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Counting rational points on del Pezzo surfaces with a conic bundle structure

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1. Introduction. Let k be a number field. A *del Pezzo surface* X over k is a non-singular projective surface defined over k, with ample anticanonical divisor $-K_X$. The degree of X is defined to be $d = (-K_X)^2$. In this paper we will be concerned with upper bounds for the number of k-rational points of bounded height on del Pezzo surfaces of small degree. The arithmetic of del Pezzo surfaces becomes harder to understand as d decreases. For $d \in \{2, 3, 4\}$ they admit the following classical description:

- an intersection of two quadrics in $\mathbb{P}^4$ when $d = 4$;
- a cubic surface in $\mathbb{P}^3$ when $d = 3$;
- a double cover of $\mathbb{P}^2$ branched over a smooth quartic plane curve when $d = 2$.

Given a del Pezzo surface X of degree d, let $U \subset X$ be the Zariski open set obtained by deleting from X the finite set of exceptional curves of the first kind. Let

$$N(U, k, B) = \#\{x \in U(k) : H_{-K_X}(x) \le B\},$$

where H_{-K_X} is the anticanonical height function on the set $X(k)$ of k-rational points on X. Our motivation is a simple form of the Batyrev–Manin conjecture [BM], which implies that we should have

$$N(U, k, B) = O_{\varepsilon,X}(B^{1+\varepsilon}) \tag{1.1}$$

for any $\varepsilon > 0$. Throughout this paper, unless otherwise indicated, we shall follow the convention that any implied constant is allowed to depend at most upon the number field k, with any further dependence explicitly indicated. In (1.1), for example, the implied constant is allowed to depend on X and the choice of ε, in addition to k.

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Recall that a *conic bundle surface* over k is defined to be a non-singular projective surface S defined over k, which is equipped with a dominant k-morphism $S \to \mathbb{P}^1$, all of whose fibres are conics. We shall focus our attention on del Pezzo surfaces of degree d which are also conic bundle surfaces. When no such restriction is made on the del Pezzo surface, the best general bound we have is due to Salberger [Sa]. Working in the special case $k = \mathbb{Q}$, he has established the estimate

$$N(U, \mathbb{Q}, B) = O_{\varepsilon,X}(B^{3/\sqrt{d}+\varepsilon}) \tag{1.2}$$

for any $\varepsilon > 0$.

Let us first consider the case of degree 4 del Pezzo surfaces $X \subset \mathbb{P}^4$ defined over k. In work presented at the conference "Higher dimensional varieties and rational points" in Budapest in 2001, Salberger noted that one can get better bounds for $N(U, k, B)$ when X contains a non-singular conic over k, in which case it has a conic bundle structure over k. For such surfaces he established (1.1) when $k = \mathbb{Q}$. The following result generalises this to arbitrary number fields.

THEOREM 1.1. *Let $\varepsilon > 0$ and let $X \subset \mathbb{P}^4$ be a del Pezzo surface of degree 4 over k, containing a non-singular conic defined over k. Then*

$$N(U, k, B) = O_{\varepsilon,X}(B^{1+\varepsilon}).$$

The implied constant is ineffective.

All of the implied constants in our results about del Pezzo surfaces are ineffective. This arises from an application of the Thue–Siegel–Roth theorem over number fields [L, §7, Thm. 1.1] (see Remark 2.5 for an indication of how effectivity can be recovered).

In the case $k = \mathbb{Q}$, de la Bretèche and Browning [dlBB] have obtained an asymptotic formula for $N(U, \mathbb{Q}, B)$, as $B \to \infty$, for a particular del Pezzo surface of degree 4 with a conic bundle structure over $\mathbb{Q}$. In general the best bound available is given by (1.2), although one can do better if one is willing to assume a standard rank hypothesis for elliptic curves over $\mathbb{Q}$ (see [Brow1, Section 7.3]).

According to Iskovskikh's k-birational classification [I], there are two possible classes of degree 4 conic bundle surfaces defined over k. When the anticanonical divisor is ample, one has a del Pezzo surface of degree 4, as considered in Theorem 1.1. When the anticanonical divisor is not ample, on the other hand, one obtains a Châtelet surface to which one can associate an analogous counting function $N(U, k, B)$. In this setting one still expects (1.1) to hold, and Browning [Brow2] has established this when $k = \mathbb{Q}$. Although we choose not to do so here, it is possible to use the results in this paper to extend this work to Châtelet surfaces defined over arbitrary number fields.

We now turn to del Pezzo surfaces of degree 3 over k. These arise as cubic surfaces $X \subset \mathbb{P}^3$. We know of no single example for which (1.1) has been proved. Cubic surfaces admit a conic bundle structure over k when one of the 27 lines contained in the surface is defined over k. The best bounds that we have for $N(U,k,B)$ arise when stronger hypotheses are placed on the configuration of lines in the surface. The following result is due to Broberg [Brob1] and is a generalisation of the case $k=\mathbb{Q}$ handled by Heath-Brown [HB1].

THEOREM 1.2. *Let $\varepsilon > 0$ and let $X \subset \mathbb{P}^3$ be a del Pezzo surface of degree* 3 *over k, containing three coplanar lines defined over k. Then*

$$N(U,k,B) = O_{\varepsilon,X}(B^{4/3+\varepsilon}).$$

The implied constant is ineffective.

Note that the implied constant is actually effective in [HB1] and [Brob1]. We will provide our own proof of Theorem 1.2, since our argument is simpler than that appearing in [Brob1], albeit at the expense of effectivity in the implied constant. In the case $k=\mathbb{Q}$, the best general bound is given by (1.2) with $d=3$. There is also further work of Heath-Brown [HB2] when $k=\mathbb{Q}$, which shows that the estimate in Theorem 1.2 holds for all cubic surfaces conditionally on the rank hypothesis mentioned previously.

Much less is known about the arithmetic of del Pezzo surfaces of degree 2 over k. These may be embedded in weighted projective space $\mathbb{P}(2,1,1,1)$ via an equation of the form

$$t^2 = f(x_1,x_2,x_3), \tag{1.3}$$

where $f \in k[x_1,x_2,x_3]$ is a non-singular form of degree 4. Taking $U \subset X$ to be the complement of the 56 exceptional curves, it was shown by Broberg [Brob2, Thm. 2] that $N(U,\mathbb{Q},B) = O_{\varepsilon,X}(B^{9/4+\varepsilon})$ for any $\varepsilon > 0$. This is improved upon by (1.2), but both bounds are rather far from the expectation in (1.1). In the spirit of the previous results it is possible to exploit conic bundle structures.

THEOREM 1.3. *Let $\varepsilon > 0$ and let $X \subset \mathbb{P}(2,1,1,1)$ be a del Pezzo surface of degree* 2*, containing a non-singular conic defined over k. Then*

$$N(U,k,B) = O_{\varepsilon,X}(B^{2+\varepsilon}).$$

The implied constant is ineffective.

One can do better when $k=\mathbb{Q}$ and one assumes that all of the 56 exceptional curves in X are defined over $\mathbb{Q}$. In this case, as announced by Salberger at the conference "Géométrie arithmétique et variétés rationnelles" at Luminy in 2007, one has the sharper bound $N(U,\mathbb{Q},B) = O_{\varepsilon,X}(B^{11/6+\varepsilon})$.

We proceed to indicate the contents of this paper. We shall use the underlying conic bundle structures to prove Theorems 1.1–1.3, closely following the notation and framework developed by Broberg [Brob1]. In Section 2 we recall some basic facts from algebraic number theory and present our main technical result, Theorem 2.2, from which our results on del Pezzo surfaces are deduced in Section 3. This is concerned with counting k-rational points of bounded height on certain "conic bundle torsors", and its proof hinges upon two further results: Theorems 2.3 and 2.4. The first of these involves counting k-rational points of bounded height on non-singular conics defined over k, which needs to be done uniformly with respect to the coefficients of the underlying equation. This is likely to be of independent interest and is proved in Section 4. The second is concerned with a certain average involving binary forms over k and is proved in Section 5.

2. Counting points on conic bundle torsors

2.1. Algebraic number theory. We begin by recalling some basic notation and facts concerning our number field k. Let $d = [k : \mathbb{Q}]$ and let $\mathfrak{o}$ be the ring of integers of k. We denote by Ω the set of places of k. We let s_k denote the number of infinite places of k. For any $\nu \in \Omega$, we let μ be its restriction to $\mathbb{Q}$ and put $d_\nu = [k_\nu : \mathbb{Q}_\mu]$. The absolute value $|\cdot|_\nu$ on k is the one which induces the normal absolute value on $\mathbb{R}$ if $\nu \mid \infty$ and the p-adic absolute value if $\nu \mid p$. The normalised absolute value is

$$\|\cdot\|_\nu = |\cdot|_\nu^{d_\nu}.$$

We denote by $\mathrm{N}_k(\mathfrak{a}) = [\mathfrak{o} : \mathfrak{a}]$ the ideal norm for any fractional ideal $\mathfrak{a}$ of $\mathfrak{o}$. We also have $\mathrm{N}_k(\langle\alpha\rangle) = |N_{k/\mathbb{Q}}(\alpha)|$, where $\langle\alpha\rangle$ is the principal ideal generated by any $\alpha \in k^*$. In this case we write $\mathrm{N}_k(\alpha)$ for short. Recall that for any $x \in k^*$ we have

$$\prod_{\nu\mid\infty} \|x\|_\nu = \mathrm{N}_k(x) \quad \text{and} \quad \prod_{\nu\in\Omega} \|x\|_\nu = 1, \tag{2.1}$$

the second equation being the *product formula.*

There is a well-defined height function $H_k : \mathbb{P}^n(k) \mapsto \mathbb{R}_{\geq 1}$, given by

$$[x_0, \dots, x_n] \mapsto \prod_{\nu\in\Omega} \sup_{0\leq i\leq n} \|x_i\|_\nu = \frac{1}{\mathrm{N}_k(\langle x_0, \dots, x_n\rangle)} \prod_{\nu\mid\infty} \sup_{0\leq i\leq n} \|x_i\|_\nu,$$

where $\langle x_0, \dots, x_n\rangle$ denotes the $\mathfrak{o}$-span of $x_0, \dots, x_n \in k^*$. We define a further distance function $\|\cdot\|_\star : k \to \mathbb{R}_{\geq 0}$ via

$$\|x\|_\star = \sup_{\nu\mid\infty} \|x\|_\nu.$$

For any $x \in k$ it is clear that

$$\mathrm{N}_k(x) \leq \|x\|_\star^{s_k}. \tag{2.2}$$

If $\mathbf{x} = (x_1, \ldots, x_n) \in k^n$, then this distance function extends via

$$\|\mathbf{x}\|_\star = \sup_{1\leq i\leq n} \|x_i\|_\star = \sup_{\substack{1\leq i\leq n\\ \nu\mid\infty}} \|x_i\|_\nu.$$

Over $\mathbb{Q}$, any $x \in \mathbb{P}^n(\mathbb{Q})$ has a representative $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ such that $\gcd(x_0, \ldots, x_n) = 1$, which easily allows one to take precisely one element from each equivalence class. Over k, an analogue arises by first fixing once and for all a set of integral ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ representing classes in the ideal class group. Then any $x \in \mathbb{P}^n(k)$ has a representative in coordinates $\mathbf{x} \in \mathfrak{o}^{n+1}$ such that the $\mathfrak{o}$-span $\langle x_0, \ldots, x_n\rangle$ is one of the ideals $\mathfrak{a}_i$. A useful consequence of Dirichlet's unit theorem is the following standard result (see [Brob1, Prop. 3] for a proof).

LEMMA 2.1. *Every point $x \in \mathbb{P}^n(k)$ has a representative $\mathbf{x} \in \mathfrak{o}^{n+1}$ such that $\langle x_0, \ldots, x_n\rangle = \mathfrak{a}_i$ for some $i \in \{1, \ldots, h\}$, and*

$$\|\mathbf{x}\|_\star \ll_n H_k(x)^{1/s_k}. \tag{2.3}$$

According to our convention the implied constant in (2.3) is allowed to depend on k in addition to n. In particular it is allowed to depend on the set of representative ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ that were fixed above. We may now define the sets

$$Z'_{n+1} = \bigcup_{1\leq i\leq h} \{(x_0, \ldots, x_n) \in \mathfrak{o}^{n+1} : \langle x_0, \ldots, x_n\rangle = \mathfrak{a}_i\},$$
$$Z_{n+1} = \{(x_0, \ldots, x_n) \in Z'_{n+1} : \text{(2.3) holds}\}.$$

Lemma 2.1 implies that associated to any element of $\mathbb{P}^n(k)$ is an element of Z_{n+1}. Note, however, that elements of the latter set do not uniquely determine elements of the former. Nonetheless this is sufficient for our purposes. It follows from Lemma 2.1 that

$$H_k(x)^{1/s_k} \leq \|\mathbf{x}\|_\star \ll H_k(x)^{1/s_k} \tag{2.4}$$

for every $x \in \mathbb{P}^n(k)$ and corresponding element $\mathbf{x} \in Z_{n+1}$.

2.2. Conic bundle torsors. Let $f_{ij} \in k[u, v]$ be binary forms for indices $1 \leq i, j \leq 3$. Let $S_1 \subset \mathbb{A}^1 \times \mathbb{P}^2$ be given by the equation

$$\sum_{i,j=1}^{3} f_{ij}(u, 1)x_i x_j = 0,$$

and let $S_2 \subset \mathbb{A}^1 \times \mathbb{P}^2$ be given by

$$\sum_{i,j=1}^{3} f_{ij}(1,v)x_i x_j = 0.$$

We shall assume that every principal 2×2 minor of the matrix $\mathbf{F} = (f_{ij})$ is a binary form of even degree and, furthermore, that $\Delta(u,v) = \det(\mathbf{F})$ is separable.

Let d_i be the degree of the cofactor of the diagonal element f_{ii} in $\mathbf{F}$ (e.g. d_1 is the degree of the bottom right 2×2 minor). We let $U_1 \subset S_1$ be the open subset given by $u \neq 0$ and we let $U_2 \subset S_2$ be the open subset given by $v \neq 0$. We obtain a conic bundle surface S by glueing U_1 and U_2 via the isomorphism

$$(u; [x_1, x_2, x_3]) \mapsto (1/u; [x_1 u^{-d_1/2}, x_2 u^{-d_2/2}, x_3 u^{-d_3/2}]).$$

The morphisms $S_i \to \mathbb{P}^1$ given by $(u; [x_1, x_2, x_3]) \mapsto [u, 1]$ for $i = 1$ and $(v; [x_1, x_2, x_3]) \mapsto [1, v]$ for $i = 2$, glue together to give a conic fibration

$$\phi : S \to \mathbb{P}^1.$$

Since $\Delta(u,v)$ is separable, it follows from [Sh, §II.6.4, Prop. 1] that S is non-singular. The singular fibres of ϕ correspond to the roots of $\Delta(u,v)$. We define the degree of S to be $(-K_S)^2 = 8 - r$, where r is the number of singular fibres of ϕ.

Consider the locally closed subvariety $\mathscr{T} \subset \mathbb{A}^5$ given by

$$\begin{cases} \sum_{i,j=1}^{3} f_{ij}(u,v)x_i x_j = 0, \\ (u,v) \neq (0,0), \quad (x_1,x_2,x_3) \neq (0,0,0). \end{cases} \tag{2.5}$$

We claim that $\mathscr{T}$ is a torsor over S under $\mathbb{G}_\mathrm{m}^2$. There exists a morphism $\pi : \mathscr{T} \to S$ as follows. Let $(u, v; \mathbf{x}) \in \mathscr{T}$. If $v \neq 0$ then

$$(u/v; [x_1 v^{-d_1/2}, x_2 v^{-d_2/2}, x_3 v^{-d_3/2}]) \in S_1,$$

while if $u \neq 0$ then

$$(v/u; [x_1 u^{-d_1/2}, x_2 u^{-d_2/2}, x_3 u^{-d_3/2}]) \in S_2.$$

It is clear that $\mathbb{G}_\mathrm{m}^2$ acts on $\mathscr{T}$ via

$$(\lambda, \mu) \mapsto (\mu, \mu; \lambda\mu^{d_1/2}, \lambda\mu^{d_2/2}, \lambda\mu^{d_3/2}),$$

and this acts on $\mathbb{A}^5$ in the natural way. This action is free and transitive on the fibres of π, and so $\mathscr{T}$ is indeed a $\mathbb{G}_\mathrm{m}^2$-torsor over S. We shall henceforth refer to varieties of the shape (2.5) as *conic bundle torsors* whenever every principal 2×2 minor of $\mathbf{F}$ is a binary form of even degree and $\Delta(u,v) = \det(\mathbf{F})$ is separable.

2.3. Counting rational points. Let $\mathscr{T}$ be a conic bundle torsor, and let $\mathscr{T}_0 \subset \mathscr{T}$ be the open subset on which $\Delta(u,v) \neq 0$. Let $\mathbf{r} \in (\mathbb{R}_{\geq 1})^{s_k}$ be a vector with components r_ν for $\nu \mid \infty$. Define

$$L(\mathbf{r}) = \{x \in \mathfrak{o} : \|x\|_\nu \leq r_\nu \text{ for } \nu \mid \infty\}.$$

Let

$$\|\mathbf{r}\| = \prod_{\nu \mid \infty} r_\nu,$$

and put $\underline{\mathbf{r}} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ for $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in (\mathbb{R}_{\geq 1})^{s_k}$. For given $\mathbf{x} \in \mathfrak{o}^3$ it will be convenient to write $\mathbf{x} \in L(\underline{\mathbf{r}}) \cap Z_3'$ if $\mathbf{x} \in Z_3'$ and $x_i \in L(\mathbf{r}_i)$ for $1 \leq i \leq 3$. For given $A \geq 1$ and $\underline{\mathbf{r}} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$, with $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in (\mathbb{R}_{\geq 1})^{s_k}$, we define the counting function

$$N_{\mathscr{T}_0}(A, \underline{\mathbf{r}}) = \#\left\{ (u,v) \in Z_2,\, \mathbf{x} \in L(\underline{\mathbf{r}}) \cap Z_3' : \begin{array}{c} (u,v;\mathbf{x}) \in \mathscr{T}_0(k), \\ A \leq H_k([u,v]) < 2A \end{array} \right\}.$$

In Section 3 we shall show that the proof of Theorems 1.1–1.3 can essentially be reduced to special cases of the following general estimate.

THEOREM 2.2. *Let $\varepsilon > 0$ and let $\mathscr{T}$ be a conic bundle torsor of the shape* (2.5), *with* $\deg \Delta(u,v) = n$. *Then*

$$N_{\mathscr{T}_0}(A, \underline{\mathbf{r}}) \ll_{\varepsilon, \mathscr{T}} A^{2+\varepsilon} \left(1 + \left(\frac{\|\mathbf{r}_1\|\,\|\mathbf{r}_2\|\,\|\mathbf{r}_3\|}{A^n} \right)^{1/3} \right).$$

The implied constant is ineffective.

We proceed to prove this theorem subject to some technical results which will be established in due course. The first, which should be of independent interest, concerns counting k-rational points on conics.

For a matrix $\mathbf{M} \in \mathrm{GL}_3(\mathfrak{o})$, let $\Delta(\mathbf{M})$ and $\Delta_0(\mathbf{M})$ be the ideals generated by the determinant of $\mathbf{M}$ and the 2×2 minors of $\mathbf{M}$, respectively. Let τ be the usual divisor function on integral ideals. Then we shall establish the following result in Section 4.

THEOREM 2.3. *Let Q be a non-singular ternary quadratic form with underlying matrix* $\mathbf{M} \in \mathrm{GL}_3(\mathfrak{o})$. *Let $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in (\mathbb{R}_{\geq 1})^{s_k}$ be given. There are*

$$\ll \left(1 + \left(\frac{\|\mathbf{r}_1\|\,\|\mathbf{r}_2\|\,\|\mathbf{r}_3\| \mathrm{N}_k(\Delta_0(\mathbf{M}))^{3/2}}{\mathrm{N}_k(\Delta(\mathbf{M}))} \right)^{1/3} \right) \tau(\Delta(\mathbf{M}))$$

elements $\mathbf{x} \in L(\underline{\mathbf{r}}) \cap Z_3'$ such that $Q(\mathbf{x}) = 0$.

This result generalises to arbitrary number fields a result of Browning and Heath-Brown [BHB, Cor. 2]. It is important to note that the implied constant in this estimate depends at most on the field k, but is uniform in the coefficients of the quadratic form Q.

Theorem 2.3 is a crucial ingredient in our proof of Theorem 2.2. Indeed, if one takes $A = 1$ in the latter result, then one obtains a version of Theorem 2.3 in which the implied constant is allowed to depend on the coefficients of Q. Alternatively, if A is large compared to $\|\mathbf{r}_1\|\,\|\mathbf{r}_2\|\,\|\mathbf{r}_3\|$, then Theorem 2.2 shows that most conics in the family contribute very few points.

The second technical result we require concerns an average involving binary forms. It will be established in Section 5 and is based on Lang's generalisation of the Thue–Siegel–Roth theorem to number fields.

THEOREM 2.4. *Let $\varepsilon > 0$ and let $F(u, v) \in \mathfrak{o}[u, v]$ be a separable form of degree n. Then*

$$\sum_{\substack{(u,v)\in\mathfrak{o}^2 \\ A^{1/s_k}\le\|(u,v)\|_\star<2A^{1/s_k} \\ F(u,v)\neq 0}} \frac{1}{(\mathrm{N}_k(F(u,v)))^{1/3}} \ll_{\varepsilon,F} A^{2-n/3+\varepsilon}. \tag{2.6}$$

The implied constant is ineffective.

We now have everything in place to establish Theorem 2.2, conditionally on the technical results. Let $\mathscr{T}$ be a conic bundle torsor of the shape (2.5). We shall proceed by counting the number of points on the fibres $C_{u,v}$ of $\mathscr{T}$ above $(u, v) \in \mathbb{A}^2$, uniformly in u, v. Given $(u, v) \in \mathbb{A}^2$ such that $\Delta(u, v) \neq 0$, let $\mathbf{M}(u, v)$ be the matrix which produces the ternary quadratic form defining $C_{u,v}$. We have $\Delta(u, v) = \Delta(\mathbf{M}(u, v))$ and we put $\Delta_0(u, v) = \Delta_0(\mathbf{M}(u, v))$. By the trivial estimate for the divisor function we have

$$\tau(\Delta(u, v)) \ll_\varepsilon (\mathrm{N}_k(\Delta(u, v)))^\varepsilon.$$

Likewise, since $\Delta(u, v)$ is separable, the proof of [Brob1, Lemma 7] shows that $\mathrm{N}_k(\Delta_0(u, v)) \ll_{\mathscr{T}} 1$ for $(u, v) \in Z_2$.

For given $(u, v) \in Z_2$ such that $\Delta(u, v) \neq 0$, we put

$$N(u, v, \underline{\mathbf{r}}) = \#\{\mathbf{x} \in L(\underline{\mathbf{r}}) \cap Z_3' \cap C_{u,v}\}.$$

It follows from Theorem 2.3 that

$$N(u, v, \underline{\mathbf{r}}) \ll_{\varepsilon,\mathscr{T}} \left(1 + \frac{R^{1/3}}{\mathrm{N}_k(\Delta(u, v))^{1/3}}\right)(\mathrm{N}_k(\Delta(u, v)))^\varepsilon$$

for any $\varepsilon > 0$, where $R = \|\mathbf{r}_1\|\,\|\mathbf{r}_2\|\,\|\mathbf{r}_3\|$. We easily obtain

$$\begin{aligned} N_{\mathscr{T}_0}(A, \underline{\mathbf{r}}) &\le \sum_{\substack{(u,v)\in Z_2 \\ A\le H_k([u,v])<2A}} N(u, v, \underline{\mathbf{r}}) \\ &\ll_{\varepsilon,\mathscr{T}} A^\varepsilon \sum_{\substack{(u,v)\in Z_2 \\ A\le H_k([u,v])<2A}} \left(1 + \frac{R^{1/3}}{\mathrm{N}_k(\Delta(u, v))^{1/3}}\right). \end{aligned}$$

Finally, recalling (2.4), an application of Theorem 2.4 yields

$$N_{\mathscr{T}_0}(A,\underline{\mathbf{r}}) \ll_{\varepsilon,\mathscr{T}} A^{2+2\varepsilon}\left(1+\frac{R^{1/3}}{A^{n/3}}\right).$$

We complete the proof of Theorem 2.2 upon redefining the choice of ε.

Remark 2.5. Although we need it for Theorem 2.2, it should be noted that Theorems 1.1–1.3 do not strictly require Theorem 2.4 and it is possible to recover effectivity with extra work. Instead one can make use of the fact that there are $O_{\varepsilon,F}(G^{1/n}A)$ points $(u,v)\in Z_2$, with $H_k([u,v])\le A$ and $\mathrm{N}_k(F(u,v))\le G$ (see [Brob1, Lemma 9]). For Theorem 2.2, however, this would only produce the desired contribution when G has order of magnitude A^n. For Theorems 1.1 and 1.2, moreover, using this alternative bound would require us to handle a subset of the fibres in a different manner (see [Brob1, Prop. 7 and Lemma 8]).

3. Counting points on del Pezzo surfaces

3.1. Heights and morphisms. We begin with some general facts about the behaviour of heights under morphisms, as described by Serre [Se, §2]. Let X be a del Pezzo surface of degree $d\in\{2,3,4\}$ over a number field k and let $U\subset X$ be the Zariski open subset obtained by deleting the exceptional curves. For a morphism $g: X\to\mathbb{P}^\ell$ we write $H_g(x)=H_k(g(x))$ for any $x\in X(k)$, where H_k is the height on $\mathbb{P}^\ell(k)$.

Suppose we are given morphisms

$$f_i: X\to\mathbb{P}^1,\qquad i=1,\dots,m.$$

Let f be the morphism

$$f: X\to\mathbb{P}^1\times\cdots\times\mathbb{P}^1=(\mathbb{P}^1)^m$$

given by $(f_1,\dots,f_m)$. Now let $\psi:(\mathbb{P}^1)^m\to\mathbb{P}^{2^m-1}$ be the multilinear Segre embedding, so then $\psi\circ f$ is a morphism. We shall assume that $\psi\circ f$ takes the shape

$$\psi\circ f(x)=[\phi_0(x),\dots,\phi_{2^m-1}(x)]$$

on U, where $\phi_0,\dots,\phi_{2^m-1}$ are homogeneous polynomials of degree e which do not simultaneously vanish on X.

Let $p=([u_1,v_1],\dots,[u_m,v_m])\in(\mathbb{P}^1(k))^m$ and $\psi(p)=[y_0,\dots,y_{2^m-1}]$. Then

$$\sup_{0\le i\le 2^m-1}\|y_i\|_\nu=\prod_{i=1}^m\sup\{\|u_i\|_\nu,\|v_i\|_\nu\}$$

for every $\nu\in\Omega$. Thus we have

$$\sum_{i=1}^m\log H_{f_i}=\log H_{\psi\circ f}.$$

Furthermore, the functoriality of heights yields

$$\log H_{\psi\circ f} = e\cdot \log H_k + O_{f,X}(1).$$

It follows that there is an absolute constant $c_1 = c_1(f,X)$ such that

$$\prod_{1\le i\le m} H_k(f_i(x)) \le c_1^m H_k(x)^e$$

for any $x\in U(k)$. Thus we have

$$H_k(f_i(x)) \le c_1 H_k(x)^{e/m}$$

for at least one $i\in\{1,\dots,m\}$.

In our work it will suffice to estimate the counting function

$$N(U,k,B) = \#\{x\in U(k) : H_k(x)\le B\}.$$

When $d=2$ and X is embedded in $\mathbb{P}(2,1,1,1)$ via an equation of the form (1.3), the height is taken to be $H_k([x_1,x_2,x_3])$ on $\mathbb{P}^2(k)$. When $d\in\{3,4\}$, it is taken to be the height on $\mathbb{P}^d(k)$ that arises through the anticanonical embedding of X in $\mathbb{P}^d$. We may now conclude as follows.

Lemma 3.1. *There exists a constant $c_1 = c_1(f,X) > 0$ such that*

$$N(U,k,B) \le \sum_{i=1}^m n_i(B),$$

where

$$n_i(B) = \#\{x\in U(k) : H_k(x)\le B \text{ and } H_k(f_i(x)) \le c_1 B^{e/m}\}.$$

For the remainder of Section 3 we shall allow all of the implied constants to depend implicitly on the number field k, the del Pezzo surface X and the small parameter $\varepsilon>0$ appearing in Theorems 1.1–1.3. Furthermore, in the light of Theorem 2.2, we shall allow the implied constants to be ineffective.

3.2. Proof of Theorem 1.1. After a change of variables we may assume that X is given by

$$x_0x_1 - x_2x_3 = 0,$$
$$Q(x_0,x_1,x_2,x_3) + ax_4^2 = 0,$$

for a quadratic form $Q\in\mathfrak{o}[x_0,x_1,x_2,x_3]$ and a non-zero element $a\in\mathfrak{o}$. Let $U\subset X$ be the subset obtained by deleting the 16 lines from X.

We will consider two conic fibrations $f_1,f_2 : U\to\mathbb{P}^1$, given by

$$f_1(x) = \begin{cases} [x_0,x_2] & \text{if } (x_0,x_2)\ne(0,0),\\ [x_3,x_1] & \text{if } (x_3,x_1)\ne(0,0),\end{cases}$$

$$f_2(x) = \begin{cases} [x_0,x_3] & \text{if } (x_0,x_3)\ne(0,0),\\ [x_2,x_1] & \text{if } (x_2,x_1)\ne(0,0).\end{cases}$$

Note that the definitions agree where the open sets intersect, so that this is a well-defined morphism. Moreover, the open sets cover X since there are no points on X with $x_0 = x_1 = x_2 = x_3 = 0$. Define $f : X \to \mathbb{P}^1 \times \mathbb{P}^1$ to be the morphism given by $x \mapsto (f_1(x), f_2(x))$. With the notation of Section 3.1, we confirm that

$$\psi \circ f(x) = [x_0, x_3, x_2, x_1]$$

for all $x \in U$. Thus we can take $e = 1$ and $m = 2$ in Lemma 3.1. Our task is then to show that $n_i(B) \ll B^{1+\varepsilon}$ for $i = 1, 2$. Without loss of generality we shall show this for $i = 1$, with

$$n_1(B) = \#\{x \in U(k) : H_k(x) \le B \text{ and } H_k(f_1(x)) \le c_1 B^{1/2}\}$$

and an appropriate constant $c_1 = c_1(X) > 0$.

For each $p \in \mathbb{P}^1(k)$ with $H_k(p) \le c_1 B^{1/2}$ we can choose a representative $(u, v) \in Z_2$, by Lemma 2.1. Let $n_1(B; u, v)$ be the number of points in $f_1^{-1}([u, v]) \cap U(k)$ with height at most B. Then

$$n_1(B) \le \sum_{\substack{(u,v)\in Z_2 \\ H_k([u,v]) \le c_1 B^{1/2}}} n_1(B; u, v).$$

Given $\mathbf{A} \in (\mathbb{R}_{\ge 1})^{s_k}$, we split the right hand side into dyadic intervals, writing

$$n_1(\mathbf{A}, B) = \sum_{\substack{(u,v)\in Z_2 \\ A_\nu \le \sup\{\|u\|_\nu, \|v\|_\nu\} < 2A_\nu}} n_1(B; u, v). \tag{3.1}$$

Let $A = \|\mathbf{A}\|$. It then follows from (2.4) that any point (u, v) in the sum satisfies $A \ll H_k([u, v]) \ll A$. We are clearly only interested in $A \ll B^{1/2}$.

Now $(ux, yv, xv, yu, z) \in f_1^{-1}([u, v])$ if and only if

$$Q(ux, yv, xv, yu) + az^2 = 0.$$

This is a conic bundle torsor $\mathscr{T}$, as in (2.5), with $\deg \Delta(u, v) = 4$. On multiplying (x, y, z) by an appropriate scalar, Lemma 2.1 ensures that we will have $\langle ux, yv, xv, yu, z\rangle = \mathfrak{a}_i$ for some $i \in \{1, \dots, h\}$, and

$$\|(ux, yv, xv, yu, z)\|_\star \le c_2 H_k([ux, yv, xv, yu, z])^{1/s_k} \tag{3.2}$$

for some constant $c_2 > 0$. We must count the number of such points which lie on $\mathscr{T}$. Moreover, it suffices to work on the open set $\mathscr{T}_0$ since we wish to avoid points lying on lines in X.

Our goal is to apply Theorem 2.2. A triple (x, y, z) satisfying the above restrictions does not necessarily have $x, y \in \mathfrak{o}$. On multiplying (x, y, z) by a suitable scalar and adjusting c_2 appropriately in (3.2), however, we can proceed under the assumption that $(x, y, z) \in Z_3'$. Thus $n_1(B; u, v)$ is at most the number of elements $(x, y, z) \in Z_3'$ with $(ux, yv, xv, yu, z) \in U(k)$

and

$$\sup\{\|ux\|_\nu, \|yv\|_\nu, \|xv\|_\nu, \|yu\|_\nu, \|z\|_\nu\} \le c_2 B^{1/s_k}. \tag{3.3}$$

We redefine $n_1(B; u, v)$ to be this cardinality.

By (3.1) and (3.3), for every point counted by $n_1(B; u, v)$, there is an absolute constant $c_3 > 0$ such that

$$\|x\|_\nu, \|y\|_\nu \le \frac{c_2 B^{1/s_k}}{\sup\{\|u\|_\nu, \|v\|_\nu\}} \le \frac{c_3 B^{1/s_k}}{A_\nu}$$

for every $\nu \mid \infty$. Moreover, we have $\|z\|_\nu \le c_2 B^{1/s_k}$ for each $\nu \mid \infty$. Hence we can apply Theorem 2.2 with $r_{1,\nu} = r_{2,\nu} = c_3 B^{1/s_k}/A_\nu$ and $r_{3,\nu} = c_2 B^{1/s_k}$ and $n = 4$. This yields the estimate

$$n_1(\mathbf{A}, B) \ll A^{2+\varepsilon} + BA^\varepsilon.$$

Summing over dyadic values of A_ν, with $A = \|\mathbf{A}\| \ll B^{1/2}$, therefore leads to the desired bound $n_1(B) \ll B^{1+\varepsilon}$.

3.3. Proof of Theorem 1.2. The argument in this section and the next is similar to the proof of Theorem 1.1, and so we shall allow ourselves to be more concise. Suppose X is a del Pezzo surface of degree 3 over k, with three coplanar lines defined over k. After a possible change of variables we may assume that $X \subset \mathbb{P}^3$ is given by

$$L_1 L_2 L_3 = x_0 Q,$$

where each $L_i \in \mathfrak{o}[x_1, x_2, x_3]$ is a linear form and $Q \in \mathfrak{o}[x_0, \dots, x_3]$ is a quadratic form. Following Broberg [Brob1], we define three conic bundle morphisms $f_1, f_2, f_3 : X \to \mathbb{P}^1$ via

$$f_1(x) = \begin{cases} [x_0, L_1] & \text{if } (x_0, L_1) \ne (0,0), \\ [L_2 L_3, Q] & \text{if } (L_2 L_3, Q) \ne (0,0), \end{cases}$$
$$f_2(x) = \begin{cases} [x_0, L_2] & \text{if } (x_0, L_2) \ne (0,0), \\ [L_1 L_3, Q] & \text{if } (L_1 L_3, Q) \ne (0,0), \end{cases}$$
$$f_3(x) = \begin{cases} [x_0, L_3] & \text{if } (x_0, L_3) \ne (0,0), \\ [L_1 L_2, Q] & \text{if } (L_1 L_2, Q) \ne (0,0). \end{cases}$$

These morphisms are all well-defined, since X is non-singular. In the notation of Section 3.1 we have

$$\psi \circ f(x) = [x_0^2, x_0 L_3, x_0 L_2, x_0 L_1, L_2 L_3, L_1 L_3, L_1 L_2, Q]$$

for all $x \in U$, so we take $e = 2$ and $m = 3$ in Lemma 3.1. We need to show that $n_i(B) \ll B^{4/3+\varepsilon}$ for $1 \le i \le 3$. Without loss of generality we shall do so for $i = 1$, with

$$n_1(B) = \#\{x \in U(k) : H_k(x) \le B \text{ and } H_k(f_1(x)) \le c_1 B^{2/3}\}.$$

After a change of variables we may assume that $L_1 = x_1$ and $L_2 = x_2$. We look at the fibres of $f_1 : X \to \mathbb{P}^1$. The fibre above a point $[u, v]$ is the set of points $[uy_1, vy_1, y_2, y_3]$ where $y_1 \neq 0$ and

$$vy_2 L_3(vy_1, y_2, y_3) = uQ(uy_1, vy_1, y_2, y_3). \tag{3.4}$$

This is a conic bundle torsor $\mathscr{T}$, as in (2.5), with $\deg \Delta(u, v) = 5$. Let $n_1(B; u, v)$ be the number of points in $f_1^{-1}([u, v]) \cap U(k)$ with height at most B. Then

$$n_1(B) \leq \sum_{\substack{(u,v)\in Z_2 \\ H_k([u,v]) \leq c_1 B^{2/3}}} n_1(B; u, v).$$

As in (3.1), we split the right hand side into dyadic intervals $n_1(\mathbf{A}, B)$ for suitable $\mathbf{A} \in (\mathbb{R}_{\geq 1})^{s_k}$ such that $A = \|\mathbf{A}\| \ll B^{2/3}$.

Using an identical argument to that leading up to (3.3), we can redefine $n_1(B; u, v)$ to be the number of points $(y_1, y_2, y_3) \in Z_3'$ such that (3.4) holds and

$$\sup\{\|uy_1\|_\nu, \|vy_1\|_\nu, \|y_2\|_\nu, \|y_3\|_\nu\} \leq c_2 B^{1/s_k}$$

for some constant $c_2 > 0$. Similarly to before, we see there is a constant $c_3 > 0$ such that Theorem 2.2 can be applied with $r_{1,\nu} = c_3 B^{1/s_k}/A_\nu$ and $r_{2,\nu} = r_{3,\nu} = c_2 B^{1/s_k}$ and $n = 5$. This shows that

$$n_1(\mathbf{A}, B) \ll A^{2+\varepsilon} + BA^\varepsilon,$$

and we obtain the desired conclusion by summing over dyadic values of A_ν, with $A \ll B^{2/3}$.

3.4. Proof of Theorem 1.3. We suppose that $X \subset \mathbb{P}(2, 1, 1, 1)$ is a del Pezzo surface of degree 2, as in (1.3), which contains a non-singular conic C defined over k. Following an argument suggested to us by Professor Per Salberger, we will show that it may be given by an equation of the form

$$t^2 = q_1 q_2 + q_3^2,$$

where $q_1, q_2, q_3 \in k[x_1, x_2, x_3]$ are quadratic forms such that $q_1 q_2 + q_3^2$ is a non-singular quartic form.

Any non-singular conic C contained in X is an irreducible curve satisfying $(C, -K_X) = 2$. Hence C is mapped isomorphically onto a conic in $\mathbb{P}^2$, under the double cover map $[t, x_1, x_2, x_3] \mapsto [x_1, x_2, x_3]$. Let us suppose that this conic in $\mathbb{P}^2$ is given by the equation $q_1 = 0$, for a non-singular ternary quadratic form q_1 defined over k. We must have one more relation between t and the six quadratic monomials in x_1, x_2, x_3. This gives a further equation $t - q_3 = 0$ on C, for a quadratic form q_3 defined over k. Substituting this into the equation for X we see that $f - q_3^2$ vanishes on the conic $q_1 = 0$ in $\mathbb{P}^2$. Hence $f = q_1 q_2 + q_3^2$ for a further quadratic form q_2 defined over k, such that $q_1 q_2 + q_3^2$ is non-singular, which thereby establishes the claim.

We may henceforth assume that q_1, q_2, q_3 are all defined over $\mathfrak{o}$ on absorbing a suitable constant into t. The 56 exceptional curves are the preimages of the 28 bitangents to the quartic plane curve $q_1q_2 + q_3^2 = 0$. We let $U \subset X$ be the open subset which avoids all of these. As indicated previously, we take our height function $H_k : \mathbb{P}(2,1,1,1)(k) \mapsto \mathbb{R}_{\geq 1}$ to be

$$[t, \mathbf{x}] \mapsto \prod_{\nu \in \Omega} \sup\{\|x_1\|_\nu, \|x_2\|_\nu, \|x_3\|_\nu\}.$$

We define the morphisms $f_1, f_2 : X \to \mathbb{P}^1$ via

$$f_1([t,\mathbf{x}]) = \begin{cases} [t - q_3, q_1] & \text{if } (t - q_3, q_1) \neq (0,0), \\ [q_2, t + q_3] & \text{if } (q_2, t + q_3) \neq (0,0), \end{cases}$$
$$f_2([t,\mathbf{x}]) = \begin{cases} [t - q_3, q_2] & \text{if } (t - q_3, q_2) \neq (0,0), \\ [q_1, t + q_3] & \text{if } (q_1, t + q_3) \neq (0,0). \end{cases}$$

These morphisms are well-defined since $q_1q_2 + q_3^2$ is non-singular. In this setting we have

$$\psi \circ f([t,\mathbf{x}]) = [t - q_3, q_2, q_1, t + q_3]$$

for all $[t, \mathbf{x}] \in U$, so we take $e = 2$ and $m = 2$ in Lemma 3.1. We wish to show that $n_i(B) \ll B^{2+\varepsilon}$ for $i = 1, 2$. Without loss of generality we shall show this for $n_1(B)$, with

$$n_1(B) = \#\{x \in U(k) : H_k(x) \leq B \text{ and } H_k(f_1(x)) \leq c_1 B\}.$$

We look at the fibres $f_1 : X \to \mathbb{P}^1$ in U. Defining $n_1(B; u, v)$ to be the number of rational points in $f_1^{-1}([u, v]) \cap U$ with height at most B, we have

$$n_1(B) \leq \sum_{\substack{(u,v) \in Z_2 \\ H_k([u,v]) \leq c_1 B}} n_1(B; u, v).$$

We shall consider the contribution $n_1(\mathbf{A}, B)$ from dyadic intervals, as in (3.1), for $\mathbf{A} \in (\mathbb{R}_{\geq 1})^{s_k}$ such that $A = \|\mathbf{A}\| \ll B$.

Suppose $[t, \mathbf{x}] \in f_1^{-1}([u, v]) \cap U$ for $uv \neq 0$. Then the point $(u, v; \mathbf{x})$ satisfies $uv \neq 0$ and $\mathbf{x} \neq (0, 0, 0)$, and is constrained to lie on the variety in $\mathbb{A}^5$ given by the equation

$$q_1(\mathbf{x})u^2 + 2q_3(\mathbf{x})uv - q_2(\mathbf{x})v^2 = 0.$$

This is a conic bundle torsor of the form (2.5), with $\deg \Delta(u, v) = 6$. Thus Theorem 2.2 can be applied directly with $r_{i,\nu} = B^{1/s_k}$ for $i = 1, 2, 3$ and $n = 6$, giving

$$n_1(A, B) \ll A^{2+\varepsilon} + BA^{\varepsilon}.$$

Summing for dyadic $A \ll B$ shows that $n_1(B) \ll B^{2+\varepsilon}$, as claimed.

4. Ternary forms. In this section we establish Theorem 2.3. The structure of the proof is similar to [Brob1, Thm. 6] (which in turn follows the proof of [HB1, Thm. 2]). The main idea is to cover the solutions to $Q(\mathbf{x}) = 0$ by a relatively small number of lattices, each of which has a large determinant. This is done in Section 4.2, after first recalling some basic facts about lattices over number fields in Section 4.1. Then, for given $\underline{\mathbf{r}} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$, we obtain in Section 4.3 a uniform estimate for the number of points $x \in \mathbb{P}^2(k)$ with representative $\mathbf{x} = (x_1, x_2, x_3) \in Z_3'$, such that $x_i \in L(\mathbf{r}_i)$ and $Q(\mathbf{x}) = 0$. This is then used to deduce Theorem 2.3 by rescaling the lattices appropriately. Throughout Section 4 we return to our convention that all of the implied constants are allowed to depend at most upon the number field k. Furthermore, all of the implied constants in this section are effective.

4.1. Lattices. We say that an $\mathfrak{o}$-module Λ in k^n is an $\mathfrak{o}$-*lattice in* k^n if it is finitely generated and contains a basis of k^n over k. We can define its determinant $\det \Lambda$ to be the index $[\mathfrak{o}^n : \Lambda]$ as an additive subgroup. If Λ is an $\mathfrak{o}$-lattice in k^n and $\nu \in \Omega$ is a finite place, then $\Lambda_\nu = \Lambda \otimes_{\mathfrak{o}} \mathfrak{o}_\nu$ is a free $\mathfrak{o}_\nu$-module in k_ν^n such that Λ_ν contains a basis for k_ν^n over k_ν, where $\mathfrak{o}_\nu$ is the ring of integers of k_ν. We say that such an $\mathfrak{o}_\nu$-module is an $\mathfrak{o}_\nu$-*lattice in* k_ν^n. The following results are standard (see [Brob1, Thm. 4] and [Brob1, Prop. 5], respectively).

Lemma 4.1. *For each finite place $\nu \in \Omega$, let L_ν be an $\mathfrak{o}_\nu$-lattice in k_ν^n such that $L_\nu = \mathfrak{o}_\nu^n$ for almost all ν. If $\Lambda = \bigcap_{\nu \nmid \infty} L_\nu \cap k^n$, then Λ is the unique $\mathfrak{o}$-lattice in k^n such that $\Lambda_\nu = L_\nu$ for all $\nu \in \Omega$.*

Lemma 4.2. *If $L \subset \Gamma$ are $\mathfrak{o}$-lattices in k^n, then there is an element $a \in k^*$ such that $\Gamma \subset aL$ and $[\Gamma : L] \ll \mathrm{N}_k(a)$.*

We define measures for the places $\nu \in \Omega$ as follows. If $\nu \mid \infty$ and $k_\nu = \mathbb{R}$, then $\mathrm{d}\mu_\nu$ is the ordinary Lebesgue measure. If $\nu \mid \infty$ and $k_\nu = \mathbb{C}$, then $\mathrm{d}\mu_\nu$ is the Lebesgue measure multiplied by 2. If $\nu \nmid \infty$, then $\mathrm{d}\mu_\nu$ is the usual ν-adic measure normalised so that $\mu_\nu(\mathfrak{o}_\nu) = \|\mathfrak{D}_\nu\|_\nu$, where $\mathfrak{D}_\nu$ is the local different of k at ν.

For each $\nu \mid \infty$, let S_ν be a non-empty, open, convex, symmetric, bounded subset of k_ν^n. For an $\mathfrak{o}$-lattice Λ in k^n, we shall identify Λ with its image in $S = \prod_{\nu \mid \infty} S_\nu$, under the diagonal embedding. We define the *ith successive minimum of Λ with respect to S* to be

$$\lambda_i = \inf\{\lambda \in \mathbb{R}_{>0} : \Lambda \cap \lambda S \text{ contains } i \text{ linearly independent vectors}\}.$$

The following result is an analogue of *Minkowski's second theorem* in the adèles due to Bombieri and Vaaler [BV] (see the corollary to [Brob1, Thm. 5] for the present formulation).

LEMMA 4.3. *If* $\lambda_1 \leq \cdots \leq \lambda_n$ *are the successive minima of* Λ *with respect to* S, *then*

$$(\lambda_1 \cdots \lambda_n)^d \prod_{\nu \mid \infty} \mathrm{vol}(S_\nu) \ll_n [\mathfrak{o}^n : \Lambda],$$

the volume $\mathrm{vol}(S_\nu)$ *being taken with respect to* $\mathrm{d}\mu_\nu$.

4.2. Points on Fermat curves. In this section we shall prove a generalisation of [Brow1, Lemma 4.9] and [BD, Thm. 3] to number fields. In fact the proof of these results contains an error and we shall take the opportunity to correct this here. Let I_k be the set of integral ideals of $\mathfrak{o}$. For each integer $t \geq 1$, define the multiplicative function on integral ideals $\delta_t : I_k \to \mathbb{Z}_{>0}$, via

$$\delta_t(\mathfrak{p}^r) = r + t - 1 \tag{4.1}$$

for each prime ideal $\mathfrak{p}$. Note that $\delta_2 = \tau$ is the usual divisor function on integral ideals.

LEMMA 4.4. *Consider the equation*

$$F(\mathbf{x}) = a_1 x_1^t + a_2 x_2^t + a_3 x_3^t = 0 \tag{4.2}$$

for $a_i \in \mathfrak{o}$ *and* $t \in \mathbb{Z}_{\geq 2}$. *Let* $\Delta(F)$ *be the principal ideal* $\langle a_1 a_2 a_3 \rangle$, *and let* $\Delta_0(F)$ *be the ideal* $\langle a_1 a_2, a_2 a_3, a_3 a_1 \rangle$. *Suppose* $\mathbf{x} \in \mathfrak{o}^3$ *is a solution of* (4.2). *Then* $\mathbf{x}$ *lies in one of at most* J $\mathfrak{o}$*-lattices* $\Gamma_1, \ldots, \Gamma_J \subset \mathfrak{o}^3$, *such that*

(i) $J \leq t^{3d} \delta_t(\Delta(F))$;
(ii) *for each* $j \leq J$ *we have* $\dim \Gamma_j = 3$ *and*

$$\det \Gamma_j \geq \frac{t^{-2d} \mathrm{N}_k(\Delta(F))^{2/t}}{\mathrm{N}_k(\Delta_0(F))^{3/t}}.$$

REMARK 4.5. When $k = \mathbb{Q}$, [Brow1, Lemma 4.9] and [BD, Thm. 3] record a version of this result with the factor $t^{\omega(a_1 a_2 a_3)}$ instead of our $\delta_t(\Delta(F)) = \delta_t(a_1 a_2 a_3)$, where $\omega(n)$ is the number of distinct prime divisors of an integer n (note that $\delta_t(p^r) = r + t - 1 \geq t = t^{\omega(p^r)}$ for any prime p and any $r \in \mathbb{Z}_{>0}$). However, there is an error in the proof of these results which invalidates this bound. In addition to providing a generalisation to arbitrary number fields, Lemma 4.4 corrects this error. Moreover, one easily shows that nothing has been lost on average, since

$$\sum_{\substack{\mathfrak{a} \subset \mathfrak{o} \\ \mathrm{N}_k(\mathfrak{a}) \leq B}} \delta_t(\mathfrak{a}) \ll B (\log B)^{t-1}$$

for any $t \geq 2$.

Proof of Lemma 4.4. Suppose $\mathfrak{p} \mid \Delta(F)$ is a prime ideal, let $\mathfrak{o}_\mathfrak{p}$ be the localisation of $\mathfrak{o}$ at $\mathfrak{p}$ and put $\mathfrak{q} = \mathfrak{p}\mathfrak{o}_\mathfrak{p}$. Suppose that $q = \mathrm{N}_k(\mathfrak{p}) = p^l$ for some rational prime p, so that $\mathfrak{o}_\mathfrak{p}/\mathfrak{q} \cong \mathbb{F}_q$. Let ν be the place associated to $\mathfrak{p}$ and suppose that π is a uniformiser of $\mathfrak{o}_\mathfrak{p} = \mathfrak{o}_\nu$. Finally, put $\gamma = 2\operatorname{ord}_\nu(t)$, and note that $\mathrm{N}_k(\mathfrak{p})^\gamma = q^{2\operatorname{ord}_\nu(t)} \mid t^{2d}$ if $\mathfrak{p}^{\operatorname{ord}_\nu(t)} \mid t$.

We suppose that

$$F(\mathbf{x}) = \epsilon_1 \pi^{\alpha_1} x_1^t + \epsilon_2 \pi^{\alpha_2} x_2^t + \epsilon_3 \pi^{\alpha_3} x_3^t$$

for units ϵ_i in $\mathfrak{o}_\mathfrak{p}$ and $\alpha_i \in \mathbb{Z}_{\geq 0}$ such that $\alpha_1 \leq \alpha_2 \leq \alpha_3$. Let a_ν and b_ν be the non-negative integers defined by $\|\Delta(F)\|_\nu = \|\pi\|_\nu^{a_\nu}$ and $\|\Delta_0(F)\|_\nu = \|\pi\|_\nu^{b_\nu}$. Then $a_\nu = \alpha_1 + \alpha_2 + \alpha_3$ and $b_\nu = \alpha_1 + \alpha_2$. Hence

$$\mathrm{N}_k(\mathfrak{p})^{(2\alpha_3 - \alpha_1 - \alpha_2)/t} = \mathrm{N}_k(\mathfrak{p})^{(2a_\nu - 3b_\nu)/t}. \tag{4.3}$$

Suppose that $\mathbf{x} \in \mathfrak{o}_\mathfrak{p}^3$, with $Q(\mathbf{x}) = 0$. We will show that there exist $\mathfrak{o}_\nu$-lattices $M_1, \dots, M_K \subset \mathfrak{o}_\mathfrak{p}^3$ of dimension 3, such that $\mathbf{x}$ belongs to M_i for some index $i \in \{1, \dots, K\}$, with

$$K \leq \begin{cases} \alpha_3 - 1 + t & \text{if } \gamma = 0, \\ (\alpha_3 - 1 + t)\mathrm{N}_k(\mathfrak{p})^{\gamma+1} & \text{if } \gamma > 0, \end{cases}$$

and

$$\det M_i \geq \mathrm{N}_k(\mathfrak{p})^{(2\alpha_3 - \alpha_1 - \alpha_2)/t - \gamma}$$

for $1 \leq i \leq K$. Using the Chinese remainder theorem, we may then deduce the result by taking the product over all prime ideals such that $\mathfrak{p} \mid \Delta(F)$ and recalling (4.3).

Suppose that $x_i = \pi^{\xi_i} u_i$ for $i = 1, 2$, with u_1, u_2 units in $\mathfrak{o}_\mathfrak{p}$. Then

$$\epsilon_1 u_1^t \pi^{\alpha_1 + t\xi_1} + \epsilon_2 u_2^t \pi^{\alpha_2 + t\xi_2} \equiv 0 \pmod{\mathfrak{q}^{\alpha_3}}.$$

We split into cases as in the proof of [Brow1, Lemma 4.9]. The oversight in that proof was that the contributions from the different cases were not added up correctly at the end, and this turns out to be fairly delicate. The $\mathbf{x}$ in which we are interested satisfy

(I) $\alpha_3 \leq \min_{i=1,2}\{\alpha_i + t\xi_i\}$, or
(II) $\alpha_3 > \max_{i=1,2}\{\alpha_i + t\xi_i\}$.

Note that it is impossible for α_3 to be between the two.

Let L_1 be the lattice x

$$L_1 = \{\mathbf{x} \in \mathfrak{o}_\mathfrak{p}^3 : x_i \in \mathfrak{q}^{\max\{0, \lceil (\alpha_3 - \alpha_i - \gamma)/t \rceil\}} \text{ for } i = 1, 2\}. \tag{4.4}$$

The determinant of L_1 is at least

$$\mathrm{N}_k(\mathfrak{p})^{\max\{0, \lceil (\alpha_3 - \alpha_2 - \gamma)/t \rceil\} + \max\{0, \lceil (\alpha_3 - \alpha_1 - \gamma)/t \rceil\}} \geq \mathrm{N}_k(\mathfrak{p})^{(2\alpha_3 - \alpha_1 - \alpha_2)/t - \gamma},$$

since $t \geq 2$. Any $\mathbf{x}$ from case (I) must lie in L_1, since $\gamma \geq 0$. Hence the points in case (I) can be covered by one lattice of the required determinant.

For the points from case (II) we have $\alpha_1 + t\xi_1 = \alpha_2 + t\xi_2 = \eta$, say. Note that there are $\lfloor (\alpha_3 - \alpha_2 - 1)/t \rfloor + 1$ possibilities for η. If $\alpha_3 - \eta \leq \gamma$, then it is easy to see that $\mathbf{x} \in L_1$, and so we are done.

Alternatively, we suppose that $\alpha_3 - \eta > \gamma$ and

$$(u_1/u_2)^t \equiv -\epsilon_2/\epsilon_1 \pmod{\mathfrak{q}^{\alpha_3-\eta}}.$$

Now, $y^t \equiv \epsilon \pmod{\mathfrak{q}}$ has at most $\gcd(t, q-1) \le t$ roots, since $\mathfrak{o}_\mathfrak{p}/\mathfrak{q} \cong \mathbb{F}_q$. Hensel's lemma tells us that the congruence $y^t \equiv \epsilon \pmod{\mathfrak{q}^{\alpha_3-\eta}}$ has the same number of solutions as the congruence $y^t \equiv \epsilon \pmod{\mathfrak{q}^\gamma}$, since $\alpha_3 - \eta > \gamma$. The total number of solutions is therefore bounded above by $t\mathrm{N}_k(\mathfrak{p})^\gamma$. It follows that there exist $r_1, \dots, r_H \in \mathfrak{o}_\mathfrak{p}/\mathfrak{q}^{\alpha_3-\eta}$, where $H \le t\mathrm{N}_k(\mathfrak{p})^\gamma$, such that

$$u_1 \equiv r_i u_2 \pmod{\mathfrak{q}^{\alpha_3-\eta}}$$

for some $i \in \{1, \dots, H\}$. Every solution $\mathbf{x} \in \mathfrak{o}_\mathfrak{p}^3$ which satisfies this congruence lies in the lattice defined by the conditions

$$x_i = \pi^{\xi_i} x_i', \qquad x_1' \equiv r_i x_2' \pmod{\mathfrak{q}^{\alpha_3-\eta}}, \tag{4.5}$$

for $x_i' \in \mathfrak{o}_\mathfrak{p}$. This has determinant

$$\mathrm{N}_k(\mathfrak{p})^{\alpha_3+\xi_1+\xi_2-\eta} \ge \mathrm{N}_k(\mathfrak{p})^{(2\alpha_3-\alpha_1-\alpha_2)/t}$$

in $\mathfrak{o}_\mathfrak{p}^3$, which is satisfactory.

Now we count up the total number of lattices. First suppose that $\gamma = 0$ and $\alpha_3 - \alpha_2 \equiv 1 \pmod{t}$. Then for each ξ_i arising in case (II) we have

$$\xi_i \le \left\lfloor \frac{\alpha_3 - \alpha_i - 1}{t} \right\rfloor = \frac{\alpha_3 - \alpha_i - 1}{t}, \tag{4.6}$$

since then $\alpha_1 \equiv \alpha_2 \pmod{t}$. In the boundary case, we have $\eta = \alpha_3 - 1$. But then, if it arises, this gives us a lattice of the form (4.5) with the exponent of $\mathfrak{q}$ being 1. Thus from (4.4) and (4.6), we see that L_1 is a subset of these lattices, so we need not include it in our count. The total number of lattices is therefore found to be at most

$$\begin{aligned} t\left(\left\lfloor \frac{\alpha_3 - \alpha_2 - 1}{t} \right\rfloor + 1\right) &\le t\left(\frac{\alpha_3 - 1}{t} + 1\right) \\ &= \alpha_3 - 1 + t, \end{aligned}$$

which is satisfactory.

Next suppose that $\gamma = 0$ and $\alpha_3 - \alpha_2 \not\equiv 1 \pmod{t}$. Either $\alpha_3 - \alpha_2 = 0$, and hence the second case cannot happen at all (so we need one lattice in total), or $\alpha_3 - \alpha_2 \ge 2$. But then, when we add L_1 to the count, the total number of lattices is at most

$$t\left(\left\lfloor \frac{\alpha_3 - \alpha_2 - 1}{t} \right\rfloor + 1\right) + 1 \le t\left(\frac{\alpha_3 - \alpha_2 - 2}{t} + 1\right) + 1 \le \alpha_3 - 1 + t,$$

which is also satisfactory.

Finally suppose that $\gamma > 0$. In this case the total number of lattices is at most

$$t\mathrm{N}_k(\mathfrak{p})^\gamma\left(\left\lfloor\frac{\alpha_3-\alpha_2-1}{t}\right\rfloor+1\right)+1 \le t\mathrm{N}_k(\mathfrak{p})^\gamma\left(\frac{\alpha_3-1}{t}+1\right)+1$$
$$= \mathrm{N}_k(\mathfrak{p})^\gamma(\alpha_3-1+t)+1$$
$$\le \mathrm{N}_k(\mathfrak{p})^{\gamma+1}(\alpha_3-1+t).$$

This too is satisfactory and so completes the proof of the lemma. ∎

We now turn to the setting of Theorem 2.3, working over each $\mathfrak{o}_\mathfrak{p}$ separately as in the proof of the last lemma. Exactly as in [Brob1, Lemma 4(b)], after diagonalisation of the quadratic form Q it suffices to analyse equations of the shape (4.2) with $t = 2$. We obtain the following result.

Corollary 4.6. *Let Q, $\Delta(\mathbf{M})$, $\Delta_0(\mathbf{M})$ be as in Theorem* 2.3. *Suppose that $\mathbf{x} \in \mathfrak{o}^3$ is a solution of $Q(\mathbf{x}) = 0$. Then $\mathbf{x}$ lies in one of at most J $\mathfrak{o}$-lattices $\Gamma_1, \dots, \Gamma_J \subset \mathfrak{o}^3$ such that*

(i) $J \ll \tau(\Delta(\mathbf{M}))$;
(ii) *for each $j \le J$ we have* $\dim \Gamma_j = 3$ *and*

$$\det \Gamma_j \gg \frac{\mathrm{N}_k(\Delta(\mathbf{M}))}{\mathrm{N}_k(\Delta_0(\mathbf{M}))^{3/2}}.$$

4.3. A uniform bound for rational points on conics. We now state and prove our generalisation of [BHB, Thm. 6] to number fields.

Theorem 4.7. *Let Q be a non-singular ternary quadratic form and suppose that we are given $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \in (\mathbb{R}_{\ge 1})^{s_k}$. Let $R = \|\mathbf{r}_1\|\,\|\mathbf{r}_2\|\,\|\mathbf{r}_3\|$ and let*

$$N(Q,\underline{\mathbf{r}}) = \#\{x = [\mathbf{x}] \in \mathbb{P}^2(k) : Q(\mathbf{x}) = 0 \text{ and } x \in L(\underline{\mathbf{r}}) \cap Z_3'\}.$$

Then $N(Q,\underline{\mathbf{r}}) \ll R^{1/3}$.

Adopting the notation from Section 1, and applying Lemma 2.1, we obtain the following immediate consequence.

Corollary 4.8. *Let $C \subset \mathbb{P}^2$ be an irreducible conic defined over a number field k. Then $N(C,k,B) = O(B)$.*

The proof of Theorem 2.3 follows on combining Corollary 4.6 with Theorem 4.7 exactly as in the proof of [Brob1, Thm. 6]. The argument is essentially a repetition of the final stages of the proof of Theorem 4.7, working instead with one of the lattices Γ_j.

Proof of Theorem 4.7. Our argument is a straightforward generalisation of [BHB, Thm. 6] to number fields. We may suppose that

$$Q(\mathbf{x}) = \sum_{1\le i\le j\le 3} a_{ij}x_ix_j,$$

with $(a_{11}, \dots, a_{33}) \in Z_6$. Let $\mathbf{M} \in \mathrm{GL}_3(\mathfrak{o})$ be the underlying matrix.

We begin by choosing integral prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_r$, with

$$cR^{1/3} \le \mathrm{N}_k(\mathfrak{p}_1) < \cdots < \mathrm{N}_k(\mathfrak{p}_r) \ll R^{1/3} \tag{4.7}$$

for a constant c and some fixed r to be specified later. This is possible because of the bounds of Chebyshev type on the number of prime ideals of $\mathfrak{o}$ of bounded norm. Note that this step would be an obstruction to proving a result in which the implied constant is only allowed to depend on the degree of the number field k. Now, either there exists some $i \in \{1, \dots, r\}$ such that $\mathfrak{p}_i \nmid \Delta(\mathbf{M})$, or else

$$\mathrm{N}_k(\Delta(\mathbf{M})) \ge \prod_{i=1}^{r} \mathrm{N}_k(\mathfrak{p}_i) \gg R^{r/3}. \tag{4.8}$$

We shall suppose that (4.8) holds.

Define the height $H(Q)$ of Q to be the height $H_k([a_{11}, \dots, a_{33}])$ and put $\|Q\|_\star = \|(a_{11}, \dots, a_{33})\|_\star$. We see that

$$\|Q\|_\star^{3s_k} \gg \|\det \mathbf{M}\|_\star^{s_k} \ge \mathrm{N}_k(\Delta(\mathbf{M}))$$

by (2.2), and $H(Q)^3 \gg \|Q\|_\star^{3s_k}$ by (2.4). Hence

$$H(Q) \gg R^{r/9} \ge B^{r/9}, \tag{4.9}$$

where $B = \prod_{\nu|\infty} \sup\{r_{1,\nu}, r_{2,\nu}, r_{3,\nu}\}$.

Next note that any solution with $x \in \mathbb{P}^2(k)$ and $x \in L(\underline{\mathbf{r}}) \cap Z_3'$ satisfies $H_k(x) \le B$. Suppose $Q = 0$ has at least five solutions of height at most B and suppose they have representatives $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(5)} \in Z_3$ such that $\|\mathbf{x}^{(i)}\|_\star \ll B^{1/s_k}$ for $1 \le i \le 5$. Consider the 5×6 matrix $\mathbf{C}$ whose ith row consists of the six possible monomials of degree 2 in the variables $x_1^{(i)}, x_2^{(i)}, x_3^{(i)}$. Then if the vector $\mathbf{f} \in \mathfrak{o}^6$ has entries which are the corresponding coefficients of Q, we will have $\mathbf{Cf} = \mathbf{0}$. Also, since $\mathrm{rank}(\mathbf{C}) \le 5$, the equation $\mathbf{Cg} = \mathbf{0}$ has a non-zero integer solution $\mathbf{g}$ constructed out of the 5×5 subdeterminants of $\mathbf{C}$. Note that each element c_{ij} of $\mathbf{C}$ has $\|c_{ij}\|_\star \ll B^{2/s_k}$, so that $\mathbf{g}$ satisfies $\|\mathbf{g}\|_\star \ll B^{10/s_k}$. Let G be the ternary quadratic form corresponding to the vector $\mathbf{g}$. By (2.4), we have $H(G) \ll B^{10}$. Note that G and Q have at least five common zeros, namely $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(5)}$. This contradicts Bézout's theorem unless G is a constant multiple of Q, since Q is non-singular. In this case, therefore, we have $H(Q) = H(G) \ll B^{10}$. Comparing this with (4.9), we obtain a contradiction for large R if we take $r > 90$. Thus we may conclude that $Q = 0$ has at most four solutions of height at most B, which is satisfactory.

We proceed to consider the case $\mathfrak{p}_i \nmid \Delta(\mathbf{M})$ for some index $i \in \{1, \dots, r\}$. Thus we may suppose that there is a prime ideal $\mathfrak{p}$ satisfying

$$cR^{1/3} \le \mathrm{N}_k(\mathfrak{p}) \ll R^{1/3},$$

with $\mathfrak{p} \nmid \Delta(\mathbf{M})$. We shall suppose that R is large enough to ensure that $\mathfrak{p} \nmid \mathfrak{a}_i$ for any $i \in \{1, \dots, h\}$. Let $\mathfrak{o}_\mathfrak{p}$ be the localisation of $\mathfrak{o}$ at $\mathfrak{p}$, and put $\mathfrak{q} = \mathfrak{p}\mathfrak{o}_\mathfrak{p}$.

We have $\mathfrak{o}_\mathfrak{p}/\mathfrak{q} \cong \mathbb{F}_q$, where $q = \mathrm{N}_k(\mathfrak{p}) = p^l$ for some rational prime p. If we look at the image $\overline{Q}$ (over $\mathbb{F}_q$) of Q (mod $\mathfrak{p}$) under this isomorphism, then $\overline{Q}$ is non-singular. The projective variety $\overline{Q} = 0$ has exactly q points over $\mathbb{F}_q$. Our goal is to show that there are at most 2 points counted by $N(Q, \underline{\mathbf{r}})$ for each of the corresponding cosets of $\mathfrak{o}_\mathfrak{p}/\mathfrak{q}$. This will complete the proof of Theorem 4.7, since we have assumed that $q \ll R^{1/3}$.

Fix a vector $\mathbf{x} \in (\mathfrak{o}_\mathfrak{p}/\mathfrak{q})^3 \setminus \{\mathbf{0}\}$, with $Q(\mathbf{x}) \equiv 0 \pmod{\mathfrak{q}}$ and

$$\nabla Q(\mathbf{x}) \not\equiv \mathbf{0} \pmod{\mathfrak{q}}. \tag{4.10}$$

We claim that there exists a vector $\mathbf{x}^{(1)} \in \mathfrak{o}_\mathfrak{p}^3$, with $\mathbf{x}^{(1)} \equiv \mathbf{x} \pmod{\mathfrak{q}}$, which satisfies $Q(\mathbf{x}^{(1)}) \equiv 0 \pmod{\mathfrak{q}^2}$ and (4.10). To see this put $\mathbf{x}^{(1)} = \mathbf{x} + \pi\mathbf{y}^{(1)}$ for some uniformiser $\pi \in \mathfrak{q}$. Then $Q(\mathbf{x}^{(1)}) \equiv 0 \pmod{\mathfrak{q}^2}$ if and only if

$$\mathbf{y}^{(1)}.\nabla Q(\mathbf{x}) \equiv -\pi^{-1} Q(\mathbf{x}) \pmod{\mathfrak{q}},$$

and this is clearly solvable for $\mathbf{y}^{(1)}$. This establishes the claim.

We shall count points $w \in \mathbb{P}^2(k)$ which have at least one representation as $\mathbf{w} \in Z_3'$ satisfying $Q(\mathbf{w}) = 0$ and $w_i \in L(\mathbf{r}_i)$, and such that there exists $\lambda \in \mathfrak{o}_\mathfrak{p}$ with $\mathbf{w} \equiv \lambda\mathbf{x}^{(1)} \pmod{\mathfrak{q}}$. Then there is a vector $\mathbf{z} \in \mathfrak{o}_\mathfrak{p}^3$ such that $\mathbf{w} = \lambda\mathbf{x}^{(1)} + \pi\mathbf{z}$. It follows that

$$\begin{aligned} 0 = Q(\mathbf{w}) &\equiv \lambda^2 Q(\mathbf{x}^{(1)}) + \pi\lambda\mathbf{z}.\nabla Q(\mathbf{x}^{(1)}) \pmod{\mathfrak{q}^2} \\ &\equiv \pi\lambda\mathbf{z}.\nabla Q(\mathbf{x}^{(1)}) \pmod{\mathfrak{q}^2}. \end{aligned}$$

Moreover, we note that $\lambda \notin \mathfrak{q}$, since otherwise $\mathbf{w} = \lambda\mathbf{x}^{(1)} \pmod{\mathfrak{q}}$ implies that the ideal which spans the elements of $\mathbf{w}$ is divisible by $\mathfrak{p}$, contradicting the fact that $\mathbf{w} \in Z_3'$ and $\mathfrak{p} \nmid \mathfrak{a}_i$. Hence we conclude that $\mathbf{z}.\nabla Q(\mathbf{x}^{(1)}) \in \mathfrak{q}$. It follows that

$$\begin{aligned} \mathbf{w}.\nabla Q(\mathbf{x}^{(1)}) &= \lambda\mathbf{x}^{(1)}.\nabla Q(\mathbf{x}^{(1)}) + \pi\mathbf{z}.\nabla Q(\mathbf{x}^{(1)}) \\ &= 2\lambda Q(\mathbf{x}^{(1)}) + \pi\mathbf{z}.\nabla Q(\mathbf{x}^{(1)}) \equiv 0 \pmod{\mathfrak{q}^2}. \end{aligned}$$

In conclusion, we have shown that any $\mathbf{w}$ as above belongs to the set

$$L_\mathfrak{p} = \left\{ \mathbf{w} \in \mathfrak{o}_\mathfrak{p}^3 : \begin{array}{l} \mathbf{w} \equiv \lambda\mathbf{x} \pmod{\mathfrak{q}} \text{ for some } \lambda \in \mathfrak{o}_\mathfrak{p} \\ \mathbf{w}.\nabla Q(\mathbf{x}^{(1)}) \equiv 0 \pmod{\mathfrak{q}^2} \end{array} \right\}.$$

A simple generalisation of the proof of [BHB, Lemma 7] shows that $L_\mathfrak{p}$ is independent of the choice of $\mathbf{x}^{(1)}$ and that it is an $\mathfrak{o}_\mathfrak{p}$-lattice of dimension 3 and determinant $\mathrm{N}_k(\mathfrak{p})^3$. We shall not give details of this argument here.

Define L_ν to be $\mathfrak{o}_\nu$ for all ν such that $\nu \nmid \infty$ and $\nu \nmid \mathfrak{p}$. Lemma 4.1 implies that there is a unique $\mathfrak{o}$-lattice Λ such that $\Lambda_\nu = L_\nu$ for all $\nu \in \Omega$, with

$$\det(\Lambda) = [\mathfrak{o}^3 : \Lambda] = \prod_{\nu\nmid\infty} [\mathfrak{o}_\nu^3 : L_\nu] = [\mathfrak{o}_\mathfrak{p}^3 : L_\mathfrak{p}] = \mathrm{N}_k(\mathfrak{p})^3.$$

For $\nu \mid \infty$, consider the sets

$$S_\nu = \{(x_1, x_2, x_3) \in k_\nu^3 : |x_i|_\nu \le r_{i,\nu}^{1/d_\nu} \text{ for } i = 1, 2, 3\},$$

and put $S = \prod_{\nu|\infty} S_\nu$. We have $\mathfrak{o}^3 \cap S = L(\mathbf{r}_1) \times L(\mathbf{r}_2) \times L(\mathbf{r}_3)$. Moreover, S is symmetric and $\mathrm{vol}(S) \gg R$.

Next we consider the successive minima $\lambda_1 \le \lambda_2 \le \lambda_3$ of Λ with respect to S. By Lemma 4.3, we know that

$$(\lambda_1\lambda_2\lambda_3)^d \,\mathrm{vol}(S) \ll [\mathfrak{o}^3 : \Lambda].$$

It follows that

$$(\lambda_1\lambda_2)^d \ll \frac{\mathrm{N}_k(\mathfrak{p})^2}{R^{2/3}}.$$

It is evident from the definitions that we can find linearly independent vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ such that $\mathbf{u}_i \in \Lambda \cap \lambda_i S$. If u_{ij} is the jth component of $\mathbf{u}_i$, then $\|u_{ij}\|_\nu \le \lambda_i^{d_\nu} r_{j,\nu}$ for $\nu \mid \infty$. Hence, if $\mathbf{w} = y_1\mathbf{u}_1 + y_2\mathbf{u}_2 + y_3\mathbf{u}_3 \in S$ for some $(y_1, y_2, y_3) \in k^3$, and $\mathbf{U}$ is the matrix with columns $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, then for each $\nu \mid \infty$ we have

$$\|y_3\|_\nu = \frac{1}{\|\det \mathbf{U}\|_\nu} \left\| \det \begin{pmatrix} u_{11} & u_{21} & w_1 \\ u_{12} & u_{22} & w_2 \\ u_{13} & u_{23} & w_3 \end{pmatrix} \right\|_\nu \ll \frac{r_{1,\nu} r_{2,\nu} r_{3,\nu} (\lambda_1\lambda_2)^{d_\nu}}{\|\det \mathbf{U}\|_\nu},$$

by Cramer's rule. We note that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is not necessarily a basis for Λ over $\mathfrak{o}$. However, if we let L be the free $\mathfrak{o}$-lattice with generators $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, then $L \subset \Lambda \subset aL$ for some $a \in k^\times$ such that $\mathrm{N}_k(a) \gg [\Lambda : L]$, by Lemma 4.2. Hence any element $\mathbf{w} \in \Lambda \cap S$ may be written as

$$y_1(a\mathbf{u}_1) + y_2(a\mathbf{u}_2) + y_3(a\mathbf{u}_3)$$

for some $(y_1, y_2, y_3) \in \mathfrak{o}^3$.

Let Q' be the quadratic form given by the matrix $\mathbf{U}^T\mathbf{M}\mathbf{U}$. Then we have shown that every point $w \in \mathbb{P}^2(k)$ which has at least one representation $\mathbf{w} \in Z_3'$ satisfying $Q(\mathbf{w}) = 0$ and $w_i \in L(\mathbf{r}_i)$, and such that there exists $\lambda \in \mathfrak{o}_\mathfrak{p}$ with $\mathbf{w} \equiv \lambda\mathbf{x}^{(1)} \pmod{\mathfrak{q}}$, gives us a solution $(y_1, y_2, y_3) \in \mathfrak{o}^3$ to $Q' = 0$, with

$$\|y_3\|_\nu \ll \frac{r_{1,\nu} r_{2,\nu} r_{3,\nu} (\lambda_1\lambda_2)^{d_\nu}}{\|a\|_\nu \|\det \mathbf{U}\|_\nu}.$$

Taking the product over all $\nu \mid \infty$ we see that

$$\mathrm{N}_k(y_3) \ll \frac{R(\lambda_1\lambda_2)^d}{\mathrm{N}_k(a)\mathrm{N}_k(\det \mathbf{U})} \ll \frac{R(\lambda_1\lambda_2)^d}{[\Lambda : L][\mathfrak{o}^3 : L]} = \frac{R(\lambda_1\lambda_2)^d}{[\mathfrak{o}^3 : \Lambda]} \ll \frac{R^{1/3}}{\mathrm{N}_k(\mathfrak{p})}.$$

Hence, on taking c in (4.7) sufficiently large, we deduce that $y_3 = 0$, whence $\mathbf{w}$ is confined to the 2-dimensional space spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$. This means that the point $w \in \mathbb{P}^2(k)$ must not only lie on the irreducible conic $Q = 0$, but also on a line. There are at most two such points, which thereby completes the proof of Theorem 4.7. ■

5. Sums involving binary forms. In this section we shall prove Theorem 2.4. Fix $\varepsilon > 0$. Suppose that $F(u,v) = \beta(u+\alpha_1 v)\cdots(u+\alpha_n v)$, for $\alpha_i \in \overline{k}$ (if $F(u,v)$ has uv as a factor, we can do a simple change of variables to reach this form). Then F is separable by hypothesis and so $\alpha_1,\dots,\alpha_n$ are distinct. We may assume that $\beta = 1$, since the implied constant in (2.6) can vary with F. Set $K = k(\alpha_1,\dots,\alpha_n)$. For each infinite place ν of k, we fix an extension of ν to K, and extend $\|\cdot\|_\nu$ likewise. We shall let K_ν denote the completion at this place. Note that for any $(u,v) \in Z_2$ we have

$$\mathrm{N}_k(F(u,v)) = \prod_{\nu\mid\infty} \|F(u,v)\|_\nu = \prod_{\nu\mid\infty}\prod_{1\le i\le n} \|u+\alpha_i v\|_\nu.$$

Let $\mathbf{A} \in (\mathbb{R}_{\ge 1})^{s_k}$ and recall the notation $\|\mathbf{A}\| = \prod_{\nu\mid\infty} A_\nu$. We will show that

$$\sum_{\substack{(u,v)\in\mathfrak{o}^2\\ A_\nu\le\sup\{\|u\|_\nu,\|v\|_\nu\}<2A_\nu\\ F(u,v)\neq 0}} \Big(\prod_{\nu\mid\infty}\prod_{1\le i\le n}\|u+\alpha_i v\|_\nu\Big)^{-1/3} \ll \|\mathbf{A}\|^{2-n/3+\varepsilon}. \tag{5.1}$$

Here, as throughout this section, we shall allow all implied constants to be ineffective, and to depend on k, F and on the choice of ε. This will clearly suffice for the proof of Theorem 2.4 on summing over dyadic values of A_ν such that $A \ll \|\mathbf{A}\| \ll A$.

Let $c_\nu = c_\nu(k,F) \ge 1$ be fixed absolute constants. On multiplying (u,v) through by a suitable scalar, it clearly suffices to assume that $A_\nu \ge c_\nu$ for each $\nu\mid\infty$ when trying to prove (5.1). Let $\mathscr{A} = \mathscr{A}(\mathbf{A})$ denote the set of $(u,v)\in\mathfrak{o}^2$ such that

$$A_\nu \le \sup\{\|u\|_\nu, \|v\|_\nu\} < 2A_\nu \quad \text{for all } \nu\mid\infty, \tag{5.2}$$

and $F(u,v)\neq 0$. It follows from [Brob1, Prop. 1] that

$$\#\mathscr{A} \le (\#L(2\mathbf{A}))^2 \ll \|\mathbf{A}\|^2.$$

Let $(u,v)\in\mathscr{A}$. Since $\alpha_1,\dots,\alpha_n$ are fixed once and for all, this implies there is a constant $C>0$ such that $\|u+\alpha_i v\|_\nu < CA_\nu$ for all indices $i\in\{1,\dots,n\}$ and all $\nu\mid\infty$.

Let

$$\theta = \varepsilon/n.$$

For any $\nu\mid\infty$ and any $(i,q)\in\{1,\dots,n\}\times\mathbb{Z}_{\ge 0}$, we define the sets

$$\mathscr{A}_\nu(i,q) = \{(u,v)\in\mathscr{A} : CA_\nu^{1-(q+1)\theta} \le \|u+\alpha_i v\|_\nu < CA_\nu^{1-q\theta}\}.$$

The larger q is, the more the factor $\|u+\alpha_i v\|_\nu$ will contribute to the sum (5.1). The idea of the proof is that the bulk of $\mathscr{A}$ is covered by intersections of sets of the form $\mathscr{A}_\nu(i,q)$, with q not too large, and we can quantify the contribution from these points very easily. In order to handle the contribution from a set $\mathscr{A}_\nu(i,q)$ with q large, we use the fact that points in such a set produce good Diophantine approximations to α_i. Appealing to a number field version of the Thue–Siegel–Roth theorem due to Lang, we can then show that the problem sets cannot contribute too much.

We begin with the following technical lemmas.

Lemma 5.1. *Let $\mathbf{B}\in(\mathbb{R}_{>0})^{s_k}$. Let $t_\nu\in K_\nu$ for each $\nu\,|\,\infty$ and let*

$$\mathscr{S}=\{u\in\mathfrak{o}:\|u-t_\nu\|_\nu<B_\nu \text{ for all } \nu\,|\,\infty\}.$$

Then $\#\mathscr{S}\ll 1+\|\mathbf{B}\|$. The implied constant does not depend on any t_ν.

Proof. We may clearly assume that $\mathscr{S}\neq\emptyset$. Let $x'\in\mathscr{S}$. Then any $x\in\mathscr{S}$ takes the form $x=x'+\gamma$, with γ belonging to the set

$$\mathscr{S}'=\{\gamma\in\mathfrak{o}:|\gamma|_\nu<2B_\nu^{1/d_\nu}\text{ for all }\nu\,|\,\infty\}.$$

But this is $L(\mathbf{r})$, with $r_\nu=2B_\nu^{1/d_\nu}$. Hence it follows from [Brob1, Prop. 1] that $\#\mathscr{S}'\ll 1+\|\mathbf{B}\|$. ∎

Lemma 5.2. *Let $(i(\nu),q(\nu))\in\{1,\dots,n\}\times\mathbb{Z}_{\geq 0}$ for each $\nu\,|\,\infty$. Then*

$$\#\Big(\bigcap_{\nu|\infty}\mathscr{A}_\nu(i(\nu),q(\nu))\Big)\ll\|\mathbf{A}\|\Big\{1+\prod_{\nu|\infty}A_\nu^{1-\theta q(\nu)}\Big\}. \tag{5.3}$$

Proof. If $(u,v)\in\mathscr{A}$, then (5.2) implies that we must have $\|v\|_\nu\ll A_\nu$ for all $\nu\,|\,\infty$. Thus, by Lemma 5.1, we have at most $O(\|\mathbf{A}\|)$ choices for v. If we fix some v, then we must count the number of solutions $u\in\mathfrak{o}$ to the inequalities $\|u-t_\nu\|_\nu<CA_\nu^{1-q(\nu)\theta}$ for $t_\nu=\alpha_{i(v)}v\in K$. Now we apply Lemma 5.1 again. ∎

The next lemma makes precise the statement that if we fix a place $\nu\,|\,\infty$, then for any $(u,v)\in\mathscr{A}$ there is at most one α_i such that $\|u-\alpha_i v\|_\nu$ is "small".

Lemma 5.3. *Let $\nu\,|\,\infty$. There exists a constant $c(k,F)>0$ such that for any integers $i_1\neq i_2$ and $q_1,q_2\geq 1$ we have*

$$\mathscr{A}_\nu(i_1,q_1)\cap\mathscr{A}_\nu(i_2,q_2)=\emptyset,$$

if $A_\nu>c(k,F)$.

Proof. This is a simple consequence of the triangle inequality. Suppose for a contradiction that $(u,v)\in\mathscr{A}_\nu(i_1,q_1)\cap\mathscr{A}_\nu(i_2,q_2)$, with $i_1\neq i_2$ and $q_1,q_2\geq 1$. We see that

$$\begin{aligned}\|\alpha_{i_1}-\alpha_{i_2}\|_\nu\|u\|_\nu&\leq\|\alpha_{i_1}u+\alpha_{i_1}\alpha_{i_2}v\|_\nu+\|\alpha_{i_2}u+\alpha_{i_1}\alpha_{i_2}v\|_\nu\\&<C(\|\alpha_{i_1}\|_\nu+\|\alpha_{i_2}\|_\nu)A_\nu^{1-\theta}.\end{aligned}$$

This implies that $\|u\|_\nu \ll A_\nu^{1-\theta}$, if A_ν is sufficiently large. Similarly, we have $\|v\|_\nu \ll A_\nu^{1-\theta}$. But then there is a constant $c(k,F) > 0$ such that this violates the restriction (5.2) if $A_\nu > c(k,F)$. ∎

As remarked in the paragraph following (5.1), we may proceed under the assumption that each A_ν exceeds $c(k,F)$, so that Lemma 5.3 applies. In particular, if we fix $(u,v) \in \mathscr{A}$ and a place $\nu \mid \infty$, then there will be at most one pair (i_ν, q_ν) with $q_\nu \ge 1$ and $(u,v) \in \mathscr{A}_\nu(i_\nu, q_\nu)$. If there is no such pair, we can put $(i_\nu, q_\nu) = (1,0)$ by default. Thus there is a well-defined map from elements of $\mathscr{A}$ to $\mathscr{I} = (\{1,\dots,n\} \times \mathbb{Z}_{\ge 0})^{s_k}$.

Now we break the sum (5.1) into sums over those (u,v) which are mapped to a particular element

$$\varpi = \prod_{\nu\mid\infty} (i_\nu, q_\nu) \in \mathscr{I}.$$

Note that there are finitely many such ϖ, the number depending only on ε, n and k. Any (u,v) which maps to ϖ must be contained in the intersection

$$\mathscr{B} = \bigcap_{\nu\mid\infty} \mathscr{A}_\nu(i_\nu, q_\nu).$$

We begin by considering the case in which ϖ is such that

$$1 \le \prod_{\nu\mid\infty} A_\nu^{1-\theta q_\nu}.$$

Then, when we estimate the cardinality of $\mathscr{B}$, the second term on the right hand side of (5.3) dominates. Lemma 5.3 implies that if $(u,v) \in \mathscr{A}_\nu(i_\nu, q_\nu)$, then $(u,v) \in \mathscr{A}_\nu(i,0)$ for every $i \ne i_\nu$.

Using Lemma 5.2 we conclude that the contribution from the elements mapping to ϖ is at most

$$\begin{aligned}\sum_{(u,v)\in\mathscr{B}} \prod_{\nu\mid\infty} &\Big(\|u+\alpha_{i_\nu} v\|_\nu^{-1/3} \prod_{i\ne i_\nu} \|u+\alpha_i v\|_\nu^{-1/3}\Big) \\ &\ll \#\mathscr{B} \prod_{\nu\mid\infty} (A_\nu^{-(1-(q_\nu+1)\theta)/3} A_\nu^{-(n-1)(1-\theta)/3)}) \\ &\ll \|\mathbf{A}\|^{2-n/3} \Big(\prod_{\nu\mid\infty} A_\nu^{\theta\{-q_\nu+(q_\nu+1)/3+(n-1)/3\}}\Big).\end{aligned}$$

But the exponent of A_ν in the last line is

$$\theta\bigg\{-q_\nu + \frac{q_\nu+1}{3} + \frac{n-1}{3}\bigg\} \le \frac{\theta n}{3} = \frac{\varepsilon}{3},$$

by the definition of θ. Thus the last line is $\ll \|\mathbf{A}\|^{2-n/3+\varepsilon}$ and each set $\mathscr{B}$ contributes a satisfactory amount.

Now we must consider the set $\mathscr{C}$ of $(u,v) \in \mathscr{A}$ which map to $\varpi \in \mathscr{I}$ with

$$1 > \prod_{\nu|\infty} A_\nu^{1-\theta q_\nu}.$$

The cardinality of this set is estimated in the following result.

LEMMA 5.4. *We have* $\#\mathscr{C} \ll \|\mathbf{A}\|$.

Proof. Consider sets $\mathscr{C}_\nu(i,q)$ given by

$$\mathscr{C}_\nu(i,q) = \{(u,v) \in \mathscr{A} : \|u + \alpha_i v\|_\nu < CA_\nu^{1-q\theta}\}.$$

We can cover $\mathscr{C}$ with finitely many sets of the form $\bigcap_{\nu|\infty} \mathscr{C}_\nu(i_\nu, q_\nu)$. The proof of Lemma 5.2 then shows that the size of each set is $O(\|\mathbf{A}\|)$. ∎

Given an element $\xi \in k$, we define its height to be

$$H_k(\xi) = \prod_{\nu\in\Omega} \sup\{1, \|\xi\|_\nu\}.$$

With this in mind, to estimate the size of the sum over $\mathscr{C}$, we shall use the following generalisation of the Thue–Siegel–Roth theorem due to Lang [L, §7, Thm. 1.1].

LEMMA 5.5 (Lang's generalisation of Thue–Siegel–Roth). *For each $\nu \mid \infty$, let α_ν be an algebraic number over k and assume that ν is extended to $k(\alpha_\nu)$. Let $\varepsilon' > 0$. Then the elements $\xi \in k$ satisfying the approximation condition*

$$\prod_{\nu|\infty} \inf\{1, \|\alpha_\nu - \xi\|_\nu\} \le \frac{1}{H_k(\xi)^{2+\varepsilon'}}$$

have bounded height.

Suppose that $\xi = u/v$ with $(u,v) \in \mathfrak{o}^2$. The product formula (2.1) implies that

$$H_k(\xi) = \prod_{\nu\in\Omega} \sup\{1, \|u/v\|_\nu\} = \prod_{\nu\in\Omega} \sup\{\|u\|_\nu, \|v\|_\nu\} = H_k([u,v]).$$

Thus Lemma 5.5 tells us that if $(u,v) \in Z_2$ and $H_k([u,v]) \gg 1$, then

$$\prod_{\nu|\infty} \inf\{1, \|\alpha_\nu - u/v\|_\nu\} > \frac{1}{H_k([u,v])^{2+\varepsilon'}}. \tag{5.4}$$

Let $(u,v) \in \mathscr{C}$ and let $\nu \mid \infty$. By the argument of Lemma 5.3, assuming A_ν is sufficiently large, there can be at most one $i = i_\nu$ such that

$$\|u - \alpha_i v\|_\nu < A_\nu^{1-\varepsilon'}.$$

Moreover, we have $N_k(v) \geq 1$ and

$$\begin{aligned}
N_k(F(u,v)) &= \prod_{\nu\mid\infty} \prod_{1\leq i\leq n} \|u - \alpha_i v\|_\nu \\
&\geq \prod_{\nu\mid\infty} (A_\nu^{1-\varepsilon'})^{n-1} \|u - \alpha_{i_\nu} v\|_\nu \\
&= \|\mathbf{A}\|^{(n-1)(1-\varepsilon')} N_k(v) \prod_{\nu\mid\infty} \|\alpha_{i_\nu} - u/v\|_\nu \geq \|\mathbf{A}\|^{n-3-n\varepsilon'},
\end{aligned}$$

the last inequality following from (5.4) with $\alpha_\nu = \alpha_{i_\nu}$. Then, letting $\varepsilon' = 3\varepsilon/n$ and applying Lemma 5.4, we conclude that

$$\sum_{(u,v)\in\mathscr{C}} \frac{1}{(N_k(F(u,v)))^{1/3}} \ll \|\mathbf{A}\|\,\|\mathbf{A}\|^{(3-n)/3+n\varepsilon'/3} = \|\mathbf{A}\|^{2-n/3+\varepsilon},$$

which is satisfactory. This completes the proof of Theorem 2.4.

Remark 5.6. Let $\gamma \in [1/3, 1]$. The proof of Theorem 2.4 can be adapted to show that

$$\|\mathbf{A}\|^{2-\gamma n} \ll \sum_{\substack{(u,v)\in\mathfrak{o}^2 \\ A_\nu \leq \sup\{\|u\|_\nu, \|v\|_\nu\} < 2A_\nu \\ F(u,v)\neq 0}} \frac{1}{(N_k(F(u,v)))^\gamma} \ll \|\mathbf{A}\|^{2-\gamma n+\varepsilon}.$$

The lower bound is trivial since $N_k(F(u,v)) \ll \|\mathbf{A}\|^n$. This shows that our result is essentially best possible.

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