ACTA ARITHMETICA
163.3 (2014)

The Selberg–Delange method in short intervals with an application

by

Z. CUI (Shanghai) and J. WU (Vandœuvre-lès-Nancy)

1. Introduction. Many number-theoretic problems lead to the study of mean values of arithmetic functions. For this purpose, between 1954 and 1971, Selberg [8] and Delange [2, 3] developed a quite general method using the analytic properties of the Dirichlet series associated to the arithmetic function under study. This is nowadays known as the *Selberg–Delange method*. We refer the readers to [10, Chapter II.5] for an excellent exposition of this theory.

Let $f(n)$ be an arithmetic function and denote its corresponding Dirichlet series by

$$\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}. \tag{1.1}$$

Suppose that $\mathcal{F}(s)$ admits the factorization

$$\mathcal{F}(s) = \mathcal{G}(s; z)\zeta(s)^z$$

for $\Re e\, s > 1$, where $\zeta(s)$ is the Riemann ζ-function and $z \in \mathbb{C}$. Under some suitable assumptions on $\mathcal{G}(s; z)$, we may apply the Selberg–Delange method to establish a very precise asymptotic formula for the summatory function

$$S_f(x) := \sum_{n \le x} f(n).$$

See [10, Theorem II.5.3]. In 2008, Hanrot, Tenenbaum & Wu [5] further extended this method to investigate the mean value of $f(n)$ over the friable integers:

$$S_f(x, y) := \sum_{\substack{n \le x \\ P(n) \le y}} f(n),$$

2010 *Mathematics Subject Classification*: Primary 11N37.
Key words and phrases: asymptotic results on arithmetic functions.

DOI: 10.4064/aa163-3-4

where $P(n)$ is the largest prime factor of n with the convention $P(1) = 1$. In particular, suppose $\zeta_{\mathbb{K}}(s)$ is the Dedekind ζ-function of the number field $\mathbb{K}$ and $\kappa_j \in \mathbb{R}$ such that $\kappa_1 + \cdots + \kappa_r > 0$. If $\mathcal{F}(s)$ factors into

$$\mathcal{F}(s) = \mathcal{G}(s; z) \prod_{1 \le j \le r} \zeta_{\mathbb{K}_j}(s)^{\kappa_j}$$

for $\Re e\, s > 1$, then Hanrot, Tenenbaum & Wu, using also the saddle-point method of [9], established in [5, Théorème 1.2] a very precise asymptotic formula for $S_f(x, y)$ in wide ranges of x and y. It is worth noting that f is not assumed to be multiplicative albeit it is a Dirichlet convolution.

In this paper, we extend the Selberg–Delange method to handle the sum $\sum f(n)$ where n ranges over a short interval, and we give an application. We shall proceed along the same line of argument as in [10, Chapter II.5]. Let $\kappa > 0$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \ge 0$, $A \ge 0$, $B > 0$, $M > 0$ be some constants. A Dirichlet series $\mathcal{F}(s)$ as in (1.1) is said to be of type $\mathcal{P}(\kappa, w, \alpha, \delta, A, B, M)$ if the following conditions are satisfied:

(a) for any $\varepsilon > 0$ we have

$$|f(n)| \ll_\varepsilon n^\varepsilon \qquad (n \ge 1); \tag{1.2}$$

(b) we have

$$\sum_{n=1}^{\infty} |f(n)| n^{-\sigma} \ll (\sigma - 1)^{-\alpha} \qquad (\sigma > 1);$$

(c) the Dirichlet series

$$\mathcal{G}(s; \kappa, w) := \mathcal{F}(s)\zeta(s)^{-\kappa}\zeta(2s)^w \tag{1.3}$$

is analytically continued to a holomorphic function in (some open set containing) $\Re e\, s \ge 1/2$ and, in this region, $\mathcal{G}(s; \kappa, w)$ satisfies the bound

$$|\mathcal{G}(s; \kappa, w)| \le M(|\tau| + 1)^{\max\{\delta(1-\sigma), 0\}} \log^A(|\tau| + 1) \qquad (s = \sigma + \mathrm{i}\tau) \tag{1.4}$$

uniformly for $0 < \kappa \le B$ and $|w| \le B$.

In order to state our result, it is necessary to introduce some more notation. From [10, Theorem II.5.1] ([1]), the function

$$Z(s; z) := \{(s - 1)\zeta(s)\}^z \quad (z \in \mathbb{C})$$

is holomorphic in the disc $|s-1| < 1$, and admits the Taylor series expansion

$$Z(s; z) = \sum_{j=0}^{\infty} \frac{\gamma_j(z)}{j!}(s - 1)^j,$$

([1]) In [10], $Z(s; z)$ is defined as $s^{-1}\{(s - 1)\zeta(s)\}^z$ but obviously the argument there works for our $Z(s; z)$.

where the $\gamma_j(z)$'s are entire functions of z that satisfy, for all $B > 0$ and $\varepsilon > 0$, the estimate

$$\gamma_j(z)/j! \ll_{B,\varepsilon} (1+\varepsilon)^j \qquad (j \geq 0,\ |z| \leq B). \tag{1.5}$$

Under our hypothesis, the function $\mathcal{G}(s;\kappa,w)\zeta(2s)^{-w}Z(s;\kappa)$ is holomorphic in the disc $|s-1| < 1/2$ and

$$|\mathcal{G}(s;\kappa,w)\zeta(2s)^{-w}Z(s;\kappa)| \ll_{A,B,\delta,\varepsilon} M \tag{1.6}$$

for $|s-1| \leq 1/2+\varepsilon$, $0 < \kappa \leq B$ and $|w| \leq B$. Thus for $|s-1| < 1/2$, we can write

$$\mathcal{G}(s;\kappa,w)\zeta(2s)^{-w}Z(s;\kappa) = \sum_{\ell=0}^{\infty} g_\ell(\kappa,w)(s-1)^\ell, \tag{1.7}$$

where

$$g_\ell(\kappa,w) := \frac{1}{\ell!}\sum_{j=0}^{\ell}\binom{\ell}{j}\frac{\partial^{\ell-j}(\mathcal{G}(s;\kappa,w)\zeta(2s)^{-w})}{\partial s^{\ell-j}}\bigg|_{s=1}\gamma_j(\kappa). \tag{1.8}$$

The following result is an analogue of Theorem II.5.3 of [10] for the mean value over short intervals.

Theorem 1.1. *Let $\kappa > 0$, $w \in \mathbb{C}$, $\alpha > 0$, $\delta \geq 0$, $A \geq 0$, $B > 0$, $M > 0$ be some constants. Suppose that*

$$\mathcal{F}(s) := \sum_{n=1}^{\infty} f(n)n^{-s}$$

is a Dirichlet series of type $\mathcal{P}(\kappa,w,\alpha,\delta,A,B,M)$. Then for any $\varepsilon > 0$, we have

$$\sum_{x<n\leq x+y} f(n) = y(\log x)^{\kappa-1}\bigg\{\sum_{\ell=0}^{N}\frac{\lambda_\ell(\kappa,w)}{(\log x)^\ell} + O(R_N(x,y))\bigg\} \tag{1.9}$$

uniformly for

$$x \geq y \geq x^{\theta(\kappa,\delta)+\varepsilon} \geq 2, \qquad N \geq 0, \qquad 0 < \kappa \leq B, \qquad |w| \leq B,$$

where

$$\lambda_\ell(\kappa,w) := \frac{g_\ell(\kappa,w)}{\Gamma(\kappa-\ell)}, \qquad \theta(\kappa,\delta) := \frac{5\kappa+15\delta+21}{5\kappa+15\delta+36},$$

$$R_N(x,y) := \frac{y}{x}\sum_{\ell=1}^{N+1}\frac{\ell|\lambda_{\ell-1}(\kappa,w)|}{(\log x)^\ell} + \frac{(c_1N+1)^{N+1}}{x^{1/2}} + M\bigg\{\bigg(\frac{c_1N+1}{\log x}\bigg)^{N+1} + \mathrm{e}^{-c_2(\log x/\log_2 x)^{1/3}}\bigg\}$$

for some constants $c_1, c_2 > 0$. The implied constant in the O-term depends only on A, B, α, δ and ε.

The proof of Theorem 1.1 is rather similar to that of [10, Theorem II.5.3]. The main new ingredient we introduce is the contour of integration as in [7]. Thanks to the hypothesis (1.2), our proof seems slightly simpler.

As an application of Theorem 1.1, we generalize the Deshouillers–Dress–Tenenbaum arcsine law on divisors to the short interval case. For each positive integer n, denote by $\tau(n)$ the number of divisors of n and define the random variable D_n to take the value $(\log d)/\log n$, as d runs through the set of the $\tau(n)$ divisors of n, with the uniform probability $1/\tau(n)$. The distribution function F_n of D_n is given by

$$F_n(t) = \mathrm{Prob}(D_n \le t) = \frac{1}{\tau(n)} \sum_{d|n,\, d\le n^t} 1 \qquad (0 \le t \le 1).$$

It is clear that the sequence $\{F_n\}_{n\ge1}$ does not converge pointwise on $[0,1]$. However Deshouillers, Dress & Tenenbaum ([4] or [10, Theorem II.6.7]) proved that its Cesàro means converge uniformly to the arcsine law, more precisely,

$$\frac{1}{x}\sum_{n\le x} F_n(t) = \frac{2}{\pi}\arcsin\sqrt{t} + O\bigg(\frac{1}{\sqrt{\log x}}\bigg) \tag{1.10}$$

uniformly for $x \ge 2$ and $0 \le t \le 1$. The error term in (1.10) is optimal. Very recently Basquin [1] considered the generalization of (1.10) for friable integers. Interestingly he showed that the limit law shifts from the arcsine law towards the Gaussian as $u := (\log x)/\log y \to \infty$.

Here we obtain an analogue of (1.10) for short intervals.

THEOREM 1.2. *Let $\varepsilon > 0$ be an arbitrarily small positive constant. Then*

$$\frac{1}{y}\sum_{x<n\le x+y} F_n(t) = \frac{2}{\pi}\arcsin\sqrt{t} + O_\varepsilon\bigg(\frac{1}{\sqrt{\log x}}\bigg) \tag{1.11}$$

uniformly for $0 \le t \le 1$, $x \ge 2$ and $x^{62/77+\varepsilon} \le y \le x$, where the implied constant depends only on ε. Further (1.11) with $y = x$ implies (1.10).

2. Proof of Theorem 1.1. Since $\mathcal{F}(s)$ is a Dirichlet series of type $\mathcal{P}(\kappa,\alpha,w,\delta,A,B,M)$, we can apply [10, Corollary II.2.2.1] with the choice of parameters $\sigma_a = 1$, $B(n) := n^\varepsilon$, $\alpha = \alpha$, $\sigma = 0$ to write

$$\sum_{x<n\le x+y} f(n) = \frac{1}{2\pi\mathrm{i}}\int_{b-\mathrm{i}T}^{b+\mathrm{i}T} \mathcal{F}(s)\frac{(x+y)^s - x^s}{s}\,\mathrm{d}s + O\bigg(\frac{x^{1+\varepsilon}}{T}\bigg),$$

where $b := 1 + 2/\log x$ and $100 \le T \le x$ such that $\zeta(\sigma + \mathrm{i}T) \ne 0$ for $0 < \sigma < 1$.

Let $\mathscr{L}$ be the boundary of the modified rectangle with vertices $1/2 \pm \mathrm{i}T$ and $b \pm \mathrm{i}T$, where

- the zeros of $\zeta(s)$ of the form $1/2 + \mathrm{i}\gamma$ with $|\gamma| < T$ are avoided by the semicircles of infinitely small radius lying to the right of the line $\Re e\, s = 1/2$,
- the zeros of $\zeta(s)$ of the form $\rho = \beta + \mathrm{i}\gamma$ with $\beta > 1/2$ and $|\gamma| < T$ are avoided by the horizontal cut drawn from the critical line inside this rectangle to $\rho = \beta + \mathrm{i}\gamma$,
- the pole of $\zeta(2s)$ at the point $s = 1/2$ is avoided by two arcs $\mathscr{L}_3$ and $\mathscr{L}_4$ with the radius $r := 1/\log x$,
- the pole of $\zeta(s)$ at the point $s = 1$ is avoided by the truncated Hankel contour Γ (its upper part is made up of an arc surrounding the point $s = 1$ with radius $r := 1/\log x$ and a line segment joining $1 - r$ to $1/2 + r$).

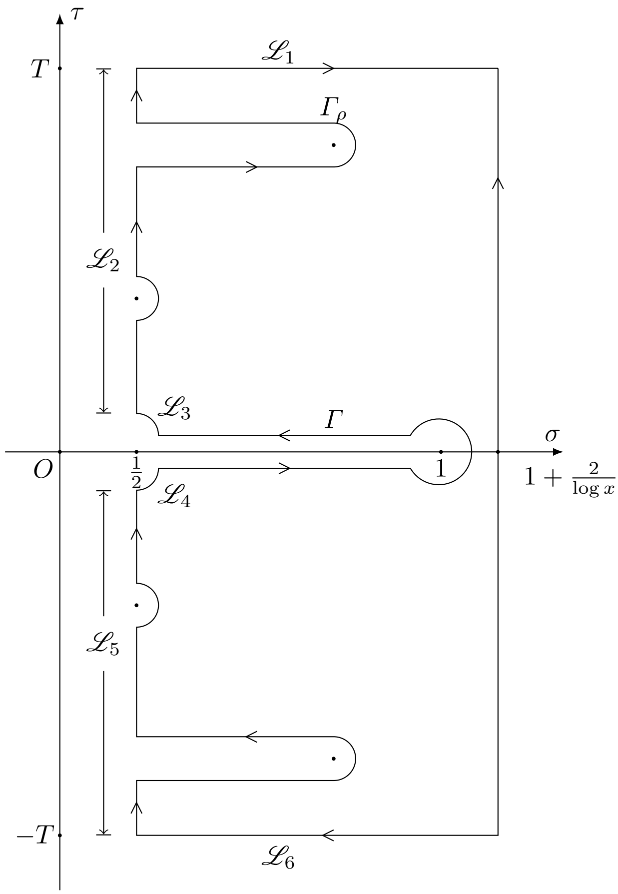

Fig. 1. Contour $\mathscr{L}$

Clearly the function $\mathcal{F}(s)$ is analytic inside $\mathscr{L}$. By the Cauchy residue theorem, we can write

$$\sum_{x<n\le x+y} f(n) = I + I_1 + \cdots + I_6 + \sum_{\beta>1/2,\, |\gamma|<T} I_\rho + O_\varepsilon\Big(\frac{x^{1+\varepsilon}}{T}\Big), \tag{2.1}$$

where

$$\begin{aligned} I &:= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} \, \mathrm{d}s, \\ I_\rho &:= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\rho} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} \, \mathrm{d}s, \\ I_j &:= \frac{1}{2\pi \mathrm{i}} \int_{\mathscr{L}_j} \mathcal{F}(s) \frac{(x+y)^s - x^s}{s} \, \mathrm{d}s. \end{aligned}$$

A. *Evaluation of* I. Let $0 < c < 1/10$ be a small constant. Since $\mathcal{G}(s;\kappa,w)\zeta(2s)^{-w}Z(s;\kappa)$ is holomorphic and $O(M)$ in the disc $|s-1| \le c$, the Cauchy formula implies that

$$g_\ell(\kappa, w) \ll Mc^{-\ell} \qquad (\ell \ge 0,\ 0<\kappa\le B,\ |w|\le B), \tag{2.2}$$

where $g_\ell(\kappa,w)$ is defined as in (1.8). From this and (1.7), it is easy to deduce that for any integer $N \ge 0$ and $|s-1| \le \frac{1}{2}c$,

$$\mathcal{G}(s;\kappa,w)\zeta(2s)^{-w}Z(s;\kappa) = \sum_{\ell=0}^{N} g_\ell(\kappa,w)(s-1)^\ell + O\big(M(|s-1|/c)^{N+1}\big).$$

Thus we have

$$I = \sum_{\ell=0}^{N} g_\ell(\kappa,w) M_\ell(x,y) + O\big(Mc^{-N} E_N(x,y)\big), \tag{2.3}$$

where

$$\begin{aligned} M_\ell(x,y) &:= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} (s-1)^{\ell-\kappa} \frac{(x+y)^s - x^s}{s} \, \mathrm{d}s, \\ E_N(x,y) &:= \int_{\Gamma} \left| (s-1)^{N+1-\kappa} \frac{(x+y)^s - x^s}{s} \right| |\mathrm{d}s|. \end{aligned}$$

Firstly we evaluate $M_\ell(x,y)$. By using the formula

$$\frac{(x+y)^s - x^s}{s} = \int_{x}^{x+y} t^{s-1} \, \mathrm{d}t \tag{2.4}$$

and Corollary II.5.2.1 of [10], we can write

$$\begin{aligned} M_\ell(x,y) &= \int_x^{x+y} \bigg(\frac{1}{2\pi \mathrm{i}} \int_\Gamma (s-1)^{\ell-\kappa} t^{s-1} \, \mathrm{d}s\bigg) \, \mathrm{d}t \\ &= \int_x^{x+y} (\log t)^{\kappa-1-\ell} \bigg\{ \frac{1}{\Gamma(\kappa-\ell)} + O\bigg(\frac{(c_1\ell+1)^\ell}{t^{1/2}}\bigg)\bigg\} \, \mathrm{d}t, \end{aligned}$$

where we have used the inequality

$$47^{|\kappa-\ell|} \Gamma(1+|\kappa-\ell|) \ll_B (c_1\ell+1)^\ell \quad (\ell \ge 0,\ 0 < \kappa \le B).$$

The constant c_1 and the implied constant depend at most on B. On the other hand, it is easy to see that, for $0 < \kappa \le B$,

$$\begin{aligned} \int_x^{x+y} (\log t)^{\kappa-1-\ell} \, \mathrm{d}t &= \int_0^y \log^{\kappa-1-\ell}(x+t) \, \mathrm{d}t \\ &= y(\log x)^{\kappa-1-\ell} \bigg\{ 1 + O_B\bigg(\frac{(\ell+1)y}{x\log x}\bigg)\bigg\}. \end{aligned}$$

Inserting this into the preceding formula, we obtain

$$\begin{aligned} (2.5)\qquad & M_\ell(x,y) \\ &= y(\log x)^{\kappa-1-\ell} \bigg\{ \frac{1}{\Gamma(\kappa-\ell)} + O_B\bigg(\frac{(\ell+1)y}{\Gamma(\kappa-\ell) x\log x} + \frac{(c_1\ell+1)^\ell}{x^{1/2}}\bigg)\bigg\} \end{aligned}$$

for $\ell \ge 0$ and $0 < \kappa \le B$.

Next we estimate $E_N(x,y)$. In view of the trivial inequality

$$\left| \frac{(x+y)^s - x^s}{s} \right| \ll y x^{\sigma-1}, \tag{2.6}$$

we deduce that

$$\begin{aligned} (2.7)\qquad E_N(x,y) &\ll \int_{1/2+1/\log x}^{1-1/\log x} (1-\sigma)^{N+1-\kappa} x^{\sigma-1} y \, \mathrm{d}\sigma + \frac{y}{(\log x)^{N+2-\kappa}} \\ &\ll \frac{y}{(\log x)^{N+2-\kappa}} \bigg(\int_{1/2}^{\infty} t^{N+1-\kappa} \mathrm{e}^{-t} \, \mathrm{d}t + 1\bigg) \\ &\ll y(\log x)^{\kappa-1} \bigg(\frac{c_1 N+1}{\log x}\bigg)^{N+1} \end{aligned}$$

uniformly for $x \ge y \ge 2$, $N \ge 0$ and $0 < \kappa \le B$, where the constant $c_1 > 0$ and the implied constant depend only on B.

Inserting (2.5) and (2.7) into (2.3) and using (2.2), we find that

$$I = y(\log x)^{\kappa-1}\bigg\{\sum_{\ell=0}^{N}\frac{\lambda_\ell(\kappa,w)}{(\log x)^\ell} + O_B(E_N^*(x,y))\bigg\}, \tag{2.8}$$

where

$$E_N^*(x,y) := \frac{y}{x}\sum_{\ell=1}^{N+1}\frac{\ell|\lambda_{\ell-1}(\kappa,w)|}{(\log x)^\ell} + \frac{(c_1N+1)^{N+1}}{x^{1/2}} + M\bigg(\frac{c_1N+1}{\log x}\bigg)^{N+1}.$$

B. *Estimations of* I_3 *and* I_4. For $s = 1/2 + \mathrm{e}^{\mathrm{i}\theta}/\log x$ with $0 < |\theta| \le \pi/2$, we have trivially

$$\mathcal{F}(s) \ll (\log x)^{|\Re e\, w|+A}, \qquad \bigg|\frac{(x+y)^s - x^s}{s}\bigg| \ll x^{1/2}.$$

Thus

$$|I_3| + |I_4| \ll x^{1/2}(\log x)^{|\Re e\, w|+A-1} \qquad (x \ge 3). \tag{2.9}$$

C. *Estimations of* I_1 *and* I_6. It is well known that

$$|\zeta(\sigma+\mathrm{i}\tau)| \ll |\tau|^{(1-\sigma)/3}\log|\tau| \qquad (1/2 \le \sigma \le 1 + \log^{-1}|\tau|,\ |\tau| \ge 2), \tag{2.10}$$

$$|\zeta(\sigma+\mathrm{i}\tau)| \gg \log^{-1}(|\tau|+3) \qquad (\sigma \ge 1 - \sigma_0(\tau),\ \tau \in \mathbb{R}), \tag{2.11}$$

where $C > 0$ is an absolute positive constant and

$$\sigma_0(t) := \frac{C}{(\log(|t|+3))^{2/3}(\log\log(|t|+3))^{1/3}}. \tag{2.12}$$

In view of (2.10), (2.11) and (1.4), we have

$$\mathcal{F}(s) \ll MT^{\max\{(1-\sigma)(\kappa/3+\delta),\,0\}}(\log T)^{|\Re e\, w|+\kappa+A}$$

for $s = \sigma \pm \mathrm{i}T$ with $1/2 \le \sigma \le b$. Thus

$$\begin{aligned} |I_1| + |I_6| &\ll \int_{1/2}^{b} MT^{(1-\sigma)(\kappa/3+\delta)}(\log T)^{|\Re e\, w|+\kappa+A}\frac{x^\sigma}{T}\,\mathrm{d}\sigma \\ &\ll \frac{x}{T}(\log T)^{|\Re e\, w|+\kappa+A} \end{aligned} \tag{2.13}$$

provided $T \le x^{1/(\kappa/3+\delta)}$.

D. *Estimations of* I_2 *and* I_5. For $s = 1/2 + \mathrm{i}\tau \ne 1/2 + \mathrm{i}\gamma$ with $\zeta(1/2+\mathrm{i}\gamma) = 0$ and $1/\log x \le |\tau| \le T$, the estimates (2.10), (2.11) and (1.4) imply that

$$\mathcal{F}(s) \ll (|\tau|+1)^{\kappa/6+\delta/2}(\log x)^{|\Re e\, w|+\kappa+A}.$$

This allows us to write

$$(2.14)\qquad |I_2| + |I_5| \ll x^{1/2}(\log x)^{|\Re e\, w|+\kappa+A} \int_0^T (\tau+1)^{-1+\kappa/6+\delta/2}\,\mathrm{d}\tau$$
$$\ll x^{1/2}(\log x)^{|\Re e\, w|+\kappa+A} T^{\kappa/6+\delta/2}.$$

E. *Estimations of the* I_ρ. As in case C, we have

$$\mathcal{F}(s) \ll M|\gamma|^{(1-\sigma)(\kappa/3+\delta)}(\log|\gamma|)^{|\Re e\, w|+\kappa+A}$$

for $s = \sigma + \mathrm{i}\gamma$ with $1/2 \le \sigma \le \beta < 1 - \sigma_0(\gamma)$. From this and (2.6) we deduce that

$$(2.15)\qquad |I_\rho| \ll \int_{1/2}^{\beta} M|\gamma|^{(1-\sigma)(\kappa/3+\delta)}(\log|\gamma|)^{|\Re e\, w|+\kappa+A} x^{\sigma-1} y\,\mathrm{d}\sigma.$$

Denote by $N(\sigma, T)$ the number of zeros of $\zeta(s)$ in the region $\Re e\, s \ge \sigma$ and $|\Im m\, z| \le T$. Summing (2.15) over $|\gamma| < T$ and interchanging the summations, we have

$$\sum_{\substack{\beta>1/2\\ |\gamma|<T}} |I_\rho| \ll My(\log x)^{|\Re e\, w|+\kappa+A} \int_{1/2}^{1-\sigma_0(T)} (T^{\kappa/3+\delta}/x)^{1-\sigma} N(\sigma, T)\,\mathrm{d}\sigma.$$

According to [6],

$$(2.16)\qquad N(\sigma, T) \ll T^{(12/5)(1-\sigma)}(\log T)^{44}$$

for $1/2 \le \sigma \le 1$ and $T \ge 2$. Thus

$$(2.17)\qquad \sum_{\substack{\beta>1/2\\ |\gamma|<T}} |I_\rho| \ll My(\log x)^{|\Re e\, w|+\kappa+A+44} \int_{1/2}^{1-\sigma_0(T)} (T^{\kappa/3+\delta+12/5}/x)^{1-\sigma}\,\mathrm{d}\sigma$$
$$\ll y(\log x)^{|\Re e\, w|+\kappa+A+44}(T^{\kappa/3+\delta+12/5}/x)^{\sigma_0(T)}$$

provided $T \le x^{1/(\kappa/3+\delta+12/5)}/2$.

Inserting (2.8), (2.9), (2.13), (2.14) and (2.17) into (2.1), we find that

$$\sum_{x<n\le x+y} f(n) = y(\log x)^{\kappa-1}\bigg\{\sum_{\ell=0}^{N} \frac{\lambda_\ell(\kappa, w)}{(\log x)^\ell} + O(E_N^*(x, y))\bigg\} + \mathcal{R}_T(x, y),$$

where

$$\mathcal{R}_T(x, y) := y(\log x)^{|\Re e\, w|+\kappa+A+44}\bigg(\frac{T^{\kappa/3+\delta+12/5}}{x}\bigg)^{\sigma_0(T)}$$
$$+ \frac{x^{1+\varepsilon}}{T} + x^{1/2}(\log x)^{|\Re e\, w|+\kappa+A} T^{\kappa/6+\delta/2}.$$

Taking $T = x^{1/(\kappa/3+\delta+12/5)-10\varepsilon}$, we obtain the required result.

3. Proof of Theorem 1.2. Firstly we establish the following lemma with the help of Theorem 1.1.

LEMMA 3.1. *For any $\varepsilon > 0$, we have*

$$\sum_{x<n\le x+y} \frac{1}{\tau(dn)} = \frac{hy}{\sqrt{(\pi \log x)}}\left\{g(d) + O_\varepsilon\left(\frac{(3/4)^{\omega(d)}}{\log x}\right)\right\}$$

uniformly for $d \ge 1$, $x \ge 2$ and $x^{47/77+\varepsilon} \le y \le x$, where $\omega(n)$ is the number of distinct prime factors of n and

$$h := \prod_p \sqrt{p(p-1)}\log(1-1/p)^{-1},$$

$$g(d) := \prod_{p^\nu \| d}\left(\sum_{j=0}^{\infty}\frac{p^{-j}}{j+\nu+1}\right)\left(\sum_{j=0}^{\infty}\frac{p^{-j}}{j+1}\right)^{-1}.$$

Proof. As usual, we denote by $v_p(n)$ the p-adic valuation of n. By using the formula

$$\tau(dn) = \prod_p (v_p(n) + v_p(d) + 1),$$

we write, for $\Re e\, s > 1$,

$$\begin{aligned}\mathcal{F}_d(s) := \sum_{n=1}^{\infty}\tau(dn)^{-1}n^{-s} &= \prod_p \sum_{j=0}^{\infty}\frac{p^{-js}}{j+v_p(d)+1}\\ &= \frac{\zeta(s)^{1/2}}{\zeta(2s)^{1/24}}\mathcal{G}_d(s;1/2,1/24),\end{aligned}$$

where

$$\begin{aligned}\mathcal{G}_d(s;1/2,1/24) := &\prod_p \sum_{j=0}^{\infty}\frac{p^{-js}}{j+1}\left(1-\frac{1}{p^s}\right)^{1/2}\left(1-\frac{1}{p^{2s}}\right)^{-1/24}\\ &\times \prod_{p^\nu\|d}\left(\sum_{j=0}^{\infty}\frac{p^{-js}}{j+\nu+1}\right)\left(\sum_{j=0}^{\infty}\frac{p^{-js}}{j+1}\right)^{-1}\end{aligned}$$

is a Dirichlet series that converges absolutely for $\Re e\, s > 1/3$. For $\Re e\, s \ge 1/2$, we easily see that

$$\left|\sum_{j=0}^{\infty}\frac{p^{-js}}{j+1}\right| = \left|\frac{\log(1-p^{-s})}{p^{-s}}\right| \ge \frac{\log(1+p^{-\sigma})}{p^{-\sigma}} \ge \frac{1}{1+p^{-1/2}}.$$

This implies

$$|\mathcal{G}_d(s;1/2,1/24)| \ll \prod_{p^\nu\|d}\left\{\frac{1}{1+\nu} + O\left(\frac{1}{\sqrt{p}}\right)\right\} \le C\left(\frac{3}{4}\right)^{\omega(d)}$$

for $\Re e\, s \ge 1/2$, where $C > 0$ is an absolute constant.

Hence, $\mathcal{F}_d(s)$ is a Dirichlet series of type $\mathcal{P}(1/2, 1/24, 1/2, 0, 0, 1/2, C(3/4)^{\omega(d)})$. Applying Theorem 1.1 with $N = 0$ and noticing that $\lambda_0(1/2) = hg(d)/\Gamma(1/2) = hg(d)/\sqrt{\pi}$, we get

$$\sum_{x<n\le x+y} \frac{1}{\tau(dn)} = \frac{hy}{\sqrt{(\pi \log x)}}\bigg\{g(d) + O_\varepsilon\bigg(\frac{g(d)y}{x\log x} + \frac{(3/4)^{\omega(d)}}{\log x}\bigg)\bigg\}$$

uniformly for $d \ge 1$, $x \ge 2$ and $x^{47/77+\varepsilon} \le y \le x$. This implies the required result since $g(d) \ll (3/4)^{\omega(d)}$ and $y \le x$. ∎

We are now ready to prove Theorem 1.2.

In view of the symmetry of the divisors of n about $\sqrt{n}$, we have

$$\begin{aligned} F_n(t) &= \mathrm{Prob}(D_n \ge 1-t) = 1 - \mathrm{Prob}(D_n < 1-t) \\ &= 1 - F_n(1-t) + O(\tau(n)^{-1}). \end{aligned}$$

Summing over $x < n \le x+y$ and applying Lemma 3.1 with $d = 1$, we find that

$$S(x, y; t) + S(x, y; 1-t) = 1 + O\bigg(\frac{1}{\sqrt{\log x}}\bigg) \qquad (0 \le t \le 1),$$

where

$$S(x, y; t) := \frac{1}{y}\sum_{x<n\le x+y} F_n(t).$$

On the other hand, we have the identity

$$\frac{2}{\pi}\arcsin\sqrt{t} + \frac{2}{\pi}\arcsin\sqrt{1-t} = 1 \qquad (0 \le t \le 1).$$

Therefore it is sufficient to prove (1.11) for $0 \le t \le 1/2$.

For $0 \le t \le 1/2$, we can write

$$S(x, y; t) = \frac{1}{y}\sum_{x<n\le x+y}\frac{1}{\tau(n)}\sum_{d\mid n,\, d\le n^t} 1 = S_1(x, y; t) - S_2(x, y; t), \tag{3.1}$$

where

$$\begin{aligned} S_1(x, y; t) &:= \frac{1}{y}\sum_{x<n\le x+y}\frac{1}{\tau(n)}\sum_{d\mid n,\, d\le (x+y)^t} 1, \\ S_2(x, y; t) &:= \frac{1}{y}\sum_{x<n\le x+y}\frac{1}{\tau(n)}\sum_{d\mid n,\, n^t<d\le (x+y)^t} 1. \end{aligned}$$

Firstly we evaluate $S_1(x, y; t)$. Changing the order of summations, we have

$$S_1(x, y; t) = \frac{1}{y}\sum_{d\le (x+y)^t}\ \sum_{x/d<m\le (x+y)/d}\frac{1}{\tau(dm)}.$$

For $d \le (x+y)^t \le (2x)^{1/2}$ and $y \ge x^{62/77+\varepsilon}$, it is easy to verify that

$$(y/d) \ge (x/d)^{47/77+\varepsilon}.$$

Thus we can apply Lemma 3.1 with $(x/d, y/d)$ in place of (x,y) to write

$$S_1(x,y;t) = \frac{h}{\sqrt{\pi}} \sum_{d\le(x+y)^t} \frac{1}{d\sqrt{\log(x/d)}}\left\{g(d) + O_\varepsilon\left(\frac{(3/4)^{\omega(d)}}{\log x}\right)\right\}$$

uniformly for $0 \le t \le 1/2$, $x \ge 2$ and $x \ge y \ge x^{62/77+\varepsilon}$. Bounding $(3/4)^{\omega(d)}$ by 1, the contribution of the error term to S_1 is $\ll 1/\sqrt{\log x}$. According to [10, Chapter II.6], we have

$$\frac{h}{\sqrt{\pi}} \sum_{d\le x^t} \frac{g(d)}{d\sqrt{\log(x/d)}} = \frac{2}{\pi}\arcsin\sqrt{t} + O\left(\frac{1}{\sqrt{\log x}}\right),$$

which implies that

$$\frac{h}{\sqrt{\pi}} \sum_{d\le (x+y)^t} \frac{g(d)}{d\sqrt{\log(x/d)}} = \frac{2}{\pi}\arcsin\sqrt{t} + O\left(\frac{1}{\sqrt{\log x}}\right),$$

since

$$\sum_{x^t<d\le(x+y)^t} \frac{g(d)}{d\sqrt{\log(x/d)}} \ll \frac{1}{\sqrt{\log x}} \sum_{x^t<d\le(x+y)^t} \frac{1}{d} \ll \frac{1}{\sqrt{\log x}}.$$

Combining these estimates, we obtain

$$S_1(x,y;t) = \frac{2}{\pi}\arcsin\sqrt{t} + O\left(\frac{1}{\sqrt{\log x}}\right) \tag{3.2}$$

uniformly for $0 \le t \le 1/2$, $x \ge 2$ and $x \ge y \ge x^{62/77+\varepsilon}$.

Next, a similar treatment leads to

$$\begin{aligned} S_2(x,y;t) &\le \frac{1}{y} \sum_{x^t<d\le(x+y)^t} \sum_{x/d<m\le(x+y)/d} \frac{1}{\tau(m)} \\ &\ll \frac{1}{\sqrt{\log x}} \sum_{x^t<d\le(x+y)^t} \frac{1}{d} \ll \frac{1}{\sqrt{\log x}}. \end{aligned} \tag{3.3}$$

Inserting (3.2) and (3.3) into (3.1), we find that

$$S(x,y;t) = \frac{2}{\pi}\arcsin\sqrt{t} + O_\varepsilon\left(\frac{1}{\sqrt{\log x}}\right)$$

uniformly for $0 \le t \le 1/2$, $x \ge 2$ and $x \ge y \ge x^{62/77+\varepsilon}$.

Finally we prove that (1.10) follows from (1.11) with $y = x$. Since $0 \le F_n(t) \le 1$, we have

$$\begin{aligned}\sum_{n\le x} F_n(t) &= \sum_{\sqrt{x}<n\le x} F_n(t) + O(\sqrt{x})\\ &= \sum_{0\le k\le(\log x)/(2\log 2)} \sum_{x/2^{k+1}<n\le x/2^k} F_n(t) + O(\sqrt{x}).\end{aligned}$$

Applying (1.11) with $y = x$ to the inner sum, we deduce that

$$\begin{aligned}&\sum_{n\le x} F_n(t)\\ &\quad = h\sum_{k=0}^{[(\log x)/(2\log 2)]}\left\{\frac{x}{2^{k+1}}\,\frac{2}{\pi}\arcsin\sqrt{t} + O\biggl(\frac{x/2^{k+1}}{\sqrt{\log(x/2^{k+1})}}\biggr)\right\} + O(\sqrt{x})\\ &\quad = x\frac{2}{\pi}\arcsin\sqrt{t} + O\biggl(\frac{x}{\sqrt{\log x}}\biggr),\end{aligned}$$

since

$$2^{[(\log x)/(2\log 2)]+1} \asymp \sqrt{x} \quad\text{and}\quad \sum_{k=0}^{[(\log x)/(2\log 2)]}\frac{1}{2^{k+1}} = 1 + O\biggl(\frac{1}{\sqrt{x}}\biggr).$$

This completes the proof of Theorem 1.2.

Acknowledgments. We are grateful to the referee for his careful reading and valuable suggestions. We are also grateful to Y.-K. Lau for his help during the preparation of this paper. This paper was written when the first author visited l'Institut Élie Cartan de l'Université de Lorraine during the academic year 2012-2013. He would like to thank the institute for the pleasant working conditions. His work is supported by the National Natural Science Foundation of China (Grant No. 11271249) and the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20120073110059). The second author is supported in part by IRT1264.

References

[1] J. Basquin, *Trois études autour de sommes de fonctions multiplicatives sur les entiers friables*, Thèse de doctorat, Université de Lorraine, 2012.
[2] H. Delange, *Sur les formules dues à Atle Selberg*, Bull. Sci. Math. 83 (1959), 101–111.
[3] H. Delange, *Sur les formules de Atle Selberg*, Acta Arith. 19 (1971), 105–146.
[4] J.-M. Deshouillers, F. Dress et G. Tenenbaum, *Lois de répartition des diviseurs, 1*, Acta Arith. 23 (1979), 273–283.
[5] G. Hanrot, G. Tenenbaum et J. Wu, *Moyennes de certaines fonctions arithmétiques sur les entiers friables, 2*, Proc. London Math. Soc. (3) 96 (2008), 107–135.

[6] M. N. Huxley, *The difference between consecutive primes*, Invent. Math. 15 (1972), 164–170.

[7] A. A. Sedunova, *On the mean values of some arithmetic functions on the short interval*, arXiv:1302.0471v1 (2013).

[8] A. Selberg, *Note on the paper by L. G. Sathe*, J. Indian Math. Soc. 18 (1954), 83–87.

[9] G. Tenenbaum, *La méthode du col en théorie analytique des nombres*, in: Séminaire de Théorie des Nombres, Paris 1986–87, Progr. Math. 75, Birkhäuser Boston, Boston, MA, 1988, 411–441.

[10] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Stud. Adv. Math. 46, Cambridge Univ. Press, Cambridge, 1995.

Z. Cui
Department of Mathematics
Shanghai Jiao Tong University
Shanghai, China
E-mail: zcui@sjtu.edu.cn

J. Wu
School of Mathematics
Shandong University
Jinan, Shandong 250100, China
and
CNRS
Institut Élie Cartan de Lorraine, UMR 7502
54506 Vandœuvre-lès-Nancy, France
Current address:
Université de Lorraine
Institut Élie Cartan de Lorraine, UMR 7502
54506 Vandœuvre-lès-Nancy, France
E-mail: jie.wu@univ-lorraine.fr

Received on 27.6.2013
and in revised form on 9.1.2014 (7499)