $(j_1 j_2 j_3)$ from Table 1 which are not of the form $(0\ldots)$ or $(\ldots 4)$. Using Proposition 2.2 and its symmetric versions, we obtain the values of F_{k-p}, F_{k-q}, F_{k-r}. Then, by Proposition 2.3 and its symmetric versions we compute F_{k-q-r}, F_{k-r-p}, F_{k-p-q}. Finally, we use Proposition 2.4 to determine $F_{k-p-q-r}$.

We assume that $\delta_{pq} + \delta_{qp} = 1$ since if $\delta_{pq} = \delta_{qp} = 0$ then $\mathcal{A}_2^r = \emptyset$ and the case is empty. The situation with δ_{rq}, δ_{rq} and δ_{rp}, δ_{pr} is analogous.

Now we can use Lemma 3.2 to compute $\mathrm{oct}(k)$ for the remaining cases $(j_1 j_2 j_3)$, of the form $(0\ldots)$ or $(\ldots 4)$. After these computations the second column of Table 1 is complete.

Next we compute $V(k)$, for which we use Proposition 2.5. If a row does not contain any δ, the computation is straightforward. The remaining cases are considered one by one. We write $\mathrm{oct}(k) \sim (\ldots)$ if equality holds up to adding an integer.

Table 1. The values of oct(k) in dependence on (a_k, b_k, c_k)

$(j_1j_2j_3)$	oct$(k) \sim$	$V(k)$
001	$(0,1,1,1,2,2,1,2)$	1
002	$(0,\delta_{pq},\delta_{qp},1,1+\delta_{qp},1+\delta_{pq},1,2)$	0
003	$(0,0,0,1,1,1,1,2)$	-1
004	$(0,0,0,1,1,1,0,1)$	0
011	$(0,1,1,1,2,1,1,1)$	1
012	$(0,\delta_{pq},\delta_{qp},1,1+\delta_{qp},\delta_{pq},1,1)$	δ_{qp}
013	$(0,0,0,1,1,0,1,1)$	0
014	$(0,0,0,1,1,0,0,0)$	0
022	$(0,\delta_{pq}+\delta_{pr}-1,\delta_{qp},\delta_{rp},\delta_{qp}+\delta_{rp},\delta_{pq},\delta_{pr},1)$	0
023	$(1,\delta_{pr},1,1+\delta_{pr},1+\delta_{pr},1,1+\delta_{pr},2)$	$-\delta_{rp}$
024	$(1,\delta_{pr},1,1+\delta_{pr},1+\delta_{pr},1,\delta_{pr},1)$	0
033	$(1,0,1,1,1,1,1,2)$	-1
034	$(1,0,1,1,1,1,0,1)$	0
044	$(1,0,1,1,1,0,0,0)$	0
111	$(0,1,1,1,1,1,1,0)$	0
112	$(0,\delta_{pq},\delta_{qp},1,\delta_{qp},\delta_{pq},1,0)$	0
113	$(0,0,0,1,0,0,1,0)$	0
114	$(1,1,1,2,1,1,1,0)$	-1
122	$(1,\delta_{pq}+\delta_{pr},1+\delta_{qp},1+\delta_{rp},$ $\delta_{qp}+\delta_{rp},1+\delta_{pq},1+\delta_{pr},1)$	$\delta_{pq}\delta_{pr}-\delta_{qp}\delta_{rp}$
123	$(1,\delta_{pr},1,1+\delta_{rp},\delta_{rp},1,1+\delta_{pr},1)$	$\delta_{pr}-\delta_{rp}$
124	$(1,\delta_{pr},1,1+\delta_{rp},\delta_{rp},1,\delta_{pr},0)$	$-\delta_{rp}$
133	$(1,0,1,1,0,1,1,1)$	0
134	$(1,0,1,1,0,1,0,0)$	0
144	$(2,1,2,2,1,1,1,0)$	-1
222	$(1,\delta_{pq}+\delta_{pr},\delta_{qr}+\delta_{qp},\delta_{rp}+\delta_{rq},$ $\delta_{qp}+\delta_{rp},\delta_{rq}+\delta_{pq},\delta_{pr}+\delta_{qr},1)$	0
223	$(1,\delta_{pr},\delta_{qr},\delta_{rp}+\delta_{rq},\delta_{rp},\delta_{rq},\delta_{pr}+\delta_{qr},1)$	$\delta_{pr}\delta_{qr}-\delta_{rp}\delta_{rq}$
224	$(1,\delta_{pr},\delta_{qr},\delta_{rp}+\delta_{rq},\delta_{rp},\delta_{rq},\delta_{pr}+\delta_{qr}-1,0)$	0
233	$(1,0,\delta_{qr},\delta_{rq},0,\delta_{rq},\delta_{qr},1)$	0
234	$(2,1,1+\delta_{qr},1+\delta_{rq},1,1+\delta_{rq},\delta_{qr},1)$	δ_{rq}
244	$(2,1,1+\delta_{qr},1+\delta_{rq},1,\delta_{rq},\delta_{qr},0)$	0
333	$(1,0,0,0,0,0,0,1)$	0
334	$(2,1,1,1,1,1,0,1)$	1
344	$(2,1,1,1,1,0,0,0)$	1

Note that by Proposition 2.5 and the inequality $|V(k)| \leq 1$, if $\mathrm{oct}(k)$ contains an even number of 0s or an even number of 2s then $V(k) = 0$. Otherwise $|V(k)| = 1$.

(002) It does not matter which one of δ_{pq}, δ_{qp} equals 1, so we assume that $\delta_{pq} = 1$. Then $\mathrm{oct}(k) = (0,1,0,1,1,2,1,2)$ and $V(k) = 0$.
(244) This case is analogous to the previous one.
(112) Again, the value of δ_{pq} is not important and for $\delta_{pq} = 1$ we have $\mathrm{oct}(k) \sim (0,1,0,1,0,1,1,0)$ and $V(k) = 0$.
(233) As before, the value of δ_{qr} has no influence on $V(k)$ and we can assume $\delta_{qr} = 1$. Then $\mathrm{oct}(k) \sim (1,0,1,0,0,0,1,1)$ and $V(k) = 0$.
(012) For $\delta_{pq} = 1$ we have $\mathrm{oct}(k) \sim (0,1,0,1,1,1,1,1)$ and $V(k) = 0$. For $\delta_{qp} = 1$ we have $\mathrm{oct}(k) = (0,0,1,1,2,0,1,1)$ and $V(k) = 1$. Thus $V(k) = \delta_{qp}$.
(234) For $\delta_{qr} = 1$ we have $\mathrm{oct}(k) \sim (2,1,2,1,1,1,1,1)$ and $V(k) = 0$. For $\delta_{rq} = 1$ we have $\mathrm{oct}(k) = (2,1,1,2,1,2,0,1)$ and $V(k) = 1$. So $V(k) = \delta_{rq}$.
(024) If $\delta_{pr} = 0$ then $\mathrm{oct}(k) \sim (1,0,1,1,1,1,0,1)$. If $\delta_{pr} = 1$ then $\mathrm{oct}(k) \sim (1,1,1,2,2,1,1,1)$. In both cases $V(k) = 0$.
(123) If $\delta_{rp} = 1$, then $\mathrm{oct}(k) = (1,0,1,2,1,1,1,1)$ and $V(k) = -1$. If $\delta_{pr} = 1$, then $\mathrm{oct}(k) = (1,1,1,1,0,1,2,1)$ and $V(k) = 1$. So we have $V(k) = \delta_{pr} - \delta_{rp}$.
(023) For $\delta_{rp} = 1$ we have $\mathrm{oct}(k) = (1,0,1,1,1,1,1,2)$ and $V(k) = -1$. For $\delta_{pr} = 1$ we have $\mathrm{oct}(k) \sim (1,1,1,2,2,1,2,2)$ and $V(k) = 0$. Thus $V(k) = -\delta_{rp}$.
(124) If $\delta_{rp} = 1$, then $\mathrm{oct}(k) = (1,0,1,2,1,1,0,0)$ and $V(k) = -1$. When $\delta_{pr} = 1$, we have $\mathrm{oct}(k) \sim (1,1,1,1,0,1,1,0)$ and $V(k) = 0$. So $V(k) = -\delta_{rp}$.
(022) Note that in this case the situation $\delta_{pq} = \delta_{pr} = 0$ is impossible, because then $F_{m-q-r} = F_{m-p} + 3$, contradicting Proposition 2.1. If $\delta_{pq} = \delta_{pr} = 1$, then $\mathrm{oct}(k) \sim (0,1,0,0,0,1,1,1)$ and $V(k) = 0$. If one of δ_{pq}, δ_{pr} equals 1, we assume that $\delta_{pq} = 1$ (it does not matter). Then we have $\mathrm{oct}(k) \sim (0,0,0,1,1,1,0,1)$ and $V(k) = 0$.
(224) This case is analogous to the previous one.
(122) We have

$$\mathrm{oct}(k) \sim \begin{cases} (1,2,1,1,0,2,2,1) & \text{if } \delta_{pq} = \delta_{pr} = 1, \\ (1,0,2,2,2,1,1,1) & \text{if } \delta_{qp} = \delta_{rp} = 1, \\ (1,1,1,2,1,2,1,1) & \text{if } \delta_{pq} = \delta_{rp} = 1, \\ (1,1,2,1,1,1,2,1) & \text{if } \delta_{qp} = \delta_{pr} = 1, \end{cases}$$

from which we obtain $V(k) = \delta_{pq}\delta_{pr} - \delta_{qp}\delta_{rp}$.

(223) Similarly to the previous case, we have

$$\operatorname{oct}(k) \sim \begin{cases} (1,1,1,0,0,0,2,1) & \text{if } \delta_{pr} = \delta_{qr} = 1, \\ (1,0,0,2,1,1,0,1) & \text{if } \delta_{rp} = \delta_{rq} = 1, \\ (1,1,0,1,0,1,1,1) & \text{if } \delta_{pr} = \delta_{rq} = 1, \\ (1,0,1,1,1,0,1,1) & \text{if } \delta_{rp} = \delta_{qr} = 1. \end{cases}$$

We conclude that $V(k) = \delta_{pr}\delta_{qr} - \delta_{rp}\delta_{rq}$.

(222) In both sequences

$$(1, \delta_{pq} + \delta_{pr}, \delta_{qr} + \delta_{qp}, \delta_{rp} + \delta_{rq}),$$
$$(\delta_{qp} + \delta_{rp}, \delta_{rq} + \delta_{pq}, \delta_{pr} + \delta_{qr}, 1)$$

we have the same number of 2s, so $\operatorname{oct}(k)$ contains an even number of 2s and thus $V(k) = 0$.

Thus we verified all cases from Table 1. ∎

Let us add that there are 125 sets of type $\mathcal{A}^p_{j_1} \times \mathcal{A}^q_{j_2} \times \mathcal{A}^r_{j_3}$. By symmetry, using Theorem 3.1, we are able to obtain all of them except two. The exceptions are $\mathcal{A}^p_0 \times \mathcal{A}^q_0 \times \mathcal{A}^r_0$ and $\mathcal{A}^p_4 \times \mathcal{A}^q_4 \times \mathcal{A}^r_4$. The next lemma justifies their absence in Table 1 by proving that these products are empty.

LEMMA 3.3. *The three inequalities*

$$q^{-1}(p) + r^{-1}(p) > p, \qquad r^{-1}(q) + p^{-1}(q) > q, \qquad p^{-1}(r) + q^{-1}(r) > r$$

cannot hold at the same time. The same is true with $>$ *replaced by* $<$.

Proof. Adding the equality

$$\frac{q^{-1}(p)}{p} + \frac{p^{-1}(q)}{q} = 1 + \frac{1}{pq}$$

to its analogues with (p,q) replaced by (q,r), respectively (r,p), we obtain

$$\frac{q^{-1}(p) + r^{-1}(p)}{p} + \frac{r^{-1}(q) + p^{-1}(q)}{q} + \frac{p^{-1}(r) + q^{-1}(r)}{r} = 3 + \frac{1}{qr} + \frac{1}{rp} + \frac{1}{pq}.$$

Using this identity, the proof is easily completed. ∎

4. A formula for J_{pqr}. To present the announced formula, we need the notation

$$\alpha_p = \min\{q^{-1}(p), r^{-1}(p), p - q^{-1}(p), p - r^{-1}(p)\}, \quad \beta_p = (\alpha_p qr)^{-1}(p);$$

we define α_q, α_r, β_q, β_r similarly. One can easily check that

$$\beta_p = \max\bigl\{\min\{q^{-1}(p), p - q^{-1}(p)\}, \min\{r^{-1}(p), p - r^{-1}(p)\}\bigr\} \ge \alpha_p,$$
$$\#\mathcal{A}^p_1 = \#\mathcal{A}^p_3 = \alpha_p, \quad \#\mathcal{A}^p_2 = \beta_p - \alpha_p, \quad \#\mathcal{A}^p_0 + \#\mathcal{A}^p_4 = p - \alpha_p - \beta_p$$

and analogous inequalities hold for $\mathcal{A}^q_j$ and $\mathcal{A}^r_j$. Let also

$$\delta_p = \delta_{pq}\delta_{pr} + \delta_{rp}\delta_{qp},$$

and define δ_q and δ_r similarly. If the first inequality of Lemma 3.3 is the only false or the only true one, then we put

$$R = \alpha_p(q - \alpha_q - \beta_q)(r - \alpha_r - \beta_r).$$

If the only true/false inequality is the second or the third one, then we define R analogously. In addition we put

$$S = \sum_{\text{cycl}} \delta_p \alpha_p (\beta_q - \alpha_q)(\beta_r - \alpha_r),$$

$$T = \sum_{\text{perm}} \delta_{qr} (\beta_p - \alpha_p)(\alpha_q \# \mathcal{A}_0^r + \alpha_r \# \mathcal{A}_4^q),$$

where

$$\sum_{\text{cycl}} f(p,q,r) = f(p,q,r) + f(r,p,q) + f(q,r,p),$$

$$\sum_{\text{perm}} f(p,q,r) = f(p,q,r) + f(r,p,q) + f(q,r,p) + f(r,q,p) + f(p,r,q) + f(q,p,r).$$

Now we are ready to present the main result of this section.

Theorem 4.1. *We have*

$$J_{pqr} = R + S + T + \sum_{cycl} \alpha_p \alpha_q (r - 2\alpha_r).$$

Proof. In order to make the notation more readable, we put

$$\sigma_{j_1 j_2 j_3}^{\text{perm}} = \sum_{\text{perm}} \# \mathcal{A}_{j_1}^p \# \mathcal{A}_{j_2}^q \# \mathcal{A}_{j_3}^r,$$

$$\sigma_{j_1 j_2 j_3}^{\text{perm}}(f(p,q,r)) = \sum_{\text{perm}} f(p,q,r) \# \mathcal{A}_{j_1}^p \# \mathcal{A}_{j_2}^q \# \mathcal{A}_{j_3}^r,$$

and analogously

$$\sigma_{j_1 j_2 j_3}^{\text{cycl}} = \sum_{\text{cycl}} \# \mathcal{A}_{j_1}^p \# \mathcal{A}_{j_2}^q \# \mathcal{A}_{j_3}^r,$$

$$\sigma_{j_1 j_2 j_3}^{\text{cycl}}(f(p,q,r)) = \sum_{\text{cycl}} f(p,q,r) \# \mathcal{A}_{j_1}^p \# \mathcal{A}_{j_2}^q \# \mathcal{A}_{j_3}^r.$$

By Theorem 3.1 we have

$$J_{pqr} = \sigma_{001}^{\text{cycl}} + \sigma_{011}^{\text{cycl}} + \sigma_{334}^{\text{cycl}} + \sigma_{344}^{\text{cycl}} + \sigma_{123}^{\text{perm}}(\delta_{pr}) + \sigma_{012}^{\text{perm}}(\delta_{qp}) + \sigma_{234}^{\text{perm}}(\delta_{rq}) + \sigma_{122}^{\text{cycl}}(\delta_{pq}\delta_{pr}) + \sigma_{223}^{\text{cycl}}(\delta_{pr}\delta_{qr}).$$

It is easy to observe that

$$\sigma_{001}^{\rm cycl}+\sigma_{344}^{\rm cycl}=\sigma_{100}^{\rm cycl}+\sigma_{344}^{\rm cycl}=\sum_{\rm cycl}\alpha_p\big(\#\mathcal{A}_0^q\#\mathcal{A}_0^r+\#\mathcal{A}_4^q\#\mathcal{A}_4^r\big)=R,$$
$$\sigma_{011}^{\rm cycl}+\sigma_{334}^{\rm cycl}=\sigma_{011}^{\rm cycl}+\sigma_{411}^{\rm cycl}=\sum_{\rm cycl}(p-\alpha_p-\beta_p)\alpha_q\alpha_r.$$

Now we consider sums containing δs. The equalities above remain true, since if $\delta_{pr}+\delta_{rp}\neq 1$, then the set $\mathcal{A}_2^q$ is empty. We have

$$\sigma_{123}^{\rm perm}(\delta_{pr})=\sigma_{123}^{\rm cycl}(\delta_{pr})+\sigma_{321}^{\rm cycl}(\delta_{rp})=\sum_{\rm cycl}\alpha_p\alpha_q(\beta_r-\alpha_r),$$
$$\sigma_{012}^{\rm perm}(\delta_{qp})+\sigma_{234}^{\rm perm}(\delta_{rq})=\sigma_{210}^{\rm perm}(\delta_{qr})+\sigma_{243}^{\rm perm}(\delta_{qr})=T.$$

Finally,

$$\sigma_{122}^{\rm cycl}(\delta_{pq}\delta_{pr})+\sigma_{223}^{\rm cycl}(\delta_{pr}\delta_{qr})=\sigma_{122}^{\rm cycl}(\delta_{pq}\delta_{pr})+\sigma_{322}^{\rm cycl}(\delta_{rp}\delta_{qp})=S.$$

By summing the values obtained, we get the conclusion. ∎

As a consequence of Theorem 4.1 we now obtain Theorem 1.1:

Proof of Theorem 1.1. We will use the fact that $ab\geq a+b-1$ for any positive integers a and b. By Theorem 4.1 and the obvious inequality $R,S,T\geq 0$, we have

$$\begin{aligned}J_{pqr}&\geq\sum_{\rm cycl}\alpha_p\alpha_q(r-2\alpha_r)=\frac{1}{2}\sum_{\rm cycl}\alpha_p\big(\alpha_q(r-2\alpha_r)+(q-2\alpha_q)\alpha_r\big)\\&\geq\frac{1}{2}\sum_{\rm cycl}\big(\alpha_q+(r-2\alpha_r)-1+(q-2\alpha_q)+\alpha_r-1\big)\\&=\frac{1}{2}\sum_{\rm cycl}(q-\alpha_q-1+r-\alpha_r-1)\geq\frac{1}{2}\sum_{\rm cycl}\big((q-1)/2+(r-1)/2\big)\\&=(p-1)/2+(q-1)/2+(r-1)/2>(p+q+r)/3>\sqrt[3]{pqr},\end{aligned}$$

which completes the proof. ∎

5. Polynomials with small J_{pqr}

Proof of Theorem 1.3. Let $q=tp-1$ where $3\leq t<q^{\varepsilon}$, and let $r=2q+1=2tp-1$. Then it is not hard to verify that

$$\begin{aligned}&q^{-1}(p)=p-1, && r^{-1}(q)=1, && p^{-1}(r)=2t,\\&r^{-1}(p)=p-1, && p^{-1}(q)=t, && q^{-1}(r)=2tp-3,\\&\alpha_p=1, && \alpha_q=1, && \alpha_r=2,\\&\beta_p=1, && \beta_q=t, && \beta_r=2tp-2t-1,\\&\mathcal{A}_4^p=\emptyset, && \mathcal{A}_0^q=\emptyset, && \mathcal{A}_4^r=\emptyset,\end{aligned}$$

and $\delta_{rp} = \delta_{pq} = 1$. The remaining δs from Theorem 4.1 equal 0. Therefore

$$\sum_{\text{cycl}} \alpha_p\alpha_q(r - 2\alpha_r) = (2tp - 5) + 2(tp - 3) + 2(p - 2) < 6q$$

and

$$R = (p - \alpha_p - \beta_p)\alpha_q(r - \alpha_r - \beta_r) = (p - 2)(2t - 2) < 2q.$$

Since $\delta_p = \delta_q = \delta_r = 0$, we have $S = 0$. It remains to evaluate T:

$$\begin{aligned} T &= (\beta_q - \alpha_q)(\alpha_r\#\mathcal{A}_0^p + \alpha_p\#\mathcal{A}_4^r) + (\beta_r - \alpha_r)(\alpha_p\#\mathcal{A}_0^q + \alpha_q\#\mathcal{A}_4^p) \\ &= (t - 1)2(p - 2) < 2q. \end{aligned}$$

By Theorem 4.1 we have $J_{pqr} < 10q$, while $pqr > q^{3-\varepsilon}$, so the proof is complete. ∎

In the slightly more general class of so-called inclusion-exclusion polynomials, the exponent 1/3 in Theorem 1.1 is the best possible. We recall that

$$\Phi_{pqr}(x) = \frac{(1 - x^{pqr})(1 - x^p)(1 - x^q)(1 - x^r)}{(1 - x^{qr})(1 - x^{rp})(1 - x^{pq})(1 - x)}.$$

If we replace the assumption that p, q, r are primes by their being pairwise coprime, then the formula above defines the inclusion-exclusion polynomial $Q_{\{p,q,r\}}$ (see [2]).

Let us denote by $J_{\{p,q,r\}}$ the number of jumping up coefficients of the polynomial $Q_{\{p,q,r\}}$. As long as $p, q, r > 2$, all results of our paper also hold for $Q_{\{p,q,r\}}$.

The numbers m, $6m - 1$, $12m - 1$ are pairwise coprime for every positive integer m. Thus we can repeat the argument from the proof of Theorem 1.3 to deduce that

$$J_{\{m,6m-1,12m-1\}} < 10(6m - 1) < 15n^{1/3},$$

where $n = m(6m - 1)(12m - 1)$ and $m \geq 3$. The construction gives infinitely many ternary inclusion-exclusion polynomials $Q_{\{p,q,r\}}$ for which $J_{\{p,q,r\}} < 15n^{1/3}$, where $n = pqr$.

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