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On the exact location of the non-trivial zeros of Riemann's zeta function

by

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1. Introduction. The functional equation of Riemann's zeta function $\zeta(s)$ implies that $\zeta(1/2+it) = Z(t)e^{-i\vartheta(t)}$, where $Z(t)$ and $\vartheta(t)$ are real valued and real analytic functions and the *phase* $-\vartheta(t)$ is a rather simple function depending only on Euler's gamma function $\Gamma(s)$. An analogous decomposition is valid for any meromorphic function. We give a formal definition of the phase of a real analytic function in Section 2.

We will define some functions related to the zeros of $\zeta(s)$ and the phase of related functions. Of course, these functions have appeared in the literature but only in an implicit way and have not been studied for their own sake. For example, Levinson and Montgomery [13] define

$$J(1/2+it) := \zeta(1/2+it) + \zeta'(1/2+it)\left[\frac{h'(1/2+it)}{h(1/2+it)} + \frac{h'(1/2-it)}{h(1/2-it)}\right]^{-1}$$

where $h(s) = \pi^{-s/2}\Gamma(s/2)$, and assert that "the determination of the number of zeros of $\zeta(s)$ in $\sigma > 1/2$ can be conveniently ascertained from the variation of $\arg J(1/2+it)$". They do not use the simplified form

$$J(1/2+it) = -e^{-2i\vartheta(t)}\frac{\zeta'(1/2-it)}{2\vartheta'(t)}.$$

With our notation we would have

$$\operatorname{ph} J(1/2+it) = \pi - 2\vartheta(t) - \operatorname{ph}\zeta'(1/2+it) = \pi/2 + \pi\kappa(t) - \vartheta(t) = 2\pi - E(t).$$

Here $\kappa(t)$ is the main function we introduce. It is closely connected with the zeros of $\zeta(s)$, and is implicitly used in Levinson [12, equation (1.6)] to prove that more than 1/3 of the zeros of $\zeta(s)$ are on the critical line.

In our paper we seldom assume the RH, and use the standard notation of the subject. Therefore, we denote the zeros of $\zeta(s)$ in the upper half-plane by $\beta_n + i\gamma_n$, where β_n and γ_n are real numbers and $0 < \gamma_1 \le \gamma_2 \le \cdots$.

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If a zero is multiple with multiplicity m, then it appears precisely m times consecutively in the above sequence [17, Chapter 9, p. 214]. We shall need to introduce another related sequence of *real* numbers $(0 <)\ \xi_1 < \xi_2 < \cdots$ defined so that $\{\xi_n : n \in \mathbb{N}\} := \{t > 0 : \zeta(1/2 + it) = 0\}$. Here only the ordinates of the zeros on the critical line appear. These ξ_n do not repeat by any circumstance.

The two sequences (ξ_n) and (γ_n) coincide if and only if the RH is true and all the zeros of $\zeta(s)$ on the critical line are simple.

Even in case the RH were not true, we will show that $\kappa(t)$ is related to the zeros of $\zeta(s)$ on the critical line. We will prove that $\kappa(\xi_n) = n$ for all natural numbers, independently of any hypothesis.

The relations between the zeros of $\zeta(s)$ and $\zeta'(s)$ have been the object of much study. Starting with Speiser [16] who showed that the RH is equivalent to $\zeta'(s)$ having no zeros in $0 < \sigma < 1/2$, Levinson and Montgomery [13] give a quantified version of Speiser's theorem. Berndt [2] gives an estimation of the number of zeros of $\zeta'(s)$ to a given height. Great interest in the zeros of $\zeta'(s)$ is related to their horizontal distribution, in which many questions remain open (see Levinson and Montgomery [13], Conrey and Ghosh [5], Soundararajan [15], Zhang [19], Garaev and Yıldırım [9], Farmer and Ki [7], Radziwiłł [14]). Here we get a new way to study these relationships by means of our function $\kappa(t)$. The number of zeros of $\zeta(s)$ on an interval of the critical line not counting multiplicities is related to the increment of $\kappa(t)$ in this interval. Assuming the RH this function will be strictly increasing, so $\kappa'(t) \geq 0$. The connection is by means of equation (6.3) which represents this function in terms of the zeros of $\zeta'(s)$.

Therefore, $\kappa(t)$ is related to the zeros of $\zeta(s)$ (Prop. 4.8), and $\kappa'(t)$ is fully determined by the zeros of $\zeta'(s)$ (Prop. 6.5). The relationship of $\kappa'(t)$ with the zeros of $\zeta(s)$ is also direct and double (Prop. 8.1 and equation (8.2)). See Figure 5 for a graphical description of these relations.

In Section 2 we give the definition and (some simple) properties of the decomposition into *phase* and *signed modulus* of a real analytic function. In particular, in Proposition 2.10 we write the phase as a convergent integral. After this we devote Section 3 to some properties of the phase $-\vartheta(t)$ of $\zeta(1/2 + it)$. Since we will use its convexity for all $t > 0$, we give a simple derivation of this fact. Section 4 is devoted to the introduction of $\kappa(t)$. The definition in Proposition 4.3:

$$e^{2\pi i\kappa(t)} = 1 + 2\vartheta'(t)\frac{\zeta(1/2 + it)}{\zeta'(1/2 + it)}, \qquad \kappa(0) = -1/2,$$

is possible because the function on the right hand side makes a circular movement for $t \in \mathbb{R}$. We study the relationship of $\kappa(t)$ with $\operatorname{ph}\zeta'(1/2 + it)$ and $\vartheta(t)$. The function $\kappa(t)$ is complicated, its behavior being connected with the RH. We show here the equation

$$\kappa(\xi_n) = n,$$

which determines the set of *real* numbers t with $\zeta(1/2 + it) = 0$.

Proposition 4.14 may come as a surprise. It relates the points where $\kappa(t)$ is half an integer with the zeros of $Z'(t)$. Assuming the RH the function $\kappa(t)$ will be strictly increasing, and between γ_n and γ_{n+1} there would be only one zero of $Z'(t)$, situated just at the point where $\kappa(t) = n + 1/2$. In Section 5 we show what of this remains true if we do not assume the RH, and see the first application of the function $\kappa'(t)$.

The main result of Section 6 is a formula for $\kappa'(t)$ in terms of the zeros of $\zeta'(s)$ (see Proposition 6.5). Therein appears a constant A which we relate in equation (6.5) with the zeros of $\zeta'(s)$. In Section 7 we obtain the value $A = \frac{1}{2}\log 2$. We give a proof relating this constant to the difference in the counting of zeros of $\zeta(s)$ given by Riemann and the one for the zeros of $\zeta'(s)$ given by Berndt. Also, we include a proof that the RH implies $\kappa'(t) > 0$ for $t > a_\kappa$.

Section 8 establishes the connection of $\kappa'(t)$ with the zeros of $\zeta(s)$. We know from Section 4 that for $n < m$ we have $\int_{\xi_n}^{\xi_m} \kappa'(t)\,dt = m - n$. We show that $\kappa'(\xi_n) = \vartheta'(\xi_n)/\omega$ where ω is the multiplicity of the zero $1/2+i\xi_n$ of $\zeta(s)$. In Proposition 8.2 we apply these relationships to give, assuming the RH, a new proof of a strengthening of a theorem of Garaev and Yıldırım [9] (which they prove unconditionally). In Section 9 we introduce a related function $E(t)$ and show its relationship with the classical function $S(t)$ and with a function $\mathrm{RH}(t)$ which counts the failures up to height t of both the RH and the simplicity of the zeros of $\zeta(s)$. This function is close to the one considered by Levinson and Montgomery.

Most of the functions appearing in the present paper were found some years ago (in 1997) by one of us (JvdL) while searching for a formula (or equation) for the exact location of the non-trivial zeros of the Riemann zeta function.

2. Phase and argument of a function. The results in this section are easy but we did not find any proper references. We include the simple proofs and introduce our notations about *phase* and *argument* of a real analytic function.

Definition 2.1. A function $f\colon \mathbb{R} \to \mathbb{C}$ is called *real analytic* if for every $t_0 \in \mathbb{R}$ there exists a convergent power series $P(z) = \sum_{k=0}^{\infty} c_k z^k$ such that $f(t) = P(t - t_0)$ for all t in a neighborhood of t_0. In other words: A function $f\colon \mathbb{R} \to \mathbb{C}$ is called real analytic if f has an analytic extension to a neighborhood of $\mathbb{R}$.

Proposition 2.2. *If $f\colon \mathbb{R}\to\mathbb{C}\smallsetminus\{0\}$ is real analytic, then there exists a real analytic function g such that $f(t)=e^{g(t)}$ for every $t\in\mathbb{R}$.*

Proof. For every $t_0\in\mathbb{R}$ let $\Delta(t_0)$ be a disk with center at t_0 such that $f(t)=P(t-t_0)$ for $t\in\Delta(t_0)\cap\mathbb{R}$, and such that $P(z-t_0)\neq 0$ for $z\in\Delta(t_0)$. The union $G=\bigcup_{t_0}\Delta(t_0)$ is a simply connected domain and f can be extended to G as an analytic function. Since $f(z)\neq 0$ for $z\in G$, there exists an analytic function g on G such that $f(z)=e^{g(z)}$ for all $z\in G$. ∎

Corollary 2.3. *If $f\colon \mathbb{R}\to\mathbb{C}\smallsetminus\{0\}$ is real analytic, then there exists a real analytic function $\varphi\colon\mathbb{R}\to\mathbb{R}$ such that $f(t)=|f(t)|e^{i\varphi(t)}$.*

We then write $\varphi(t)=\arg f(t)$. This is an analytic (and hence continuous) determination of the argument of f. Two such functions differ by an integral multiple of 2π.

Proposition 2.4. *If $f\colon\mathbb{R}\to\mathbb{C}$ is real analytic, then there are two real analytic functions $U\colon\mathbb{R}\to\mathbb{R}$ and $\varphi\colon\mathbb{R}\to\mathbb{R}$ such that*

$$f(t)=U(t)e^{i\varphi(t)}.$$

Given two such representations, $f=U_1e^{i\varphi_1}$ and $f=U_2e^{i\varphi_2}$, we have either $U_1=U_2$ and $\varphi_1-\varphi_2=2k\pi$ or $U_1=-U_2$ and $\varphi_1-\varphi_2=(2k+1)\pi$ for some integer k.

Proof. If f does not vanish, then $|f|$ is real analytic and by Corollary 2.3 there exists a real analytic function $\varphi\colon\mathbb{R}\to\mathbb{R}$ such that $f|f|^{-1}=e^{i\varphi}$, and we can take $U=|f|$ in this case.

Now assume that f has real zeros. Let a_n be the real zeros of $f(t)$ listed with multiplicities. We may assume that $a_1=\cdots=a_m=0$ and all the others are non-zero. By Weierstrass' factorization theorem there exists an entire function

$$g(z)=z^m\prod_{n>m}E_{n-1}(z/a_n),\qquad z\in\mathbb{C}$$

whose zeros are the numbers a_n, and the $E_n(z)=(1-z)e^{z+z^2/2+\cdots+z^n/n}$ are the canonical factors. Observe also that this function is real for real $z=t$. By the previous argument there exist real analytic functions h and φ such that $f/g=he^{i\varphi}$. Thus $f=(gh)e^{i\varphi}$, and $U=gh$. This proves that the claimed decomposition exists.

Finally, if $f=U_1e^{i\varphi_1}=U_2e^{i\varphi_2}$, then U_1/U_2 is a real analytic function without zeros. Also $|U_1/U_2|=|e^{i(\varphi_2-\varphi_1)}|=1$ and it follows that U_1/U_2 is either equal to 1 or to -1. In the first case $e^{i(\varphi_2-\varphi_1)}=1$ and $\varphi_2=2k\pi+\varphi_1$ for some integer k. The other case may be treated similarly. ∎

Definition 2.5. Given a real analytic function $f\colon\mathbb{R}\to\mathbb{C}$ we define a *phase* of f to be any real analytic function $\operatorname{ph} f\colon\mathbb{R}\to\mathbb{R}$ such that $f(t)=U(t)e^{i\operatorname{ph} f(t)}$ with $U\colon\mathbb{R}\to\mathbb{R}$ a real analytic function.

If g_1 and g_2 are two such functions there exists an integer k such that $g_1(t) = g_2(t) + k\pi$ for every $t \in \mathbb{R}$.

Observe that the above definition is not standard. We are making use of the word *phase* with a peculiar mathematical meaning.

The main difference between the phase of a real analytic function and its argument is that for some $t \in \mathbb{R}$ the value $\operatorname{ph} f(t)$ may not be equal to one of the arguments of the complex number $f(t)$. We will only have $\operatorname{ph} f(t)$ equal to this argument modulo π.

EXAMPLE 2.6. It is easy to check that

$$\cos \tfrac{\pi}{2}(1/2 + it) = \frac{1}{\sqrt{2}}\sqrt{\cosh \pi t}\, e^{-i \arctan(\tanh \frac{\pi t}{2})}. \tag{2.1}$$

EXAMPLE 2.7. One of the most interesting examples is that of the zeta function on the critical line. In this case we have (see Edwards [6, p. 119])

$$\zeta(1/2 + it) = Z(t)e^{-i\vartheta(t)} \tag{2.2}$$

where $Z\colon \mathbb{R} \to \mathbb{R}$ and $\vartheta\colon \mathbb{R} \to \mathbb{R}$ are real analytic. $Z(t)$ is the Riemann-Siegel function (sometimes called Hardy's Z-function [10]).

EXAMPLE 2.8. The phase $-\vartheta(t)$ in Example 2.7 is related to the phase of $\Gamma(1/4 + it/2)$ by

$$\Gamma(1/4 + it/2) = |\Gamma(1/4 + it/2)|e^{i(\vartheta(t)+\frac{t}{2}\log\pi)}. \tag{2.3}$$

(For more details see [17, (4.17.2)].)

EXAMPLE 2.9. We have not found any reference for our next example:

$$\Gamma(1/2+it) = \sqrt{\frac{\pi}{\cosh \pi t}}\exp\Bigl\{i\Bigl(2\vartheta(t)+t\log(2\pi)+\arctan\tanh\frac{\pi t}{2}\Bigr)\Bigr\}. \tag{2.4}$$

This may be shown using only properties of $\Gamma(s)$ but we present a proof based on the functional equation of $\zeta(s)$.

Let $\Phi(s) = \frac{1}{2}\zeta(s)\zeta(1 - s)$. Then, by the functional equation,

$$\Phi(s) = \cos\Bigl(\frac{\pi s}{2}\Bigr)(2\pi)^{-s}\Gamma(s)\zeta(s)^2. \tag{2.5}$$

Substituting (2.1) and (2.2) into this equation, with $s = 1/2 + it$ we get

(2.6)

$$\frac{1}{2}Z(t)^2 = \sqrt{\frac{\cosh(\pi t)}{2}}e^{-i\arctan(\tanh\frac{\pi t}{2})}\cdot\frac{1}{\sqrt{2\pi}}e^{-it\log(2\pi)}\cdot\Gamma(s)\cdot e^{-2i\vartheta(t)}Z(t)^2$$

from which (2.4) follows for $Z(t) \neq 0$. But since the argument in (2.4) is real analytic, the formula is true for all t.

PROPOSITION 2.10. *If f is a non-constant real analytic function, then for every $t \in \mathbb{R}$ we have*

$$\operatorname{ph} f(t) = \operatorname{ph} f(0) + \int_0^t \operatorname{Im} \frac{f'(x)}{f(x)}\, dx. \tag{2.7}$$

Proof. The function $\operatorname{ph} f(t)$ is real analytic, so that

$$\operatorname{ph} f(t) = \operatorname{ph} f(0) + \int_0^t (\operatorname{ph} f)'(x)\, dx.$$

There exists a real analytic function U such that $f(t) = U(t)e^{i \operatorname{ph} f(t)}$. Therefore, if $f(x) \neq 0$ then

$$\frac{f'(x)}{f(x)} = \frac{U'(x)}{U(x)} + i(\operatorname{ph} f)'(x)$$

so that

$$(\operatorname{ph} f)'(x) = \operatorname{Im} \frac{f'(x)}{f(x)}.$$

It follows that $\operatorname{Im} \frac{f'(x)}{f(x)}$ is in fact a real analytic function, the possible singularities at the points where $f(x) = 0$ being removable. ∎

EXAMPLE 2.11. By Examples 2.7 and 2.8 we have

$$\begin{aligned} \vartheta(t) &= -\int_0^t \operatorname{Re} \frac{\zeta'(1/2+ix)}{\zeta(1/2+ix)}\, dx \\ &= -\frac{t}{2}\log \pi + \frac{1}{2}\int_0^t \operatorname{Re} \frac{\Gamma'(1/4+ix/2)}{\Gamma(1/4+ix/2)}\, dx. \end{aligned} \tag{2.8}$$

3. The function $\vartheta(t)$. In this section we recall some properties of the function $\vartheta(t)$ introduced in Example 2.7.

We need to prove that $\vartheta(t) = 0$ has only one solution for $t > 0$. To this end we give explicit formulae for $\vartheta(t)$ for *small* t, which are seldom considered.

Indeed, after introducing $\vartheta(t)$, most authors immediately start discussing its asymptotic expansion (compare Edwards [6, p. 119] and Gabcke [8, p. 4]).

PROPOSITION 3.1. *For $\vartheta(t)$ we have the following series expansion* (*convergent for all $t \in \mathbb{R}$*), *γ being Euler's constant:*

$$\vartheta(t) = -\frac{1}{2}\left(\gamma + \log \pi + 3\log 2 + \frac{\pi}{2}\right)t + \sum_{n=0}^{\infty}\left(\frac{2t}{4n+1} - \arctan \frac{2t}{4n+1}\right). \tag{3.1}$$

Proof. From the Weierstrass product for $\Gamma(s)$ we obtain

$$\begin{aligned}
-&\operatorname{Im}\log\Gamma\big(1/4+it/2\big)\\
&=\arg\Big\{(1/4+it/2)e^{\gamma(1/4+it/2)}\prod_{n=1}^{\infty}\Big(1+\frac{1/4+it/2}{n}\Big)e^{-(1/4+it/2)/n}\Big\}+2\ell\pi\\
&=\arctan(2t)+\frac{\gamma}{2}t+\sum_{n=1}^{\infty}\Big(\arctan\frac{2t}{4n+1}-\frac{t}{2n}\Big)-2k\pi
\end{aligned}$$

for some $\ell, k\in\mathbb{Z}$.

Since $\vartheta(t)=\arg\Gamma(1/4+it/2)-(t/2)\log\pi$ we have

$$\vartheta(t)=-\frac{\gamma+\log\pi}{2}\,t-\arctan(2t)-\sum_{n=1}^{\infty}\Big(\arctan\frac{2t}{4n+1}-\frac{t}{2n}\Big)+2k\pi$$

so that, taking $t=0$, we find that $k=0$. We rewrite the last series as follows:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\Big(\arctan\frac{2t}{4n+1}-\frac{t}{2n}\Big)\\
&\qquad=-\frac{t}{2}\sum_{n=1}^{\infty}\Big(\frac{1}{n}-\frac{1}{n+1/4}\Big)+\sum_{n=1}^{\infty}\Big(\arctan\frac{2t}{4n+1}-\frac{2t}{4n+1}\Big).
\end{aligned}$$

The first series on the right hand side can be summed explicitly:

$$\begin{aligned}
\sum_{n=1}^{\infty}\Big(\frac{1}{n}-\frac{1}{n+1/4}\Big)&=\sum_{n=1}^{\infty}\Big(\int_0^1 u^{n-1}\,du-\int_0^1 u^{n-1+1/4}\,du\Big)=\int_0^1\frac{1-u^{1/4}}{1-u}\,du\\
&=4\int_0^1\frac{1-v}{1-v^4}\,v^3\,dv=\int_0^1\frac{4v^3\,dv}{(1+v)(1+v^2)}=4-3\log 2-\frac{\pi}{2}.
\end{aligned}$$

Combining these equations we obtain (3.1). ∎

Proposition 3.2. *For every $t\in\mathbb{R}$ we have*

$$\vartheta'(t)=-\tfrac{1}{2}(\gamma+\log\pi)-\frac{2}{1+4t^2}-\sum_{k=1}^{\infty}\Big(\frac{2(4k+1)}{(4k+1)^2+4t^2}-\frac{1}{2k}\Big).\tag{3.2}$$

Corollary 3.3. *The function $\vartheta(t)$ is convex on $(0,\infty)$, and there exists a unique positive real number a_ϑ where $\vartheta'(a_\vartheta)=0$.*

By differentiation of (3.2) we get

$$\vartheta''(t)=16t\sum_{k=0}^{\infty}\frac{4k+1}{((4k+1)^2+4t^2)^2},\tag{3.3}$$

from which the corollary follows.

This corollary has also been proved in [11, Lemmas 11, 12]. We have

$$a_\vartheta = 6.28983\,59888\,36902\,77966\,50901\,00821\,85339\,66583\,12945\,\ldots.$$

4. The function $\kappa(t)$. The next proposition is included in Titchmarsh [17, p. 291], but we give the proof below, because we are also interested in the formulas used.

Proposition 4.1. *If $\zeta'(1/2+ia)=0$ for a real a, then $\zeta(1/2+ia)=0$.*

Proof. We start from $\zeta(1/2+it)=e^{-i\vartheta(t)}Z(t)$. Differentiation with respect to t yields

$$i\zeta'(1/2+it) = -i\vartheta'(t)e^{-i\vartheta(t)}Z(t)+e^{-i\vartheta(t)}Z'(t).$$

Multiplying this by $-ie^{i\vartheta(t)}$ we get

$$e^{i\vartheta(t)}\zeta'(1/2+it) = -\vartheta'(t)Z(t)-iZ'(t) \tag{4.1}$$

and taking real parts we obtain

$$-\vartheta'(t)Z(t) = \operatorname{Re}\{e^{i\vartheta(t)}\zeta'(1/2+it)\} \tag{4.2}$$

which may also be written as

$$-2\vartheta'(t)Z(t) = e^{i\vartheta(t)}\zeta'(1/2+it)+e^{-i\vartheta(t)}\zeta'(1/2-it). \tag{4.3}$$

Let us assume that $\zeta'(1/2+it)=0$ for some real t. Since $\zeta'(1/2-it)=0$ we may assume that $t>0$ and we get $\vartheta'(t)Z(t)=0$. Since $\vartheta'(t)=0$ only for $t=a_\vartheta\approx 6.29$ where $\zeta'(1/2+ia_\vartheta)\neq 0$, we get $Z(t)=0$. Therefore, $\zeta'(1/2+it)=0$ implies $\zeta(1/2+it)=0$. ∎

Recall that we denote, as usual, by $\beta_n+i\gamma_n$ the non-trivial zeros of $\zeta(s)$, ordered in such a way that $(0<)\ \gamma_1\le\gamma_2\le\cdots$, repeating each term according to its multiplicity. We will need another related sequence. Let $(0<)\ \xi_1<\xi_2<\cdots$ be the *real* numbers t such that $\zeta(1/2+it)=0$, counted without multiplicities. Hence, the ξ_n *only denote zeros on the critical line*. If we assume the RH and the simplicity of the zeros, we would, of course, have $\xi_n=\gamma_n$.

Proposition 4.2. *For every real $t\neq\pm\xi_n$ we have*

$$1+2\vartheta'(t)\frac{\zeta(1/2+it)}{\zeta'(1/2+it)} = -e^{-2i\vartheta(t)}\frac{\zeta'(1/2-it)}{\zeta'(1/2+it)}. \tag{4.4}$$

Proof. Multiplying (4.3) by $e^{-i\vartheta(t)}$ we get

$$-2\vartheta'(t)\zeta(1/2+it) = \zeta'(1/2+it)+e^{-2i\vartheta(t)}\zeta'(1/2-it).$$

Since $t\neq\pm\xi_n$, using Proposition 4.1 we have $\zeta'(1/2+it)\neq 0$, so that we can divide by $\zeta'(1/2+it)$ and obtain our result. ∎

PROPOSITION 4.3. *There exists a unique real analytic function* $\kappa\colon \mathbb{R}\to\mathbb{R}$ *such that*

$$e^{2\pi i\kappa(t)} = 1 + 2\vartheta'(t)\frac{\zeta(1/2+it)}{\zeta'(1/2+it)}, \qquad \kappa(0) = -1/2. \tag{4.5}$$

Proof. By Proposition 4.2 the function $f\colon \mathbb{R}\to\mathbb{C}$ defined by

$$f(t) = 1 + 2\vartheta'(t)\frac{\zeta(1/2+it)}{\zeta'(1/2+it)}$$

satisfies $|f(t)| = 1$ for $t \neq \xi_n$. By definition, and Proposition 4.1, f is real analytic and satisfies $|f(\xi)| = 1$, so that there exists a real analytic $\kappa\colon \mathbb{R}\to\mathbb{R}$ such that $f(t) = e^{2\pi i\kappa(t)}$. This function is uniquely determined by its value at any point. Since $\vartheta(0) = 0$ we have $f(0) = -1$ (see (4.4)) and we can take $\kappa(0) = -1/2$. ∎

Applying Proposition 2.4 to $\zeta'(1/2+it)$ we arrive at two real analytic functions $\rho\colon \mathbb{R}\to\mathbb{R}$ and $\operatorname{ph}\zeta'(1/2+it)$. Observing that $\zeta'(1/2) < 0$ we may choose

$$\zeta'(1/2+it) = \rho(t)e^{i\operatorname{ph}\zeta'(1/2+it)}, \qquad \rho(0) = |\zeta'(1/2)|, \qquad \operatorname{ph}\zeta'(1/2) = \pi.$$

If we assume that $\zeta(s)$ has no multiple zero on the critical line, then $\zeta'(1/2+it) \neq 0$ and we will have $\rho(t) = |\zeta'(1/2+it)|$ and $\operatorname{ph}\zeta'(1/2+it) = \arg\zeta'(1/2+it)$ (where $\arg\zeta'(1/2+it)$ is meant to be a continuous function of t in $\mathbb{R}$).

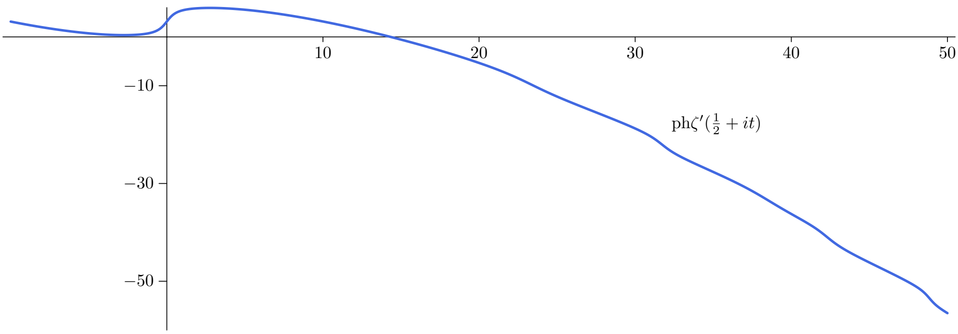

Fig. 1. $\operatorname{ph}\zeta'(1/2+it)$

The relative minimum of $\operatorname{ph}\zeta'(1/2+it)$ at $t = -2.756488\ldots$ is equal to $0.358872\ldots$. Therefore, $\operatorname{ph}\zeta'(1/2+it) > 0$ for $t < 0$.

PROPOSITION 4.4. *For all* $t\in\mathbb{R}$,

$$\kappa(t) = \frac{1}{2} - \frac{1}{\pi}(\vartheta(t) + \operatorname{ph}\zeta'(1/2+it)). \tag{4.6}$$

Proof. By the definition of $\kappa(t)$ and (4.4) we have

$$\exp(2\pi i\kappa(t)) = -e^{-2i\vartheta(t)}\frac{\zeta'(1/2-it)}{\zeta'(1/2+it)}$$
$$= \exp(\pi i - 2i\vartheta(t) - 2i\,\mathrm{ph}\,\zeta'(1/2+it)).$$

Hence, there exists an integer n such that

$$2\pi i\kappa(t) = \pi i - 2i\vartheta(t) - 2i\,\mathrm{ph}\,\zeta'(1/2+it) + 2\pi i n.$$

For $t = 0$ we get $n = 0$ and (4.6) follows. ∎

Corollary 4.5. *For every real t we have*

$$\kappa(t) = -\frac{1}{2} - \frac{1}{\pi}\int_0^t\left(\vartheta'(x) + \mathrm{Re}\,\frac{\zeta''(1/2+ix)}{\zeta'(1/2+ix)}\right)dx. \tag{4.7}$$

Proof. In formula (4.6), we replace $\mathrm{ph}\,\zeta'(1/2+it)$ by the integral expression given by (2.7). ∎

Observing that $-\vartheta(t)$ is the phase of $\zeta(1/2+it)$ we also obtain (see (2.8))

$$\kappa(t) = -\frac{1}{2} + \frac{1}{\pi}\int_0^t \mathrm{Re}\left(\frac{\zeta'(1/2+ix)}{\zeta(1/2+ix)} - \frac{\zeta''(1/2+ix)}{\zeta'(1/2+ix)}\right)dx. \tag{4.8}$$

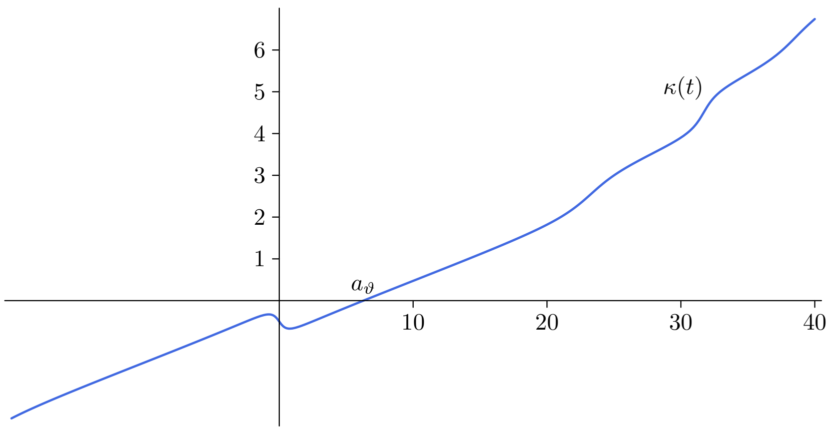

Fig. 2. $\kappa(t)$

From (4.8) we see that $\kappa(t) + 1/2$ is an odd function.

Proposition 4.6. *Choosing the phase of the real analytic function $t \mapsto Z'(t) - iZ(t)\vartheta'(t)$ to be $= \pi/2$ at $t = 0$ we will have*

$$\kappa(t) = -\frac{1}{\pi}\,\mathrm{ph}(Z'(t) - iZ(t)\,\vartheta'(t)). \tag{4.9}$$

Proof. Choosing appropriately the phase of $Z' - iZ\vartheta'$ we get from (4.1) that

$$\vartheta(t) + \operatorname{ph}\zeta'(1/2+it) = \pi/2 + \operatorname{ph}(Z'(t) - iZ(t)\vartheta'(t)).$$

Thus, from (4.6),

$$\kappa(t) = -\frac{1}{\pi}\operatorname{ph}(Z'(t) - iZ(t)\vartheta'(t)).$$

(We have $Z'(0) - iZ(0)\vartheta'(0) = -i\,3.92264\ldots$, so that we must take the phase of $(Z' - iZ\vartheta')$ equal to $\pi/2$ at $t=0$.) ∎

PROPOSITION 4.7. *We have* $\kappa(a_\vartheta) = 0$.

Proof. For z not equal to zero let, as usual, $\operatorname{Arg} z$ be the determination of the argument of z with $-\pi < \operatorname{Arg} z \le \pi$.

By Proposition 4.6 for every interval I on which $Z(t)\vartheta'(t) \ne 0$ there will exist an integer n_I such that

$$\kappa(t) = -\frac{1}{\pi}\operatorname{Arg}(Z'(t) - iZ(t)\vartheta'(t)) + n_I.$$

In particular this applies to the interval $I = (0, a_\vartheta)$. To determine n_I in this case observe that $\kappa(0) = -1/2$, $Z'(0) = 0$, $Z(0) < 0$ and $\vartheta'(0) < 0$, and it follows that $n_I = -1$.

Then choose $\varepsilon > 0$ small enough. At the point $t = a_\vartheta - \varepsilon$ we have $Z'(t) < 0$, $Z(t) < 0$, and $\vartheta'(a_\vartheta) = 0$. Since $\kappa(t)$ is continuous and $\vartheta'(t) < 0$ we get $\operatorname{Arg}(Z'(t) - iZ(t)\vartheta'(t))$ near $-\pi$ when $t = a_\vartheta - \varepsilon$. Taking limits for $\varepsilon \to 0^+$ we get $\kappa(a_\vartheta) = 0$, as asserted. ∎

PROPOSITION 4.8. *For each natural number* n *we have* $\kappa(\xi_n) = n$.

Proof. Assuming that $\kappa(\xi_k) = k$ for $k \le n$ we will show that $\kappa(\xi_{n+1}) = n+1$. The case $n = 0$ is slightly different, but similar. We assume now that $n \ge 1$.

In the interval $I_n = (\xi_n, \xi_{n+1})$ we have $Z(t)\vartheta'(t) \ne 0$. Therefore,

$$\operatorname{sgn}(Z(t)\vartheta'(t)) = \operatorname{sgn}(Z(t)) = \nu.$$

By Proposition 4.6 there is an integer m such that

$$\kappa(t) = m - \frac{1}{\pi}\operatorname{Arg}(Z'(t) - iZ(t)\vartheta'(t)), \qquad t \in I_n.$$

For $y \ne 0$ we have

$$\operatorname{Arg}(x - iy) = \begin{cases} -\arccos\dfrac{x}{\sqrt{x^2+y^2}}, & y > 0,\\ \arccos\dfrac{x}{\sqrt{x^2+y^2}}, & y < 0. \end{cases}$$

Therefore,

$$\kappa(t) = m + \frac{\nu}{\pi}\arccos\frac{Z'(t)}{\sqrt{Z'(t)^2 + Z(t)^2\vartheta'(t)^2}}, \qquad t \in I_n. \tag{4.10}$$

Then if $\mu = \operatorname{sgn}(Z'(t))$ we will have

$$\kappa(t) = m + \frac{\nu}{\pi}\arccos\frac{\mu}{\sqrt{1+\frac{Z(t)^2}{Z'(t)^2}\vartheta'(t)^2}}.$$

For $t > \xi_n$ and $t \to \xi_n$ we have, for some $A > 0$, $C > 0$ and an integer $\omega \geq 1$,

$$\begin{aligned} Z(t) &= \nu A(t-\xi_n)^\omega + \mathrm{O}((t-\xi_n)^{\omega+1}), \\ Z'(t) &= \nu\omega A(t-\xi_n)^{\omega-1} + \mathrm{O}((t-\xi_n)^{\omega}), \\ \frac{Z(t)}{Z'(t)} &= \frac{1}{\omega}(t-\xi_n) + \mathrm{O}((t-\xi_n)^2), \\ \frac{Z(t)^2}{Z'(t)^2}\vartheta'(t)^2 &= \frac{C^2}{\omega^2}(t-\xi_n)^2 + \mathrm{O}((t-\xi_n)^3). \end{aligned}$$

Therefore, in a small interval to the right of ξ_n the sign of $Z(t)$ is the same as the sign of $Z'(t)$, so that $\mu = \nu$. Hence, for $\xi_n < t < \xi_n + \delta$ we have

$$\kappa(t) = m + \frac{\nu}{\pi}\arccos\Big\{\nu\Big(1-\frac{1}{2}\frac{C^2}{\omega^2}(t-\xi_n)^2 + \mathrm{O}((t-\xi_n)^3)\Big)\Big\}. \tag{4.11}$$

Observe that for small $x > 0$ we have

$$\arccos(1-x) = \sqrt{2}\sqrt{x} + \mathrm{O}(x^{3/2}), \qquad \arccos(-1+x) = \pi - \sqrt{2}\sqrt{x} + \mathrm{O}(x^{3/2}).$$

It follows that for $\nu = 1$,

$$\kappa(t) = m + \frac{1}{\pi}\frac{C}{\omega}(t-\xi_n) + \mathrm{O}((t-\xi_n)^2),$$

and for $\nu = -1$,

$$\kappa(t) = m - 1 + \frac{1}{\pi}\frac{C}{\omega}(t-\xi_n) + \mathrm{O}((t-\xi_n)^2).$$

Taking limits for $t \to \xi_n^+$ we get

$$n = \kappa(\xi_n) = \begin{cases} m & \text{when } \nu = 1, \\ m-1 & \text{when } \nu = -1. \end{cases}$$

Having determined m we move t to the other extreme of the interval I_n in (4.10). Therefore, now $\xi_{n+1} - \delta < t < \xi_{n+1}$ with δ small enough. We still have $\operatorname{sgn}(Z(t)) = \nu$, so that $Z(t) = \nu B(\xi_{n+1} - t)^\varpi$ with $B > 0$. As before we will get

$$\frac{Z(t)}{Z'(t)} = \frac{1}{\varpi}(t - \xi_{n+1})$$

but in this case this means that $\operatorname{sgn}(Z'(t)) = -\operatorname{sgn}(Z(t))$ so that $\mu = -\nu$, where now μ is the sign of $Z'(t)$ for $\xi_{n+1} - \delta < t < \xi_{n+1}$. Hence, in this case the analogue of (4.11) is

$$\kappa(t) = m + \frac{\nu}{\pi}\arccos\Big\{-\nu\Big(1-\frac{1}{2}\frac{C'^2}{\varpi^2}(t-\xi_{n+1})^2 + \mathrm{O}((t-\xi_{n+1})^3)\Big)\Big\}. \tag{4.12}$$

It follows that for $\nu = 1$,

$$\kappa(t) = m + 1 - \frac{1}{\pi}\frac{C'}{\varpi}(\xi_{n+1} - t) + \mathrm{O}((t - \xi_{n+1})^2).$$

Taking limits for $t \to \xi_{n+1}$ we get

$$\kappa(\xi_{n+1}) = m + 1 = n + 1 = \kappa(\xi_n) + 1,$$

and for $\nu = -1$,

$$\kappa(t) = m - \frac{1}{\pi}\frac{C'}{\varpi}(\xi_{n+1} - t) + \mathrm{O}((t - \xi_{n+1})^2)$$

so that in this case

$$\kappa(\xi_{n+1}) = m = n + 1 = \kappa(\xi_n) + 1. \blacksquare$$

Corollary 4.9. *The function $\kappa(t)$ takes integer values only in the following cases: $\kappa(a_\vartheta) = 0$, $\kappa(-a_\vartheta) = -1$, $\kappa(\xi_n) = n$, $\kappa(-\xi_n) = -n - 1$ for all natural numbers n.*

Proof. Since $\kappa(t) + 1/2$ is an odd function we get $\kappa(-t) = -\kappa(t) - 1$, so that $\kappa(-a_\vartheta) = -1$ and $\kappa(-\xi_n) = -n - 1$.

Assuming that $\kappa(t) \in \mathbb{Z}$, by (4.5) we must have $\vartheta'(t)\zeta(1/2 + it) = 0$ (recall that if $\zeta'(1/2 + it) = 0$ then $\zeta(1/2 + it) = 0$ so that the quotient $\zeta(1/2+it)/\zeta'(1/2+it)$ is equal to 0 in this case). By Corollary 3.3, for $t > 0$, we have $\vartheta'(t) = 0$ only for $t = a_\vartheta$. By definition the positive real numbers t such that $\zeta(1/2 + it) = 0$ are the numbers ξ_n. This proves that $\kappa(t)$ is an integer only at the points indicated. $\blacksquare$

Corollary 4.10. *For $n = 1, 2, \dots$ the number ξ_n is the unique solution of the equation $\kappa(t) = n$.*

If we assume the RH and that the zeros are simple, we find that γ_n is the only solution of the equation $\kappa(t) = n$.

Define $\xi_0 = a_\vartheta$, $\xi_{-1} = -a_\vartheta$, $\xi_{-n} = -\xi_{n-1}$, so that for all integers $n \in \mathbb{Z}$ we have $\kappa(\xi_n) = n$. With these notations we state

Proposition 4.11. *For any integer $n \in \mathbb{Z}$ and t with $\xi_n < t < \xi_{n+1}$ we have $\kappa(\xi_n) = n < \kappa(t) < n + 1 = \kappa(\xi_{n+1})$.*

Proof. Since $t \neq \xi_m$ the value $\kappa(t)$ is not an integer. If $\kappa(t) < n$, since $\kappa(x)$ is continuous, there will exist $t < t' < \xi_{n+1}$ with $\kappa(t') = n$, in contradiction with Corollary 4.9. A similar reasoning rules out the possibility that $\kappa(t) > n + 1$. $\blacksquare$

Proposition 4.12. *For $t > a_\vartheta$, let $N_{00}(t) := \operatorname{card}\{n \in \mathbb{N} : \xi_n \le t\}$ be the number of real numbers $0 < \xi \le t$ such that $\zeta(1/2 + i\xi) = 0$ counted without multiplicity. Then*

$$N_{00}(t) = \lfloor \kappa(t) \rfloor, \qquad t > a_\vartheta. \tag{4.13}$$

Proof. Since $t > a_\vartheta = \xi_0$ there is an integer $n \ge 0$ such that $\xi_n \le t < \xi_{n+1}$. By definition $N_{00}(t) = n$, and by Proposition 4.11, $n \le \kappa(t) < n+1$ so that $\lfloor \kappa(t) \rfloor = n$. ∎

REMARK 4.13. It is known [3] that $N_0^*(T)$, the number of *simple zeros on the critical line* to height T, satisfies $\liminf_{T\to\infty} N_0^*(T)/N(T) \ge 0.4058$, where $N(T)$, as usual, denotes the number of zeros $\beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma < T$ counted with their multiplicities. Since $\kappa(t) \ge N_0^*(t)$ we deduce that $\liminf_{t\to\infty} \kappa(t)/N(t) \ge 0.4058$. In [4], assuming the RH (but not the simplicity of the zeros) this has been improved to

$$\liminf_{t\to\infty} \kappa(t)/N(t) \ge 0.84665. \tag{4.14}$$

PROPOSITION 4.14. *For any real t we have $\kappa(t) = (2k+1)/2$ with $k \in \mathbb{Z}$ if and only if $Z'(t) = 0$ and $Z(t) \ne 0$.*

Proof. The function $\vartheta'(t)$ only vanishes at $t = \pm a_\vartheta$ and at these points the function $Z'(t)$ does not vanish ($Z'(a_\vartheta) = -Z'(-a_\vartheta) = -0.18838\ldots$). Hence, $Z'(t) - iZ(t)\vartheta'(t) = 0$ only at a point where $Z(t) = Z'(t) = 0$. Since $Z(t) = 0$ there exists n with $t = \xi_n$. By Corollary 4.9 we know that at this point $\kappa(t) \in \mathbb{Z}$.

Let t be a point where $Z(t) \ne 0$ but $Z'(t) = 0$; then $Z'(t) - iZ(t)\vartheta'(t) \in i\mathbb{R}^*$ and by (4.9) we have $\kappa(t) = -(1/\pi)\operatorname{ph}(Z'(t) - iZ(t)\vartheta'(t)) = k + 1/2$ for some $k \in \mathbb{Z}$.

If, on the other hand, we assume $\kappa(t) = (2k+1)/2$, then again by (4.9), $\operatorname{ph}(Z'(t) - iZ(t)\vartheta'(t)) = -(2k+1)\pi/2$, so that $Z'(t) - iZ(t)\vartheta'(t) \in i\mathbb{R}$, and we will certainly have $Z'(t) = 0$ and as we have seen $Z(t) \ne 0$. ∎

5. Hypothesis P and its consequences. One may verify that $\kappa'(0)$ is negative ($= -0.444016\ldots$). In fact $\kappa'(t)$ is negative for all t with

$$|t| < a_\kappa = 0.77985\,35753\,38836\,03051\,82092\,08122\,53710\,71856\,73276\,\ldots.$$

We will prove in Proposition 7.6 that, assuming the RH, $\kappa'(t) > 0$ for $t > a_\kappa$. But we are unable to prove the RH assuming $\kappa'(t) > 0$ for $t > a_\kappa$. However, this appears to be a realistic hypothesis (weaker than the RH):

HYPOTHESIS P. *$\kappa'(t) \ge 0$ for $t > a_\kappa$.*

Some of our propositions will depend on this hypothesis. We will attach to them the symbol P.

PROPOSITION 5.1. (P) *For each integer $n \in \mathbb{Z}$, with $n \ge 0$, there is a unique real number η_{n+2} such that $\xi_n < \eta_{n+2} < \xi_{n+1}$, and $Z'(\eta_{n+2}) = 0$. The number η_{n+2} is the unique solution to the equation $\kappa(t) = n + 1/2$.*

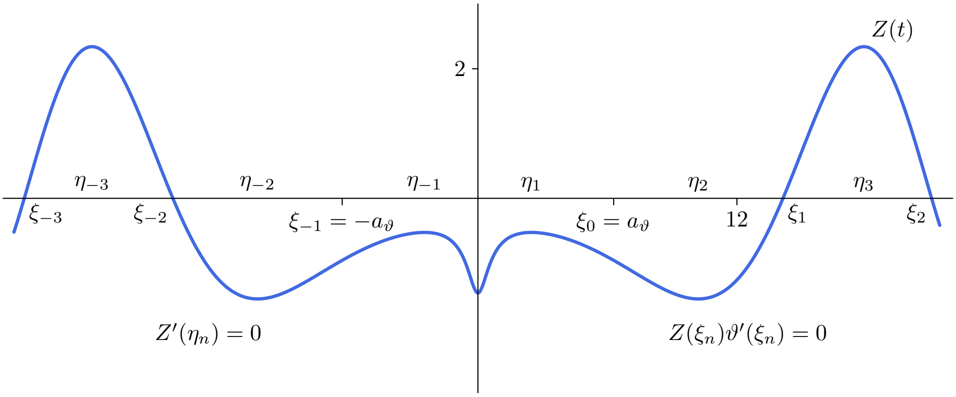

Fig. 3. $Z(t)$ near the origin

Proof. Since $a_\kappa < a_\vartheta$ and $\kappa(t)$ is real analytic, Hypothesis P implies that $\kappa(t)$, being analytic, is strictly increasing for $t > \xi_0 := a_\vartheta$. Therefore, for $n \geq 0$ the function $\kappa(t)$ is strictly increasing in the interval (ξ_n, ξ_{n+1}), so that there is only one solution to the equation $\kappa(t) = n + 1/2$.

By Proposition 4.14 the solution $t = \eta_{n+2}$ to the above equation is the only possible solution to the equation $Z'(t) = 0$ in this interval. ■

For $n = -1$ we may check numerically that $t = 0$ and $t = \pm 2.4757266\ldots$ are solutions to $Z'(t) = 0$ in the interval $(\xi_{-1}, \xi_0) = (-a_\vartheta, a_\vartheta)$.

Using the above it is easy to see that the points where $\kappa(t) = n + 1/2$ are the following:

(a) Three points in the interval $(\xi_{-1}, \xi_0) = (a_{-\vartheta}, a_\vartheta)$. These are $\eta_1 = -\eta_{-1} = 2.47572\ldots$ and $\eta_0 = 0$ at which $\kappa(\eta_{-1}) = \kappa(\eta_0) = \kappa(\eta_1) = -1/2$.

(b) A point $\eta_2 \in (\xi_0, \xi_1)$, namely $\eta_2 = 10.21207\ldots$, at which $\kappa(\eta_2) = 1/2$, and its symmetrical $\eta_{-2} = -\eta_2$, at which $\kappa(-\eta_2) = -3/2$.

(c) For each integer $n \geq 1$ a unique point $\eta_{n+2} \in (\xi_n, \xi_{n+1})$ at which $\kappa(\eta_{n+2}) = n + 1/2$, and its symmetrical $\eta_{-n-2} \in (\xi_{-n-2}, \xi_{-n-1})$ with $\eta_{-n-2} = -\eta_{n+2}$ and $\kappa(\eta_{-n-2}) = -(2n+3)/2$.

One may verify that the minimal value a_γ of $\kappa(t)$ is

$$a_\gamma := \kappa(a_\kappa) = -0.67025\,97987\,68599\,50288\,39164\,11968\,66744\,74803\,\ldots.$$

Since κ is strictly increasing on (a_κ, ∞) with values in (a_γ, ∞) we may define $\gamma(u)$ for $u > a_\gamma$ as the inverse function of $\kappa(t)$. Then $\gamma(u)$ is a real analytic function on (a_γ, ∞) and we will have

$$\gamma(n) = \xi_n, \quad \gamma(n + 1/2) = \eta_{n+2}, \qquad n \geq 0, \qquad \text{assuming P.}$$

Of course, assuming the RH with simple zeros we will have $\gamma(n) = \gamma_n$.

Proposition 5.2. *For $t \in \mathbb{R}$ not a multiple zero of $Z(t)$ we have*

$$\kappa' = \frac{1}{\pi}\,\frac{ZZ'\vartheta'' + (Z')^2\vartheta' - ZZ''\vartheta'}{(Z')^2 + (Z\vartheta')^2}, \tag{5.1}$$

where for short we have written κ' for $\kappa'(t)$, Z for $Z(t)$, etc. Therefore,

$$\mathrm{P} \;\Leftrightarrow\; ZZ'\vartheta'' + (Z')^2\vartheta' - ZZ''\vartheta' \geq 0 \text{ for } t > a_\kappa. \tag{5.2}$$

Proof. For $\xi_n < t < \xi_{n+1}$ we have (4.10) for some constant m. Differentiating and simplifying we get (5.1). Since κ' is real analytic the equality is true because we are not dividing by 0. ∎

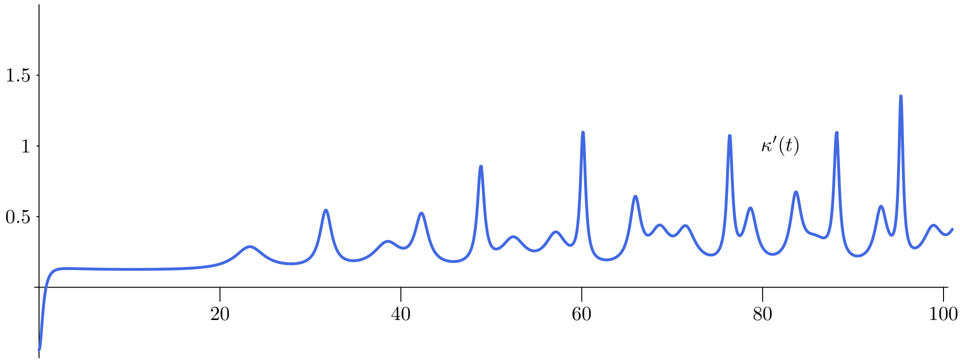

Fig. 4. $\kappa'(t)$

6. Connection of $\kappa'(t)$ with the zeros of $\zeta'(s)$. We will need some known facts (see [16], [13, Theorem 9], [2] and [17, Theorem 11.5(C)]) about the zeros of $\zeta'(s)$.

Proposition 6.1.

(a) *For $n \geq 1$ there is a unique real solution a_n of $\zeta'(s) = 0$ such that $-2n-2 < a_n < -2n$, and there are no other zeros of $\zeta'(s)$ in $\sigma \leq 0$.*

(b) *Let $\rho' = \beta' + i\gamma'$ denote the non-real zeros of $\zeta'(s)$, and let $N_1(T)$ denote the number of non-real zeros of $\zeta'(s)$ with $0 < \gamma < T$. Then*

$$N_1(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + \mathrm{O}(\log T).$$

(c) *We have $0 < \beta' \leq E$ where $E \leq 3$ is a constant. The Riemann Hypothesis is equivalent to $\zeta'(s)$ having no zeros in $0 < \sigma < 1/2$.*

The value of the constant $E = 2.81301\,40202\ldots$ has been computed in [1], where also some additional properties of this constant have been proved.

We will use $\rho' = \beta' + i\gamma'$ to denote a typical complex zero of $\zeta'(s)$. Sometimes we prefer to denote by $\rho'_n = \beta'_n + i\gamma'_n$ the sequence of zeros with $\gamma'_n > 0$ numbered in such a way that $0 < \gamma'_1 \le \gamma'_2 \le \cdots$, with the understanding that the ordinate of a zero of multiplicity m appears m times consecutively in this sequence.

PROPOSITION 6.2. *We have the following Mittag-Leffler expansion:*

$$\frac{\zeta''(s)}{\zeta'(s)} = a - \frac{2}{s-1} + \sum_{n=1}^{\infty}\left(\frac{1}{s-a_n} + \frac{1}{a_n}\right) + \sum_{\rho'}\left(\frac{1}{s-\rho'} + \frac{1}{\rho'}\right), \tag{6.1}$$

where the a_n *are the real zeros of* $\zeta'(s)$, *the* ρ'_n *are the complex ones, and* $a = 0.18334\ldots$ *is a constant* $(= -2 + \zeta''(0)/\zeta'(0))$.

Proof. The entire function $f(s) = (s-1)^2\zeta'(s)$ has the same order as $(s-1)\zeta(s)$ so that $f(s)$ is an entire function of order 1.

From the above results about the zeros of $\zeta'(s)$ it follows easily that the exponent of convergence of the zeros of $f(s)$ is 1. Also, the series $\sum_{n=1}^{\infty} 1/|a_n|$ is divergent. Thus we have

$$\zeta'(s) = e^{as+b}(s-1)^{-2}\prod_{n=1}^{\infty}\left(1 - \frac{s}{a_n}\right)e^{s/a_n}\prod_{\rho'}\left(1 - \frac{s}{\rho'}\right)e^{s/\rho'} \tag{6.2}$$

for some constants a and b.

Now we take logarithms and differentiate to get (6.1). At the point $s = 0$ we obtain the equality

$$\frac{\zeta''(0)}{\zeta'(0)} = a + 2,$$

from which we get the numerical value for a given in the statement. ∎

REMARK 6.3. It can be proved that

$$\frac{\zeta''(0)}{\zeta'(0)} = \frac{\pi^2}{12\log 2\pi} - \frac{\gamma^2 + 2\gamma_1}{\log 2\pi} + \log 2\pi,$$

where γ (the Euler constant) and γ_1 are the Stieltjes constants appearing as coefficients in the Mittag-Leffler expansion of $\zeta(s)$ at the point $s = 1$.

REMARK 6.4. The constant b in equation (6.2) is determined by $e^b = \zeta'(0) = -\frac{1}{2}\log(2\pi)$. So b is complex.

PROPOSITION 6.5. *We set*

$$\pi\kappa'(t) = A + f(t) + \sum_{\rho'=\beta'+i\gamma'} \frac{\beta' - 1/2}{(1/2-\beta')^2 + (t-\gamma')^2} \tag{6.3}$$

where A *is a constant and* $f(t)$ *is a bounded continuous function such that* $f(t) = \mathrm{O}(t^{-1})$ *as* $t \to \infty$.

REMARK 6.6. The exact definition of $f(t)$ is given in (6.6).

REMARK 6.7. In Corollary 7.5 we will prove that $A = \frac{1}{2}\log 2$.

Proof of Proposition 6.5. From (4.7) we get

$$\pi\kappa'(t) = -\vartheta'(t) - \operatorname{Re}\frac{\zeta''(1/2+it)}{\zeta'(1/2+it)}. \tag{6.4}$$

Now in (6.1) we put $s = 1/2 + it$ and take real parts:

$$\operatorname{Re}\frac{\zeta''(1/2+it)}{\zeta'(1/2+it)} = a + \frac{4}{1+4t^2} + \sum_{n=1}^{\infty}\Bigl(\frac{1/2-a_n}{(1/2-a_n)^2+t^2} + \frac{1}{a_n}\Bigr)$$
$$+ \sum_{\rho'=\beta'+i\gamma'}\Bigl(\frac{1/2-\beta'}{(1/2-\beta')^2+(t-\gamma')^2} + \frac{\beta'}{\beta'^2+\gamma'^2}\Bigr).$$

Hence, from (6.4) and (3.2) we get

$$\pi\kappa'(t) = A - \frac{2}{1+4t^2} + \sum_{n=1}^{\infty}\Bigl(\frac{2(4n+1)}{(4n+1)^2+4t^2} - \frac{1/2-a_n}{(1/2-a_n)^2+t^2}\Bigr)$$
$$+ \sum_{\rho'=\beta'+i\gamma'}\frac{\beta'-1/2}{(1/2-\beta')^2+(t-\gamma')^2}$$

where

$$A = \frac{1}{2}(\gamma+\log\pi) - a - \sum_{n=1}^{\infty}\frac{2\beta_n'}{{\beta_n'}^2+\gamma_n'^2} - \sum_{n=1}^{\infty}\Bigl(\frac{1}{2n}+\frac{1}{a_n}\Bigr). \tag{6.5}$$

We define

$$f(t) = -\frac{2}{1+4t^2} + \sum_{n=1}^{\infty}\Bigl(\frac{2(4n+1)}{(4n+1)^2+4t^2} - \frac{1/2-a_n}{(1/2-a_n)^2+t^2}\Bigr). \tag{6.6}$$

Now observe that the terms of the sum can be written as

$$\frac{(2n+1/2)}{(2n+1/2)^2+t^2} - \frac{1/2-a_n}{(1/2-a_n)^2+t^2} = \int_{2n+1/2}^{1/2-a_n}\frac{x^2-t^2}{(x^2+t^2)^2}\,dx.$$

The intervals $(2n+1/2, 1/2-a_n)$ do not intersect, so that for $|t| < T$ the absolute values of the terms of the sum are bounded by

$$\sum_{n=1}^{\infty}\Bigl|\frac{2(4n+1)}{(4n+1)^2+4t^2} - \frac{1/2-a_n}{(1/2-a_n)^2+t^2}\Bigr| \le \int_{5/2}^{\infty}\frac{|x^2-t^2|}{(x^2+t^2)^2}\,dx$$
$$\le \int_{5/2}^{\infty}\frac{T^2+x^2}{x^4}\,dx < \infty.$$

This proves that $f(t)$ is a continuous function.

Also for $t > 1$ we have

$$|f(t)| \le \frac{2}{1+4t^2} + \int_0^{\infty}\frac{|x^2-t^2|}{(x^2+t^2)^2}\,dx = \frac{2}{1+4t^2} + \frac{1}{t}. \tag{6.7}$$ ∎

REMARK 6.8. It can be shown that the zero a_{n-1} contained in the interval $(-2n, -2n+2)$ satisfies $2n + a_{n-1} \sim 1/\log(n/\pi)$ (see Yıldırım [18]). This can be used to deduce that $f(t) = \mathrm{O}(1/t \log t)$. In this way we may improve the error term in (6.9) from $\mathrm{O}(\log t)$ to $\mathrm{O}(\log\log t)$.

We introduce some notation: If $t > 0$ and $n \geq 1$ let $\varphi(t, \rho'_n) = \varphi_n(t)$ be the angle at ρ'_n of the triangle with vertices at $\rho'_n = \beta'_n + i\gamma'_n$, $1/2 - it$ and $1/2 + it$. We consider this angle expressed in radians to be positive if $\beta'_n > 1/2$ and negative if $\beta'_n < 1/2$, and we set $\varphi_n(t) = 0$ when $\beta'_n = 1/2$. In other words, with $s = 1/2 + it$ and $\rho' = \beta' + i\gamma'$ we have

$$\begin{aligned}\varphi(t, \beta' + i\gamma') &= \arctan\frac{t - \gamma'}{\beta' - 1/2} + \arctan\frac{t + \gamma'}{\beta' - 1/2} \\ &= \operatorname{Arg}\frac{\overline{s} - \rho'}{s - \rho'} \qquad (\beta' \neq 1/2).\end{aligned} \tag{6.8}$$

PROPOSITION 6.9. *For $t > 0$ we have*

$$\pi\kappa(t) = At + \sum_{n=1}^{\infty} \varphi_n(t) + \mathrm{O}(\log t) \tag{6.9}$$

where the sum is extended over all zeros $\rho'_n = \beta'_n + i\gamma'_n$ of $\zeta'(s)$ with $\gamma'_n > 0$.

Proof. By (4.7) and (6.3) we have

$$\pi\kappa(t) = -\frac{\pi}{2} + At + \int_0^t f(x)\,dx + \int_0^t \sum_{\rho' = \beta' + i\gamma'} \frac{\beta' - 1/2}{(1/2 - \beta')^2 + (x - \gamma')^2}\,dx.$$

Observe that if $\beta' = 1/2$ then the corresponding term does not contribute to the sum.

Thus

$$\pi\kappa(t) = -\frac{\pi}{2} + At + \int_0^t f(x)\,dx + \sum_{\rho' = \beta' + i\gamma'} \left\{\arctan\frac{t - \gamma'}{\beta' - 1/2} + \arctan\frac{\gamma'}{\beta' - 1/2}\right\},$$

where the terms with $\beta'_n = 1/2$ should be omitted. It is easy to see that the terms corresponding to $\rho'_n = \beta'_n + i\gamma'_n$ and $\overline{\rho'_n} = \beta'_n - i\gamma'_n$ add up to exactly $\varphi_n(t)$. (This is the reason for our convention about the sign of $\varphi_n(t)$.) Thus we arrive at

$$\pi\kappa(t) = -\frac{\pi}{2} + At + \int_0^t f(x)\,dx + \sum_{n=1}^{\infty} \varphi_n(t).$$

Now, since $f(t) = \mathrm{O}(t^{-1})$ we can write this as

$$\pi\kappa(t) = At + \sum_{n=1}^{\infty} \varphi_n(t) + \mathrm{O}(\log t). \ \blacksquare$$

7. Counting the zeros of $\zeta(s)$. The exact value of the constant A in (6.9) can be obtained in two ways. One is by computing the constants in the Mittag-Leffler expansion of related functions, and the second, more interesting for us, by comparing two different counts of the number of zeros of $\zeta(s)$. We will present this second proof. We need some definitions: Let

$$\begin{aligned} N_-(T) &= \#\{\rho = \beta + i\gamma : \zeta(\rho) = 0,\ \beta < 1/2,\ 0 < \gamma \le T\}, \\ N_0(T) &= \#\{\rho = 1/2 + i\gamma : \zeta(\rho) = 0,\ 0 < \gamma \le T\}, \\ N'(T) &= \#\{\rho' = \beta' + i\gamma' : \zeta'(\rho') = 0,\ 0 < \gamma' \le T\}, \\ N'_-(T) &= \#\{\rho' = \beta' + i\gamma' : \zeta'(\rho') = 0,\ \beta' < 1/2,\ 0 < \gamma' \le T\}, \\ N'_0(T) &= \#\{\rho' = 1/2 + i\gamma' : \zeta'(\rho') = 0,\ 0 < \gamma' \le T\}, \\ N'_+(T) &= \#\{\rho' = \beta' + i\gamma' : \zeta'(\rho') = 0,\ \beta' > 1/2,\ 0 < \gamma' \le T\}. \end{aligned}$$

In all these cases, as usual, we count the zeros with their multiplicities. But we also need to consider another count $N_{00}(T)$, which is the number of real numbers $0 < \xi \le T$ such that $\zeta(1/2 + i\xi) = 0$, but in this case we do not count multiplicities.

Taking account of Proposition 4.1 we get

$$N_0(T) - N_{00}(T) = N'_0(T), \tag{7.1}$$

which equals the number of zeros of $\zeta'(s)$ on the critical line with $0 < \gamma' \le T$.

We know some relations between these counts:

(i) Backlund, refining previous work of von Mangoldt (see Edwards [6, Section 6.7]), gave a complete proof of Riemann's assertion

$$N(T) = N_0(T) + 2N_-(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \mathrm{O}(\log T). \tag{7.2}$$

(ii) Berndt [2] proved the corresponding result for $\zeta'(s)$:

$$N'(T) = N'_-(T) + N'_0(T) + N'_+(T) = \frac{T}{2\pi}\log\frac{T}{4\pi} - \frac{T}{2\pi} + \mathrm{O}(\log T). \tag{7.3}$$

(iii) Levinson and Montgomery [13] showed that

$$N_-(T) = N'_-(T) + \mathrm{O}(\log T). \tag{7.4}$$

(iv) From our Proposition 4.12 we get

$$\kappa(t) = N_{00}(t) + \mathrm{O}(1). \tag{7.5}$$

Finally, in Proposition 7.3 we will prove a new relation (7.6). First we prove two lemmas about the zeros of $\zeta'(s)$.

Lemma 7.1. *For $t > 0$ we have*

$$\sum_{0<\gamma'_n<t-1} \frac{1}{t-\gamma'_n} = \mathrm{O}(\log^2 t).$$

Proof. Put

$$N'(t) = \frac{t}{2\pi}\log\frac{t}{4\pi} - \frac{t}{2\pi} + R(t)$$

where $R(t) = \mathrm{O}(\log t)$ by Berndt's theorem. The first zero of $\zeta'(s)$ is $\rho' \approx 2.46316 + i\,23.29832$ so that $N'(4\pi) = 0$ and consequently $R(4\pi) = 2$. We have

$$\sum_{0<\gamma'_n<t-1} \frac{1}{t-\gamma'_n} = \int_{4\pi}^{t-1} \frac{dN'(x)}{t-x} = \frac{1}{2\pi}\int_{4\pi}^{t-1} \frac{\log(x/4\pi)}{t-x}\,dx + \int_{4\pi}^{t-1} \frac{dR(x)}{t-x}$$

$$\leq \log(t/4\pi)\frac{\log(t-4\pi)}{2\pi} + R(t-1) + \int_{4\pi}^{t-1} \frac{|R(x)|}{(t-x)^2}\,dx.$$

Since $R(x) = \mathrm{O}(\log x)$ all the above terms are $\mathrm{O}(\log^2 t)$. ∎

LEMMA 7.2. *For $t \to \infty$ we have*

$$\sum_{\gamma'_n > t+1} \frac{1}{\gamma'^2_n - t^2} = \mathrm{O}\Big(\frac{\log^2 t}{t}\Big).$$

Proof. Using the notations of the previous lemma we have

$$\sum_{\gamma'_n>t+1} \frac{1}{\gamma'^2_n - t^2} = \int_{t+1}^{\infty} \frac{dN'(x)}{x^2-t^2} = \frac{1}{2\pi}\int_{t+1}^{\infty} \frac{\log(x/4\pi)}{x^2-t^2}\,dx + \int_{t+1}^{\infty} \frac{dR(x)}{x^2-t^2}.$$

For $t > 4\pi$ the first integral is less than or equal to $C\log^2 t/t$:

$$\frac{1}{2\pi}\int_{t+1}^{\infty} \frac{\log(x/4\pi)}{x^2-t^2}\,dx \leq \frac{2\log t}{2\pi}\int_{t+1}^{t^2} \frac{dx}{x^2-t^2} + \int_{t^2}^{\infty} \frac{\sqrt{x}}{x^2-t^2}\,dx$$

$$= \frac{\log t}{2\pi t}\Big(\log\frac{t-1}{t+1} + \log(2t+1)\Big) + t^{-1/2}\int_{t}^{\infty} \frac{\sqrt{y}}{y^2-1}\,dy \leq \frac{(\log t)^2}{\pi t} + \frac{4}{t}.$$

Now we bound the second integral. First we observe that

$$\int_{t+1}^{\infty} \frac{dR(x)}{x^2-t^2} = -\frac{R(t+1)}{2t+1} + \int_{t+1}^{\infty} \frac{R(x)}{(x^2-t^2)^2}2x\,dx.$$

For $x > t+1$ we have $x/(x^2-t^2) < 1$, and $|R(x)| \leq C\log x$. Thus

$$\int_{t+1}^{\infty} \frac{dR(x)}{x^2-t^2} \leq c_1\frac{\log t}{t} + c_2\int_{t+1}^{\infty} \frac{\log x}{x^2-t^2}\,dx.$$

Finally, this integral is bounded exactly as the first integral. ∎

PROPOSITION 7.3. *For $t \to \infty$,*

$$\pi\kappa(t) = At + \pi N'_+(t) - \pi N'_-(t) + \mathrm{O}(\log^2 t). \tag{7.6}$$

Proof. By (6.9) we have to show that

$$\sum_{\gamma_n'>0} \varphi_n(t) = \pi N_+'(t) - \pi N_-'(t) + \mathrm{O}(\log^2 t).$$

To this end we will prove that

$$\sum_{\substack{\gamma_n'>0\\ \beta_n'>1/2}} \varphi_n(t) = \pi N_+'(t) + \mathrm{O}(\log^2 t), \qquad \sum_{\substack{\gamma_n'>0\\ \beta_n'<1/2}} \varphi_n(t) = -\pi N_-'(t) + \mathrm{O}(\log^2 t).$$

To simplify the notation we will write $\sum^+$ to denote a sum restricted to $\beta_n' > 1/2$ and $\sum^-$ for a sum restricted to $\beta_n' < 1/2$.

We split the sums into three terms:

$$\sum_{\gamma_n'>0}^{+} \varphi_n(t) = \sum_{0<\gamma_n'<t-1}^{+} \varphi_n(t) + \sum_{|\gamma_n'-t|\le 1}^{+} \varphi_n(t) + \sum_{\gamma_n'>t+1}^{+} \varphi_n(t).$$

The middle sum is $\mathrm{O}(\log t)$ because each term is (in absolute value) less than π and the number of terms is $\mathrm{O}(\log t)$. In the first sum the summands are approximately π (or $-\pi$). Thus we arrive at

$$\sum_{\gamma_n'>0}^{+} \varphi_n(t) = \pi N_+'(t) + \sum_{0<\gamma_n'<t-1}^{+} \{\varphi_n(t) - \pi\} + \sum_{\gamma_n'>t+1}^{+} \varphi_n(t) + \mathrm{O}(\log t),$$

$$\sum_{\gamma_n'>0}^{-} \varphi_n(t) = -\pi N_-'(t) + \sum_{0<\gamma_n'<t-1}^{-} \{\varphi_n(t) + \pi\} + \sum_{\gamma_n'>t+1}^{-} \varphi_n(t) + \mathrm{O}(\log t).$$

It follows that

$$\sum_{\gamma_n'>0} \varphi_n(t) = \pi N_+'(t) - \pi N_-'(t) + \sum_{0<\gamma_n'<t-1} \{\varphi_n(t) \pm \pi\} + \sum_{\gamma_n'>t+1} \varphi_n(t) + \mathrm{O}(\log t)$$

where we use the + sign when $\beta_n' < 1/2$ and the − sign when $\beta_n' > 1/2$.

Now for $0 < \gamma_n' < t-1$ and $\beta_n' > 1/2$ we have

$$0 < \pi - \varphi_n(t) < \arctan\frac{\beta_n' - 1/2}{t-\gamma_n'} + \arctan\frac{\beta_n' - 1/2}{t+\gamma_n'} < 2\arctan\frac{\beta_n' - 1/2}{t-\gamma_n'} < \frac{6}{t-\gamma_n'}$$

and in the case $\beta_n' < 1/2$ analogously

$$0 < \pi + \varphi_n(t) < \arctan\frac{1/2 - \beta_n'}{t-\gamma_n'} + \arctan\frac{1/2 - \beta_n'}{t+\gamma_n'} < 2\arctan\frac{1/2 - \beta_n'}{t-\gamma_n'} < \frac{1}{t-\gamma_n'}.$$

Also, for $\gamma'_n > t+1$ and $\beta'_n > 1/2$,

$$0 < \varphi_n(t) = \arctan \frac{2(\beta'_n - 1/2)t}{(\beta'_n - 1/2)^2 + \gamma'^2_n - t^2} < \frac{6t}{\gamma'^2_n - t^2}$$

and for $\beta'_n < 1/2$ the absolute value $|\varphi_n(t)|$ is bounded by the same quantity.

Hence, applying the above two lemmas we find that

$$\sum_{\gamma'_n > 0} \varphi_n(t) = \pi N'_+(t) - \pi N'_-(t) + \mathrm{O}(\log^2 t). \ \blacksquare$$

COROLLARY 7.4. *For $t \to \infty$,*

$$N_{00}(t) = \frac{A}{\pi} t + N'_+(t) - N'_-(t) + \mathrm{O}(\log^2 t). \tag{7.7}$$

Proof. Combine (7.6) with (7.5). ■

COROLLARY 7.5. *The constant A is equal to $\frac{1}{2}\log 2$.*

Proof. Write $f(t) \doteq g(t)$ to denote that $f(t) - g(t) = \mathrm{O}(\log^2 t)$. (In the same way as congruences we can operate with $\doteq$ as if it were an equality sign between equivalence classes). With this notation we have

$$\begin{aligned}
N(t) &\doteq \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} && \text{by (7.2)}\\
&\doteq N_0(t) + 2N_-(t) && \text{trivially}\\
&\doteq N_0(t) + 2N'_-(t) && \text{by (7.4)}\\
&\doteq N_0(t) - N_{00}(t) + \frac{A}{\pi}t + N'_+(t) + N'_-(t) && \text{by (7.7)}\\
&\doteq N_0(t) - N_{00}(t) + \frac{A}{\pi}t - N'_0(t) + \frac{t}{2\pi}\log\frac{t}{4\pi} - \frac{t}{2\pi} && \text{by (7.3)}\\
&\doteq \frac{A}{\pi}t + \frac{t}{2\pi}\log\frac{t}{4\pi} - \frac{t}{2\pi} && \text{by (7.1)}
\end{aligned}$$

Hence,

$$\frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} \doteq \frac{A}{\pi}t + \frac{t}{2\pi}\log\frac{t}{4\pi} - \frac{t}{2\pi},$$

from which we deduce

$$\frac{A}{\pi}t \doteq \frac{t}{2\pi}\log 2.$$

Hence, $A = \frac{1}{2}\log 2$. ■

PROPOSITION 7.6. *The Riemann hypothesis implies Hypothesis* P.

Proof. The Riemann hypothesis is equivalent to $\beta' > 1/2$ for every zero $\rho' = \beta' + i\gamma'$, and it follows by (6.3) that if the Riemann hypothesis is true, then $\pi\kappa'(t) > A + f(t)$. Since $A = \frac{1}{2}\log 2$, applying (6.7) we easily see that $\kappa'(t) > 0$ for $t > 3.4$ if we assume the RH. It is clear that there is an $a_\kappa \geq 0$ such that $\kappa'(t) > 0$ for $t > a_\kappa$, and $a_\kappa < 3.4$. ■

8. Connections between the zeros of $\zeta(s)$ and $\zeta'(s)$

PROPOSITION 8.1. *Let $1/2+i\xi$ be a zero of $\zeta(s)$ of multiplicity ω on the critical line. Then*

$$\kappa'(\xi)=\frac{1}{\pi\omega}\vartheta'(\xi). \tag{8.1}$$

Proof. Since $\zeta(1/2+it)=e^{-i\vartheta(t)}Z(t)$ the function $Z(t)$ has a zero of multiplicity ω at $t=\xi$. Hence, $\lim_{t\to\xi}\frac{Z(t)}{Z'(t)}=0$, and for $\omega\ge 2$,

$$\lim_{t\to\xi}\frac{Z(t)}{Z'(t)}\frac{Z''(t)}{Z'(t)}=\frac{\omega-1}{\omega}.$$

When $\omega=1$ this second limit is equal to $0=(\omega-1)/\omega$.

Hence, for $0<|t-\xi|<\delta$ we have $Z(t),Z'(t),Z''(t)\ne 0$ and by (5.1) we deduce

$$\lim_{t\to\xi}\kappa'(t)=\lim_{t\to\xi}\frac{1}{\pi}\frac{\frac{Z}{Z'}\vartheta''+\vartheta'-\frac{Z}{Z'}\frac{Z''}{Z'}\vartheta'}{1+\left(\frac{Z}{Z'}\vartheta'\right)^2}=\frac{1}{\pi}\left(\vartheta'(\xi)-\frac{\omega-1}{\omega}\vartheta'(\xi)\right)=\frac{1}{\pi\omega}\vartheta'(\xi). \quad\blacksquare$$

Assuming the RH and the simplicity of zeros we have

$$\int_{a_\vartheta}^{\gamma_n}\kappa'(t)\,dt=n.$$

Hence, the mean value of $\kappa'(t)$ in $[0,t]$ is $N(t)/t$, which is approximately equal to $\vartheta'(t)/\pi$. The above proposition says that, assuming only the simplicity of the zeros, at the points ξ_n the value $\kappa'(\xi_n)$ is just equal to this density.

Figure 5 illustrates two ways in which the zeros of $\zeta'(s)$ determine the ξ_n (assuming only simplicity of the zeta zeros). First, ξ_n is determined from $\kappa'(t)$ by the equation

$$\int_{\xi_0}^{\xi_n}\kappa'(t)\,dt=n\quad\text{or}\quad\int_{\xi_{n-1}}^{\xi_n}\kappa'(t)\,dt=1. \tag{8.2}$$

Second, the points ξ_n are intersections of the two curves $\kappa'(t)$ and $\vartheta'(t)/\pi$. But, as we see in Figure 5, not all these intersections correspond to points ξ_n.

We can see how two close ξ_n correspond to a peak in the graph of $\kappa'(t)$ which, according to (6.3), will be produced by one or more zeros $\beta'+i\gamma'$ of $\zeta'(s)$ with a relatively small $\beta'-1/2$. Observe that equation (6.3) shows that $\kappa'(t)$ is fully determined by the zeros of $\zeta'(s)$.

Following these ideas we may improve (but assuming the RH) a theorem due to M. Z. Garaev and C. Y. Yıldırım [9]. For any given zero $\rho'=\beta'+i\gamma'$ of $\zeta'(s)$ let γ_c be, of all ordinates of zeros of $\zeta(s)$, the one for which $|\gamma_c-\gamma'|$ is the smallest (if there are more than one such zero of $\zeta(s)$, take γ_c to be the imaginary part of any one of them). Garaev and Yıldırım prove unconditionally that $|\gamma_c-\gamma'|\ll|\beta'-1/2|^{1/2}$.

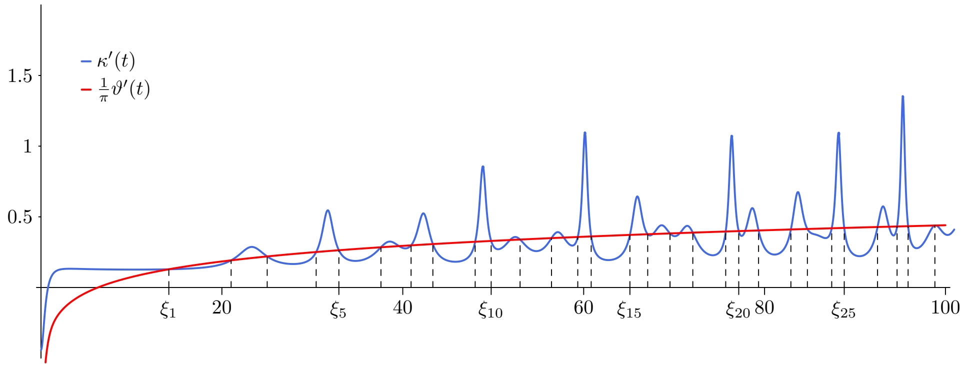

Fig. 5. $\kappa(t)$

PROPOSITION 8.2. (RH) *Assuming the RH we have, for any zero* $\beta'+i\gamma'$ *of* $\zeta'(s)$,

$$|\gamma_c-\gamma'|\le 1.8\,|\beta'-1/2|^{1/2}.$$

Proof. Assuming the RH, $\beta'>1/2$ so that by (6.3) we have

$$\kappa'(t)\ge\frac{1}{2\pi}\log 2+\frac{f(t)}{\pi}+\frac{1}{\pi}\frac{\beta'-1/2}{(\beta'-1/2)^2+(t-\gamma')^2}.$$

We find an $a>0$ such that

$$\kappa(\gamma'+a)-\kappa(\gamma'-a)\ge\frac{a\log 2}{\pi}+\frac{2}{\pi}\arctan\frac{a}{\beta'-1/2}+\frac{1}{\pi}\int_{\gamma'-a}^{\gamma'+a}f(t)\,dt>1.$$

Then there is a point $\xi\in[\gamma'-a,\gamma'+a]$ such that $\kappa(\xi)\in\mathbb{Z}$. Hence, by Corollary 4.9, $\zeta(1/2+i\xi)=0$, so that the ordinate γ_c of the nearest zero of $\zeta(s)$ will satisfy $|\gamma_c-\gamma'|\le a$.

Table 1. First non-trivial zeros of $\zeta(s)$ and $\zeta'(s)$

$\beta_n+i\gamma_n$	$\beta'_n+i\gamma'_n$
$0.5+i\,14.13472\,51417$	$2.46316\,18694+i\,23.29832\,04927$
$0.5+i\,21.02203\,96387$	$1.28649\,68222+i\,31.70825\,00831$
$0.5+i\,25.01085\,75801$	$2.30757\,00637+i\,38.48998\,31730$
$0.5+i\,30.42487\,61258$	$1.38276\,36057+i\,42.29096\,45545$
$0.5+i\,32.93506\,15877$	$0.96468\,56227+i\,48.84715\,99050$
$0.5+i\,37.58617\,81588$	$2.10169\,99009+i\,52.43216\,12451$
$0.5+i\,40.91871\,90121$	$1.89595\,97624+i\,57.13475\,31990$

Using the data in Table 1, we can easily prove our proposition for the first three zeros of $\zeta'(s)$. So, we may assume in what follows that $\gamma'>42$. Then for any $a>0$ with $a/\gamma'<1/2$ we get $\gamma'-a>\gamma'/2>21$.

By (6.7), for $t > 20$ we have $|f(t)| < 41/40t$ so that for $a/\gamma' < 1/2$,

$$\left|\frac{1}{\pi}\int_{\gamma'-a}^{\gamma'+a} f(t)\,dt\right| \le \frac{41}{40\pi}\log\frac{\gamma'+a}{\gamma'-a} \le \frac{41}{40\pi}\,\frac{8}{3}\,\frac{a}{\gamma'} \le \frac{a}{\gamma'},$$

because $\log\frac{1+x}{1-x} \le 8x/3$ for $|x| \le 1/2$.

Therefore, we want to choose a such that

$$\kappa(\gamma'+a) - \kappa(\gamma'-a) \ge \frac{a\log 2}{\pi} + \frac{2}{\pi}\arctan\frac{a}{\beta'-1/2} - \frac{a}{\gamma'} > 1$$

or

$$a\frac{\log 2}{2} - \frac{\pi a}{2\gamma'} \ge \frac{\pi}{2} - \arctan\frac{a}{\beta'-1/2} = \arctan\frac{\beta'-1/2}{a}.$$

It suffices to have

$$a\left(\frac{\log 2}{2} - \frac{\pi}{2\gamma'}\right) \ge \frac{\beta'-1/2}{a}.$$

Since $\gamma' \ge 42$ it is enough to take

$$a = 1.8\sqrt{\beta'-1/2} \ge \left(\frac{\log 2}{2} - \frac{\pi}{2\gamma'}\right)^{-1/2}\sqrt{\beta'-1/2}.$$

Since always $\beta' < 3$ and $\gamma' > 42$ this a satisfies $a/\gamma' < 1/2$, as used above. ∎

9. The functions $E(t)$ and $S(t)$. In the theory of the zeta function we consider the function

$$S(t) = \pi^{-1}\arg\zeta(1/2+it),$$

where the argument is obtained by its continuous variation along the straight lines joining 2, $2+it$, $1/2+it$ starting with the value 0. If t is the ordinate of a zero, $S(t)$ is taken equal to $S(t+0)$ (see [17, Section 9.3]). This function satisfies (see Edwards [6, p. 173])

$$S(t) = N(t) - 1 - \frac{1}{\pi}\vartheta(t). \tag{9.1}$$

If we assume the RH and the simplicity of the zeros, we will have $N(t) = N_{00}(t) = \lfloor\kappa(t)\rfloor$ (see Proposition 4.12).

We introduce a real analytic version of $S(t)$ that we will call $E(t)$:

$$E(t) := \pi + 2\vartheta(t) + \operatorname{ph}\zeta'(1/2+it). \tag{9.2}$$

By (4.6) this is equivalent to

$$E(t) = 3\frac{\pi}{2} + \vartheta(t) - \pi\kappa(t) \tag{9.3}$$

with $E(0) = 2\pi$.

If $1/2 + i\xi_n$ is a simple zero of $\zeta(s)$ we will have $E'(\xi_n) = 0$ by Proposition 8.1. The converse is not true. For example at $t_0 = 39.587\,127\,340\ldots$

the function $E(t)$ has a local minimum with $E(t_0) = 0.151\,790\,437\ldots$. It is also easy to show that $E(t) - 2\pi$ is a real analytic odd function.

In fact $E'(t) = \vartheta'(t) - \pi\kappa'(t)$ so that the zeros of $E'(t)$ are just the points where the graphs of $(1/\pi)\vartheta'(t)$ and $\kappa'(t)$ intersect (see Figure 5). By (5.1) for $Z'(t)^2 + (Z(t)\vartheta'(t))^2 \neq 0$ we have

$$E' = \vartheta' - \pi\kappa' = Z \cdot \frac{Z\vartheta'^3 - Z'\vartheta'' + Z''\vartheta'}{(Z')^2 + (Z\vartheta')^2}. \tag{9.4}$$

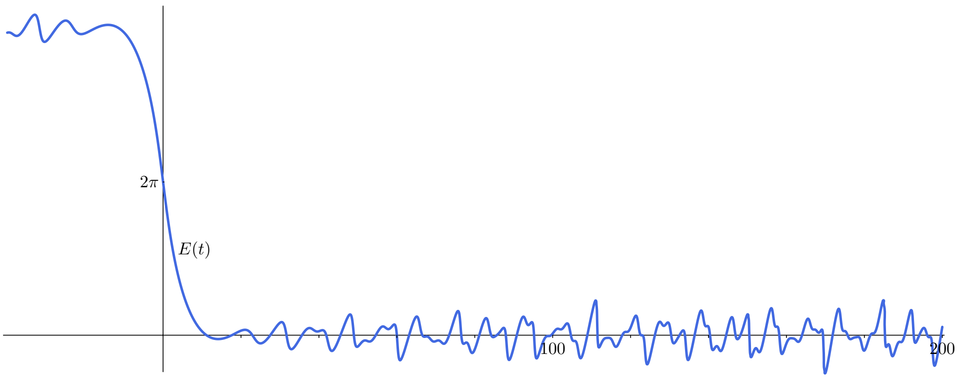

Fig. 6. $E(t)$

For the next proposition we need a measure of the possible failure of the RH.

Definition 9.1. For any $t > 0$ we define $\mathrm{RH}(t)$ by

$$\mathrm{RH}(t) := N(t) - N_{00}(t). \tag{9.5}$$

That is, $\mathrm{RH}(t)$ is equal to the number of zeros $\beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq t$ and $\beta \neq 1/2$, plus the number of zeros $\beta' + i\gamma'$ of $\zeta'(s)$ with $\beta' = 1/2$ and $0 < \gamma' \leq t$, all of them counted with their multiplicities. By Proposition 4.1 these zeros of $\zeta'(s)$ will be multiple zeros of $\zeta(s)$ on the critical line. We have $\mathrm{RH}(t) = 0$ if and only if the zeros $\beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma \leq t$ are all on the critical line and are simple.

Proposition 9.2. *We have*

$$-1/2 + \mathrm{RH}(t) < S(t) + \frac{1}{\pi}E(t) \leq 1/2 + \mathrm{RH}(t), \qquad t > a_\vartheta. \tag{9.6}$$

Proof. By (9.1) and (9.3) we have

$$\begin{aligned} S(t) + \frac{1}{\pi}E(t) &= N(t) - 1 - \frac{1}{\pi}\vartheta(t) + \frac{1}{\pi}(3\pi/2 + \vartheta(t) - \pi\kappa(t)) \\ &= N(t) + 1/2 - \kappa(t) = \mathrm{RH}(t) + 1/2 - \kappa(t) + N_{00}(t) \end{aligned}$$

so that by (4.13) for $t > a_\vartheta$ we get

$$S(t) + \frac{1}{\pi}E(t) = \mathrm{RH}(t) - (\kappa(t) - \lfloor \kappa(t) \rfloor - 1/2), \qquad t > a_\vartheta \tag{9.7}$$

from which the result follows. ∎

Corollary 9.3. *Assuming the RH and the simplicity of the zeros we will have*

$$-1/2 < S(t) + \frac{1}{\pi}E(t) \le 1/2, \qquad t > a_\vartheta. \tag{9.8}$$

Indeed, the hypotheses are equivalent to $\mathrm{RH}(t) = 0$. By the well known Fourier series of $\widetilde{B}_1(x) = x - \lfloor x \rfloor - 1/2$ we get from (9.7), under the assumptions of the corollary,

$$S(t) + \frac{1}{\pi}E(t) = 2\sum_{n=1}^{\infty} \frac{\sin(2\pi n \kappa(t))}{2\pi n}, \qquad t > a_\vartheta. \tag{9.9}$$

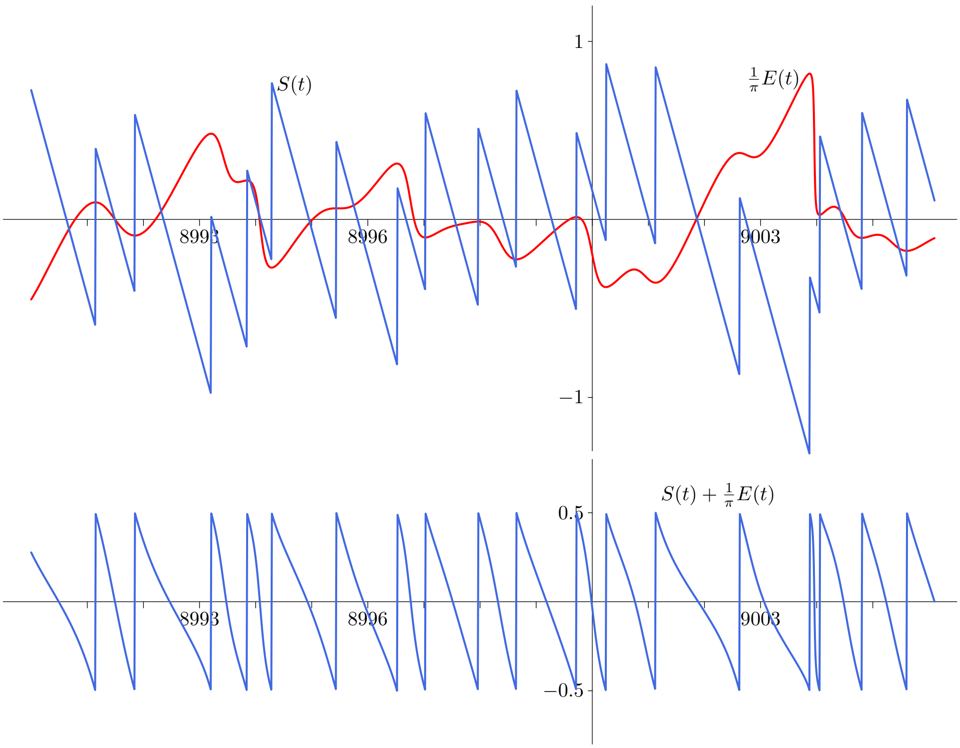

Fig. 7. Plots of $S(t)$, $\frac{1}{\pi}E(t)$ and $S(t) + \frac{1}{\pi}E(t)$ for t in $(8990, 9006)$

10. Extension to other L-functions. Most of the formulas and functions defined in this paper for $\zeta(s)$ can be generalized to other functions, including the Selberg class. The main thing we need is a functional equation. So let us assume that we have a Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which can be extended as a meromorphic function to the plane $\mathbb{C}$ in such a way that there exist numbers $Q > 0$, $\alpha_n > 0$ and $r_n \in \mathbb{C}$ with $\operatorname{Re}(r_n) \geq 0$ such that

$$\Phi(s) := Q^s f(s) \prod_{n=1}^{d} \Gamma(\alpha_n s + r_n) \quad \text{satisfies} \quad \Phi(s) = w\overline{\Phi(1-\overline{s})}$$

where w is a complex number of modulus $|w| = 1$. In this way all Dirichlet series for a primitive character, and the Dirichlet series $f(s)$ considered by Titchmarsh [17, Section 10.25], which has no Euler product and does not satisfy an RH, will be included.

Setting $s = 1/2 + it$ we see that the functional equation leads to

$$\frac{f(1/2+it)}{\overline{f(1/2+it)}} = wQ^{-2it} \prod_{n=1}^{d} \frac{\overline{\Gamma(\alpha_n(1/2+it)+r_n)}}{\Gamma(\alpha_n(1/2+it)+r_n)}.$$

Therefore, if we define

$$\vartheta(f,t) := -\frac{\arg w}{2} + t\log Q + \sum_{n=1}^{d} \operatorname{ph}\Gamma(\alpha_n(1/2+it)+r_n),$$

this will be a real analytic function and $\operatorname{ph} f(1/2+it) = -\vartheta(f,t)$ so that

$$f(1/2+it) = e^{-i\vartheta(f,t)} Z(f,t)$$

where $Z(f,t)$ is a real valued real analytic function of the real variable t. It is not difficult to define functions $\kappa(f,t)$, $E(f,t)$, and so on.

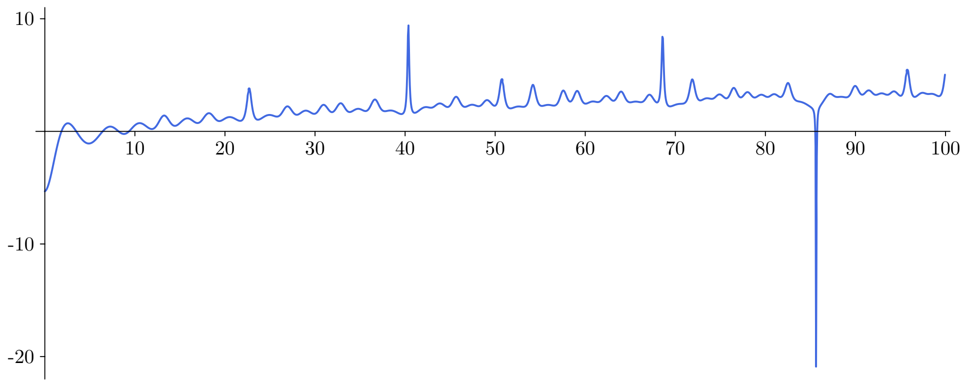

Fig. 8. Plot of $\kappa'(f,t)$ for the Titchmarsh function $f(s)$ mentioned above. This Dirichlet series has a zero at the point $\rho \approx 0.80851718 + i\,85.69934848$.

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