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Additive problems with prime numbers of special type

by

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1. Introduction and statement of the results. In 1937 I. M. Vinogradov [28] proved that for every sufficiently large odd integer n the equation

$$p_1 + p_2 + p_3 = n \tag{1.1}$$

has a solution in prime numbers. It is still not known whether every sufficiently large even integer n can be represented as

$$p_1 + p_2 = n, \tag{1.2}$$

where p_1, p_2 are primes. Denote by $E(N)$ the number of even integers not exceeding N and not representable in the form (1.2). Many researchers have worked to obtain non-trivial upper bounds for this quantity. The most important result belongs to Montgomery and Vaughan [19]. They proved in 1975 that there exists an effective constant $\delta > 0$ such that $E(N) \ll N^{1-\delta}$.

Another important approach for studying the equation (1.2) is by the use of sieve methods. The strongest result in this direction belongs to Chen [3]. Denote, as usual, by P_r any integer with no more than r prime factors, counted according to multiplicity. In 1973 Chen proved that every sufficiently large even n can be represented as a sum of a prime and a P_2. He also proved that there are infinitely many primes p such that $p + 2 = \mathrm{P}_2$.

In 1938 Hua studied the equation

$$p_1^2 + p_2^2 + p_3^2 = n \tag{1.3}$$

for solvability in prime numbers. By elementary considerations one may see that necessary conditions for the solvability of (1.3) are $n \equiv 3 \pmod{24}$ and $n \not\equiv 0 \pmod 5$. Denote by $E_1(N)$ the number of integers $n \le N$ satisfying these congruences and which are not representable in the form (1.3). Hua [8] proved the existence of a constant $B > 0$ such that $E_1(N) \ll N(\log N)^{-B}$. Schwarz [22] proved this estimate with arbitrarily large $B > 0$. In 1993

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M.-C. Leung and M.-C. Liu [14] showed that $E_1(N) \ll N^{1-\delta}$ for some $\delta > 0$. Short-interval versions of this problem were considered by J. Liu and T. Zhan [15] and Mikawa [16].

As a corollary to his theorem Hua established that the equation

$$p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 = n \tag{1.4}$$

is solvable in primes provided that n is sufficiently large and satisfies $n \equiv 5 \pmod{24}$.

In 1939 van der Corput [26] established that there exist infinitely many arithmetic progressions of three different primes. The corresponding question for progressions of four or more primes is still open. In 1981, however, Heath-Brown [6] proved that there exist infinitely many arithmetic progressions of four different terms, three of which are primes and the fourth is P_2.

The work of Heath-Brown motivated the author to study additive problems with primes p such that $p + 2$ is almost-prime. In [21] Peneva and the author proved that there exist infinitely many arithmetic progressions of three different primes p_1, p_2, p_3 such that $(p_1 + 2)(p_2 + 2) = \mathrm{P}_9$. Later the author used some ideas of Brüdern and Fouvry [1] and Heath-Brown and was able to impose a multiplicative restriction on $p_3 + 2$ as well. It was proved in [23] that there exist infinitely many arithmetic progressions of three different primes p_1, p_2, $p_3 = \frac{1}{2}(p_1 + p_2)$ such that $p_1 + 2 = \mathrm{P}_5$, $p_2 + 2 = \mathrm{P}'_5$, $p_3 + 2 = \mathrm{P}_8$. Peneva [20] used the method of [23] to consider the corresponding problem for the equation (1.1).

Recently the author considered the equation (1.4) for solvability in primes of the type described above. It was established in [24] that if n is a sufficiently large integer satisfying $n \equiv 5 \pmod{24}$ then (1.4) has a solution in primes $p_1, \ldots, p_5$ such that each of the numbers $p_1 + 2$, $p_2 + 2$, $p_3 + 2$, $p_4 + 2$ is P_6 and $p_5 + 2 = \mathrm{P}_7$. We should also mention the earlier result [13] of Laporta and the author, which is somewhat related to [24].

In the present paper we study the equations (1.2) and (1.3) with variables prime numbers of the type mentioned above. We prove that they are solvable for almost all n satisfying some natural congruence conditions. The following theorems hold:

THEOREM 1. *Denote by $\mathcal{K}$ the set of integers n for which the equation* (1.3) *has a solution in primes p_1, p_2, p_3 such that $p_1 + 2 = \mathrm{P}_5$, $p_2 + 2 = \mathrm{P}'_5$, $p_3 + 2 = \mathrm{P}_8$. Consider the set*

$$\mathcal{F} = \{n \le N : n \equiv 3 \pmod{24},\ n \not\equiv 0 \pmod 5\} \setminus \mathcal{K}$$

and let $\mathcal{Y}(N)$ be its cardinality. Then for arbitrarily large $B > 0$ we have

$$\mathcal{Y}(N) \ll N(\log N)^{-B}.$$

THEOREM 2. *Denote by $\mathcal{K}_0$ the set of integers n for which the equation* (1.2) *has a solution in different primes p_1, p_2 such that $p_1 + 2 = \mathrm{P}_5$, $p_2 + 2 = \mathrm{P}_7$. Consider the set*

$$\mathcal{F}_0 = \{n \le N : n \equiv 4 \pmod 6\} \setminus \mathcal{K}_0$$

and let $\mathcal{Y}_0(N)$ be its cardinality. Then for arbitrarily large $B > 0$ we have

$$\mathcal{Y}_0(N) \ll N(\log N)^{-B}.$$

From Theorem 1 we easily obtain

COROLLARY 1. *For every sufficiently large integer $n \equiv 5 \pmod{24}$ the equation* (1.4) *has a solution in prime numbers $p_1, \dots, p_5$ such that $p_1 + 2 = \mathrm{P}_2$, $p_2 + 2 = \mathrm{P}'_2$, $p_3 + 2 = \mathrm{P}_5$, $p_4 + 2 = \mathrm{P}'_5$, $p_5 + 2 = \mathrm{P}_8$.*

Proof. Consider the sets of primes

$$\mathfrak{A} = \left\{p \le \tfrac{1}{2}\sqrt{n} : p \equiv 11 \pmod{30},\ p + 2 = \mathrm{P}_2\right\}$$

and

$$\mathfrak{A}' = \left\{p \le \tfrac{1}{2}\sqrt{n} : p \equiv 17 \pmod{30},\ p + 2 = \mathrm{P}_2\right\}.$$

Applying the arguments of Chen we establish that the cardinalities of $\mathfrak{A}$ and $\mathfrak{A}'$ are $\gg \sqrt{n}(\log n)^{-2}$.

Suppose that $n \not\equiv 2 \pmod 5$. Consider the set $\{n - p^2 - q^2 : p, q \in \mathfrak{A}\}$. It is not difficult to see that it contains $\gg n(\log n)^{-9}$ distinct integers k satisfying $k \equiv 3 \pmod{24}$, $k \not\equiv 0 \pmod 5$. It remains to apply Theorem 1.

If $n \equiv 2 \pmod 5$ then we consider the set $\{n - p^2 - q^2 : p \in \mathfrak{A},\ q \in \mathfrak{A}'\}$ and then we proceed as in the first case.

Similarly, from Theorem 2 we obtain the following corollaries:

COROLLARY 2. *For every sufficiently large integer $n \equiv 3 \pmod 6$ the equation* (1.1) *has a solution in prime numbers p_1, p_2, p_3 such that $p_1 + 2 = \mathrm{P}_2$, $p_2 + 2 = \mathrm{P}_5, p_3 + 2 = \mathrm{P}_7$.*

COROLLARY 3. *There are infinitely many arithmetic progressions of three different primes $p_1, p_2, p_3 = \frac{1}{2}(p_1 + p_2)$ such that $p_3 + 2 = \mathrm{P}_2$, $p_1 + 2 = \mathrm{P}_5$, $p_2 + 2 = \mathrm{P}_7$.*

To prove the theorems we apply the method of [20], [23] and [24]. In many places we omit the calculations because they are similar to those in the papers mentioned above. We present only the proof of Theorem 1. The proof of Theorem 2 is simpler and it was briefly explained in [25].

In Section 2 we introduce the notations and state a Proposition, which is of some independent interest. It asserts that the expected asymptotic formula for the number of the solutions of (1.3) in primes from arithmetic progressions is valid "on average".

In Section 3 we prove Theorem 1. We consider the sum Γ defined by (3.3) and we show that it is not large. On the other hand, we estimate it from below using the vector sieve of Iwaniec [10] and Brüdern–Fouvry [1]. We find that if the cardinality $\mathcal{Y}(N)$ of the set $\mathcal{F}$ were large then the lower bound for Γ would be considerably larger than Γ, which is not possible. This proves the theorem.

In Sections 4 and 5 we prove the Proposition by means of the circle method. We consider the minor arcs in Section 4. The crucial point is formula (4.4) which gives a non-trivial estimate for a double exponential sum. The idea is due to Heath-Brown, who pointed out to the author that non-trivial estimates exist for such kind of sums. We also find an estimate for the mean value of the same sum.

To treat the major arcs we work as in [13], [21], [23], [24]. We find asymptotic formulae for exponential sums over primes lying in arithmetic progressions. It appears that the error terms of these formulae are small "on average" and applying the Bombieri–Vinogradov theorem we find that their contribution is negligible. The computations are presented in Section 5.

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2. Notations and statement of the Proposition. The letter p is reserved for prime numbers. Lower case Latin letters (except x, y, z and p) denote integers. Other letters denote real or complex numbers and the meaning is always clear from the context. As usual, $\mu(n)$, $\varphi(n)$, $\Lambda(n)$, $\nu(n)$ denote the Möbius function, Euler's function, von Mangoldt's function and the number of distinct prime factors of n, respectively; $\tau_k(n)$ denotes the number of solutions of the equation $m_1 \ldots m_k = n$ in integers $m_1, \ldots, m_k$; $\tau(n) = \tau_2(n)$. We denote by $(m_1, \ldots, m_k)$ and $[m_1, \ldots, m_k]$ the greatest common divisor and least common multiple of $m_1, \ldots, m_k$, respectively. For real y, z, however, (y, z) denotes the open interval on the real line with endpoints y and z. Instead of $m \equiv n \pmod k$ we sometimes write for simplicity $m \equiv n\ (k)$. As usual $\|y\|$ denotes the distance from y to the nearest integer and $e(y) = \exp(2\pi i y)$. We write $p^l \,\|\, n$ if $p^l \mid n$ and $p^{l+1} \nmid n$. The Legendre symbol is denoted by $\left(\frac{\cdot}{p}\right)$. For positive U and V we write $U \asymp V$ instead of $U \ll V \ll U$.

Suppose that $A \geq 10000$ is a constant. If not explicitly specified, constants in $\mathcal{O}$-terms and Vinogradov's symbols are absolute or depend only on A. Let N be sufficiently large and put $X = \sqrt{N}$ and $\mathcal{L} = \log X$.

A central point in our paper is the study of the sum

$$I(n;k_1,k_2,k_3)=\sum_{\substack{p_1^2+p_2^2+p_3^2=n\\ p_i+2\equiv 0\,(\mathrm{mod}\,k_i)\\ i=1,2,3}}\log p_1\log p_2\log p_3, \tag{2.1}$$

where k_1,k_2,k_3 are odd squarefree numbers and $n\le N$. It is clear that

$$I(n;k_1,k_2,k_3)=\int_0^1 S_{k_1}(\alpha)S_{k_2}(\alpha)S_{k_3}(\alpha)e(-n\alpha)\,d\alpha,$$

where

$$S_k(\alpha)=\sum_{\substack{p\le X\\ p+2\equiv 0\,(k)}}\log p\,e(\alpha p^2). \tag{2.2}$$

Define

$$Q=\mathcal{L}^{1000A},\qquad \tau=X^2\mathcal{L}^{-2000A}, \tag{2.3}$$

$$E_1=\bigcup_{q<Q}\bigcup_{\substack{a=0\\(a,q)=1}}^{q-1}\left(\frac{a}{q}-\frac{1}{q\tau},\frac{a}{q}+\frac{1}{q\tau}\right),\qquad E_2=\left(-\frac{1}{\tau},1-\frac{1}{\tau}\right)\setminus E_1. \tag{2.4}$$

We have

$$I(n;k_1,k_2,k_3)=I_1+I_2, \tag{2.5}$$

where

$$I_j=\int_{E_j} S_{k_1}(\alpha)S_{k_2}(\alpha)S_{k_3}(\alpha)e(-n\alpha)\,d\alpha,\qquad j=1,2. \tag{2.6}$$

Define

$$s_k(a,q)=\frac{\varphi((k,q))}{\varphi(q)}\sum_{\substack{1\le m\le q\\(m,q)=1\\ m+2\equiv 0\,((k,q))}}e\left(\frac{am^2}{q}\right), \tag{2.7}$$

$$\begin{aligned} t(q)&=t(q;n;k_1,k_2,k_3)\\ &=\sum_{\substack{0\le a\le q-1\\(a,q)=1}}s_{k_1}(a,q)s_{k_2}(a,q)s_{k_3}(a,q)e\left(-n\frac{a}{q}\right). \end{aligned} \tag{2.8}$$

The function $t(q)$ is multiplicative with respect to q. Using the definition (2.7) of $s_k(a,q)$ and the properties of the Gauss sum (see, for example, Hua [9], Chapter 7) it is not difficult to compute $t(p^l)$.

We find that if $n\equiv 3\ (\mathrm{mod}\,8)$ and k_1,k_2,k_3 are odd integers then

$$t(2)=1,\quad t(4)=2,\quad t(8)=4,\quad t(2^l)=0\quad\text{for } l>3. \tag{2.9}$$

Define

$$h_0(p)=\begin{cases}\dfrac{\left(\frac{-n}{p}\right)p^2+\left(3\left(\frac{n}{p}\right)+3\left(\frac{-1}{p}\right)\right)p+1}{(p-1)^3} & \text{if } p\nmid n,\\ \dfrac{-3\left(\frac{-1}{p}\right)p-1}{(p-1)^2} & \text{if } p\mid n,\end{cases}\tag{2.10}$$

$$h_1(p)=\begin{cases}\dfrac{\left(-2\left(\frac{n-4}{p}\right)-\left(\frac{-1}{p}\right)\right)p-1}{(p-1)^2} & \text{if } p\nmid n-4,\\ \dfrac{\left(\frac{-1}{p}\right)p+1}{p-1} & \text{if } p\mid n-4,\end{cases}\tag{2.11}$$

$$h_2(p)=\begin{cases}\dfrac{\left(\frac{n-8}{p}\right)p+1}{p-1} & \text{if } p\nmid n-8,\\ -1 & \text{if } p\mid n-8,\end{cases}\tag{2.12}$$

$$h_3(p)=\begin{cases}-1 & \text{if } p\nmid n-12,\\ p-1 & \text{if } p\mid n-12.\end{cases}\tag{2.13}$$

If $p>2$ and k_1,k_2,k_3 are squarefree integers then we have

$$\begin{gathered}t(p)=\begin{cases}h_0(p) & \text{if } p\nmid k_1k_2k_3,\\ h_1(p) & \text{if } p\,\|\,k_1k_2k_3,\\ h_2(p) & \text{if } p^2\,\|\,k_1k_2k_3,\\ h_3(p) & \text{if } p^3\,\|\,k_1k_2k_3,\end{cases}\\ t(p^l)=0\quad \text{if } l>1.\end{gathered}\tag{2.14}$$

We leave the calculations to the reader.

Define

$$\mathfrak{S}=\mathfrak{S}(n;Q;k_1,k_2,k_3)=8\prod_{2<p<Q}(1+t(p;n;k_1,k_2,k_3)).\tag{2.15}$$

We write

$$I(n;k_1,k_2,k_3)=\frac{\pi}{4}\sqrt{n}\,\frac{\mathfrak{S}(n;Q;k_1,k_2,k_3)}{\varphi(k_1)\varphi(k_2)\varphi(k_3)}+\mathfrak{R}(n;Q;k_1,k_2,k_3).\tag{2.16}$$

The first summand arises from the application of the circle method. We cannot find a non-trivial estimate for the remainder $\mathfrak{R}$ for individual n, k_1, k_2, k_3, but we prove that it is small on average. We have:

PROPOSITION. *Suppose that*

$$K_1,K_2\le X^{1/2}\mathcal{L}^{-20000A},\qquad K_3\le X^{1/3}\mathcal{L}^{-20000A}\tag{2.17}$$

and let $\beta_i(k_i), k_i\le K_i, i=1,2,3,$ *be complex numbers satisfying*

$$\beta_i(k)=0\quad \text{if } 2\mid k \text{ or } \mu(k)=0;\qquad |\beta_i(k)|\le\tau(k).\tag{2.18}$$

Then for

$$\mathcal{U} = \sum_{\substack{n\le N\\ n\equiv 3\,(24)\\ n\not\equiv 0\,(5)}} \Big| \sum_{\substack{k_i\le K_i\\ i=1,2,3}} \beta_1(k_1)\beta_2(k_2)\beta_3(k_3)\mathfrak{R}(n;Q;k_1,k_2,k_3)\Big|$$

we have

$$\mathcal{U} \ll X^3\mathcal{L}^{-A}. \tag{2.19}$$

For brevity we will write $\sum^{*}_{n\le N}$ to emphasize that the summation is taken over the integers n satisfying $n\equiv 3 \pmod{24}$ and $n\not\equiv 0 \pmod 5$.

To prove the Proposition we consider

$$\mathcal{U}_1 = \sum_{n\le N}^{*} \Big| \sum_{\substack{k_i\le K_i\\ i=1,2,3}} \beta_1(k_1)\beta_2(k_2)\beta_3(k_3) \times \Big(I_1 - \frac{\pi}{4}\sqrt{n}\,\frac{\mathfrak{S}(n;Q;k_1,k_2,k_3)}{\varphi(k_1)\varphi(k_2)\varphi(k_3)}\Big)\Big|, \tag{2.20}$$

$$\mathcal{U}_2 = \sum_{n\le N} \Big| \sum_{\substack{k_i\le K_i\\ i=1,2,3}} \beta_1(k_1)\beta_2(k_2)\beta_3(k_3) I_2\Big|. \tag{2.21}$$

Obviously

$$\mathcal{U} \ll \mathcal{U}_1 + \mathcal{U}_2. \tag{2.22}$$

We study $\mathcal{U}_2$ in Section 4 and $\mathcal{U}_1$ in Section 5 and we prove that

$$\mathcal{U}_1, \mathcal{U}_2 \ll X^3\mathcal{L}^{-A}. \tag{2.23}$$

The estimate (2.19) is a consequence of (2.22) and (2.23).

Note that only in the proof of the inequality (4.4) do we need the tight restriction on K_3 imposed by (2.17). So the validity of (4.4) for larger values of K_3 would certainly imply an improvement of Theorem 1.

3. Proof of Theorem 1.

Let $\mathcal{F}$ be the set defined in Theorem 1. We put

$$Q_0 = \mathcal{L}^{0.6}, \qquad z_1 = z_2 = X^{0.167}, \qquad z_3 = X^{0.116}. \tag{3.1}$$

Let $\mathfrak{R} = \{p \ge 11 : p\nmid n-4\} \cup \{p\ge 11 : p\,|\,n-4,\ p\equiv 1 \pmod 4\}$. We define

$$\mathcal{B}_0 = \prod_{3\le p<Q_0} p, \quad \mathcal{P}_0 = \prod_{\substack{Q_0\le p<Q\\ p\in\mathfrak{R}}} p, \quad \mathcal{P}_i = \prod_{Q\le p<z_i} p, \quad i=1,2,3. \tag{3.2}$$

Consider the sum

$$\Gamma = \sum_{n\in\mathcal{F}} \sum_{\substack{p_1^2+p_2^2+p_3^2=n \\ (p_i+2,\,\mathcal{B}_0\mathcal{P}_0\mathcal{P}_i)=1 \\ i=1,2,3}} \log p_1 \log p_2 \log p_3 = \sum_{n\in\mathcal{F}} w(n), \tag{3.3}$$

say. Suppose that $w(n) > 0$ for some $n \in \mathcal{F}$. Then there exist primes p_1, p_2, p_3 satisfying the conditions imposed in the inner sum of formula (3.3). For one of them, p_1 say, we should have $(p_1 + 2, \prod_{p<z_1} p) > 1$, otherwise we would have $(p_i + 2, \prod_{p<z_i} p) = 1$ for $i = 1, 2, 3$, which would contradict the definitions of $\mathcal{F}$ and z_i.

If $p_1 = 2$ then $w(n) \ll \mathcal{L}^3 \sum_{n=m_1^2+m_2^2+4} 1$.

If $p_1 > 2$ then $p_1 + 2$ would have a prime factor $p > 2$ such that $p \mid n-4$ and $p \equiv 3 \pmod 4$. Hence $p_2^2 + p_3^2 \equiv 0 \pmod p$, which implies $p_2 = p_3 = p$ and, therefore $w(n) \ll \mathcal{L}^3 \sum_{p\mid n-4} 1$.

Consequently,

$$\Gamma \ll \mathcal{L}^3\Big(\sum_{m_1^2+m_2^2+4\le N} 1 + \sum_{n\le N} \tau(n-4)\Big) \ll X^2\mathcal{L}^4. \tag{3.4}$$

Now we will use the vector sieve to estimate Γ from below. First we get rid of the summands corresponding to integers n such that $n-4$ has many distinct prime factors. From this point onwards $\sum^{\#}$ stands for a sum over n such that $\nu(n-4) \le A \log \mathcal{L}$. For technical reasons we sieve separately by the primes from the intervals $[3, Q_0)$, $[Q_0, Q)$ and $[Q, \infty)$. From the basic property of Möbius' function we get

$$\Gamma \ge \sum_{n\in\mathcal{F}}{}^{\#} \sum_{p_1^2+p_2^2+p_3^2=n} \log p_1 \log p_2 \log p_3\, \Phi_1\Phi_2\Phi_3\Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6, \tag{3.5}$$

where

$$\Phi_i = \sum_{d\mid (p_i+2,\mathcal{B}_0)} \mu(d), \qquad i = 1, 2, 3;$$

$$\Lambda_i = \begin{cases} \sum_{d\mid (p_i+2,\mathcal{P}_i)} \mu(d) & \text{for } i = 1, 2, 3, \\ \sum_{d\mid (p_{i-3}+2,\mathcal{P}_0)} \mu(d) & \text{for } i = 4, 5, 6. \end{cases} \tag{3.6}$$

Define

$$\begin{aligned} D_1 &= D_2 = X^{1/2} \exp(-4\mathcal{L}^{0.6}), \\ D_3 &= X^{1/3} \exp(-4\mathcal{L}^{0.6}), \qquad D_0 = \exp(\mathcal{L}^{0.6}). \end{aligned} \tag{3.7}$$

By $\lambda_i^{\pm}(d)$ we denote Rosser's weights of order D_i, $0 \le i \le 3$ (see Iwaniec [11], [12] for the definition). In particular, we have

$$|\lambda_i^{\pm}(d)| \le 1, \qquad \lambda_i^{\pm}(d) = 0 \quad \text{for } d \ge D_i, \qquad 0 \le i \le 3. \tag{3.8}$$

Denote

$$\Lambda_i^{\pm} = \begin{cases} \sum\limits_{d|(p_i+2,\mathcal{P}_i)} \lambda_i^{\pm}(d) & \text{for } i = 1,2,3, \\ \sum\limits_{d|(p_{i-3}+2,\mathcal{P}_0)} \lambda_i^{\pm}(d) & \text{for } i = 4,5,6. \end{cases} \tag{3.9}$$

By the properties of Rosser's weights (see Iwaniec [11], [12]) we have $\Lambda_i^- \le \Lambda_i \le \Lambda_i^+$, $1 \le i \le 6$. We apply the inequality

$$\begin{aligned}\Lambda_1\Lambda_2\Lambda_3\Lambda_4\Lambda_5\Lambda_6 \ge{}& \Lambda_1^-\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^+\Lambda_6^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+\Lambda_4^+\Lambda_5^+\Lambda_6^+ \\ &+ \Lambda_1^+\Lambda_2^+\Lambda_3^-\Lambda_4^+\Lambda_5^+\Lambda_6^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^-\Lambda_5^+\Lambda_6^+ \\ &+ \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^-\Lambda_6^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^+\Lambda_6^- \\ &- 5\Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^+\Lambda_6^+.\end{aligned}$$

The proof is the same as in Lemma 13 of [1]. Using this inequality and (3.5) we get

$$\Gamma \ge \sum_{i=1}^{6} \Gamma_i - 5\Gamma_7, \tag{3.10}$$

where

$$\Gamma_1 = \sum_{n\in\mathcal{F}}{}^{\#} \sum_{p_1^2+p_2^2+p_3^2=n} \log p_1 \log p_2 \log p_3\, \Phi_1\Phi_2\Phi_3\Lambda_1^-\Lambda_2^+\Lambda_3^+\Lambda_4^+\Lambda_5^+\Lambda_6^+.$$

The definition of the other sums Γ_i is clear. We change the order of summation to get

$$\begin{aligned}\Gamma_1 = \sum_{n\in\mathcal{F}}{}^{\#} \sum_{\substack{\nu_i|\mathcal{B}_0,\, \delta_i|\mathcal{P}_0 \\ d_i|\mathcal{P}_i,\, i=1,2,3}} & \mu(\nu_1)\mu(\nu_2)\mu(\nu_3)\lambda_1^-(d_1)\lambda_2^+(d_2)\lambda_3^+(d_3) \\ &\times \lambda_0^+(\delta_1)\lambda_0^+(\delta_2)\lambda_0^+(\delta_3) I(n;\nu_1\delta_1 d_1, \nu_2\delta_2 d_2, \nu_3\delta_3 d_3),\end{aligned}$$

where $I(n;k_1,k_2,k_3)$ is defined by (2.1).

Using formula (2.16) we split Γ_1 into two parts:

$$\Gamma_1 = \Gamma_1' + \Gamma_1'', \tag{3.11}$$

where Γ_1' and Γ_1'' are the contributions from the main term and error term of the formula (2.16) respectively.

Consider Γ_1''. We write it in the form

$$\Gamma_1'' = \sum_{n\in\mathcal{F}}{}^{\#} \sum_{\substack{k_i \le \mathcal{B}_0 D_0 D_i \\ i=1,2,3}} \gamma_1(k_1)\gamma_2(k_2)\gamma_3(k_3)\mathfrak{R}(n;Q;k_1,k_2,k_3),$$

where

$$\gamma_1(k) = \sum_{\substack{\nu|\mathcal{B}_0,\, \delta|\mathcal{P}_0,\, d|\mathcal{P}_1 \\ \nu\delta d=k}} \mu(\nu)\lambda_0^+(\delta)\lambda_1^-(d),$$

$$\gamma_i(k) = \sum_{\substack{\nu \mid \mathcal{B}_0,\, \delta \mid \mathcal{P}_0,\, d \mid \mathcal{P}_i \\ \nu\delta d = k}} \mu(\nu)\lambda_0^+(\delta)\lambda_i^+(d) \quad \text{for } i = 2, 3.$$

Now we use (3.1), (3.2), (3.7), (3.8) and apply the Proposition to find that

$$\Gamma_1'' \ll X^3\mathcal{L}^{-A}. \tag{3.12}$$

Consider Γ_1'. Using the definitions (2.8) and (2.15) of $t(q)$ and $\mathfrak{S}$, respectively, we find that if $\nu_i \mid \mathcal{B}_0$, $\delta_i \mid \mathcal{P}_0$ and $d_i \mid \mathcal{P}_i$, $i = 1, 2, 3$, then

$$\begin{aligned}&\mathfrak{S}(n; Q; \nu_1\delta_1 d_1, \nu_2\delta_2 d_2, \nu_3\delta_3 d_3) \\ &\quad = 8 \prod_{3 \le p < Q_0} (1 + t(p; n; \nu_1, \nu_2, \nu_3)) \prod_{Q_0 \le p < Q} (1 + t(p; n; \delta_1, \delta_2, \delta_3)).\end{aligned}$$

So, after some calculations we find that

$$\Gamma_1' = 2\pi \sum_{n \in \mathcal{F}}{}^{\#} \sqrt{n} \Big(\prod_{3 \le p < Q_0} \mathcal{V}_p(n) \Big) \mathcal{H}^+(n) \mathcal{G}_1^- \mathcal{G}_2^+ \mathcal{G}_3^+, \tag{3.13}$$

where

$$\mathcal{V}_p(n) = \sum_{\nu_1, \nu_2, \nu_3 \mid p} \frac{\mu(\nu_1)\mu(\nu_2)\mu(\nu_3)}{\varphi(\nu_1)\varphi(\nu_2)\varphi(\nu_3)} (1 + t(p; n; \nu_1, \nu_2, \nu_3)),$$

$$\mathcal{H}^\pm(n) = \sum_{\delta_1, \delta_2, \delta_3 \mid \mathcal{P}_0} \frac{\lambda_0^\pm(\delta_1)\lambda_0^+(\delta_2)\lambda_0^+(\delta_3)}{\varphi(\delta_1)\varphi(\delta_2)\varphi(\delta_3)} \prod_{Q_0 \le p < Q} (1 + t(p; n; \delta_1, \delta_2, \delta_3)),$$

$$\mathcal{G}_i^\pm = \sum_{d \mid \mathcal{P}_i} \frac{\lambda_i^\pm(d)}{\varphi(d)}, \qquad i = 1, 2, 3.$$

We treat the sums $\Gamma_i, 2 \le i \le 7$, in the same manner and we find formulas similar to (3.11)–(3.13). Then we apply (3.10) to get

$$\begin{aligned}\Gamma \ge\ & 2\pi \sum_{n \in \mathcal{F}}{}^{\#} \sqrt{n} \Big(\prod_{3 \le p < Q_0} \mathcal{V}_p(n) \Big) \\ & \times (\mathcal{H}^+(n)(\mathcal{G}_1^- \mathcal{G}_2^+ \mathcal{G}_3^+ + \mathcal{G}_1^+ \mathcal{G}_2^- \mathcal{G}_3^+ + \mathcal{G}_1^+ \mathcal{G}_2^+ \mathcal{G}_3^- - 5\mathcal{G}_1^+ \mathcal{G}_2^+ \mathcal{G}_3^+) \\ & + 3\mathcal{H}^-(n) \mathcal{G}_1^+ \mathcal{G}_2^+ \mathcal{G}_3^+) + \mathcal{O}(X^3\mathcal{L}^{-A}).\end{aligned} \tag{3.14}$$

Using (2.7), (2.8) we establish that

$$\mathcal{V}_p(n) = \frac{p}{(p-1)^3} \sum_{\substack{1 \le m_1, m_2, m_3 \le p-1 \\ m_1, m_2, m_3 \ne p-2 \\ m_1^2 + m_2^2 + m_3^2 \equiv n \ (p)}} 1.$$

This formula gives $0.001 \le \mathcal{V}_p(n) \le 3$ for $p = 3, 5, 7$ and 11. By the definition of $\mathcal{V}_p(n)$ and (2.14) we find another expression:

$$\mathcal{V}_p(n) = 1 + h_0(p) - 3\frac{1 + h_1(p)}{p - 1} + 3\frac{1 + h_2(p)}{(p-1)^2} - \frac{1 + h_3(p)}{(p-1)^3}.$$

Using this formula and (2.10)–(2.13) we find that $1-9(p-1)^{-1}\leq\mathcal{V}_p(n)\leq 1+9(p-1)^{-1}$ for $p>11$. From the observations above and the definition (3.1) of Q_0 we obtain

$$(\log\mathcal{L})^{-9}\ll\prod_{3\leq p<Q_0}\mathcal{V}_p(n)\ll(\log\mathcal{L})^9. \tag{3.15}$$

We leave the computations to the reader.

Consider the other quantities included in formula (3.14). Obviously

$$\mathcal{G}_i^{\pm}\ll\mathcal{L},\qquad i=1,2,3. \tag{3.16}$$

We have $\log D_0/\log Q\to\infty$ as $X\to\infty$. Hence we may expect that the sums $\mathcal{H}^{\pm}(n)$ can be approximated by

$$\mathcal{H}_0(n)=\sum_{\delta_1,\delta_2,\delta_3|\mathcal{P}_0}\frac{\mu(\delta_1)\mu(\delta_2)\mu(\delta_3)}{\varphi(\delta_1)\varphi(\delta_2)\varphi(\delta_3)}\prod_{Q_0\leq p<Q}(1+t(p;n;\delta_1,\delta_2,\delta_3)).$$

More precisely, we will prove that uniformly for $n\in\mathcal{F}$ satisfying $\nu(n-4)\leq A\log\mathcal{L}$ the following formula holds:

$$\mathcal{H}^{\pm}(n)=\mathcal{H}_0(n)+\mathcal{O}(\mathcal{L}^{-2A}). \tag{3.17}$$

We present the proof of (3.17) at the end of this section.

The sum $\mathcal{H}_0(n)$ is much more easy to deal with. We use (2.8)–(2.14) and after some elementary considerations we represent it as a product:

$$\mathcal{H}_0(n)=\prod_{\substack{Q_0\leq p<Q\\ p\nmid\mathcal{P}_0}}(1+h_0(p))\prod_{p|\mathcal{P}_0}\mathcal{V}_p(n).$$

Now we are able to verify that

$$(\log\mathcal{L})^{-14}\ll\mathcal{H}_0(n)\ll(\log\mathcal{L})^{14}. \tag{3.18}$$

Using (3.14)–(3.17) we get

$$\varGamma\geq 2\pi\sum_{n\in\mathcal{F}}{}^{\#}\sqrt{n}\Big(\prod_{3\leq p<Q_0}\mathcal{V}_p(n)\Big)\mathcal{H}_0(n)\mathfrak{N}+\mathcal{O}(X^3\mathcal{L}^{-A}), \tag{3.19}$$

where

$$\mathfrak{N}=\mathcal{G}_1^-\mathcal{G}_2^+\mathcal{G}_3^+ +\mathcal{G}_1^+\mathcal{G}_2^-\mathcal{G}_3^+ +\mathcal{G}_1^+\mathcal{G}_2^+\mathcal{G}_3^- -2\mathcal{G}_1^+\mathcal{G}_2^+\mathcal{G}_3^+.$$

Arguing as in Section 8 of [23] we get

$$\mathfrak{N}\gg(\log\mathcal{L})^3\mathcal{L}^{-3}. \tag{3.20}$$

Therefore using (3.15), (3.18)–(3.20) we find that

$$\varGamma\geq\mathcal{L}^{-5}\sum_{n\in\mathcal{F}}{}^{\#}\sqrt{n}+\mathcal{O}(X^3\mathcal{L}^{-A}).$$

We combine the last estimate with (3.4) to obtain

$$\sum_{n\in\mathcal{F}}{}^{\#}\sqrt{n} \ll X^3\mathcal{L}^{5-A}.$$

Denote by $\mathcal{Y}^{\#}(N)$ the cardinality of the set $\{n \in \mathcal{F} : \nu(n-4) \le A\log\mathcal{L}\}$. From the last formula we get

$$\mathcal{Y}^{\#}(N) \ll X^2\mathcal{L}^{5-A/2}.$$

It remains to notice that $\mathcal{Y}(N) - \mathcal{Y}^{\#}(N) \ll X^2\mathcal{L}^{-A\log A+A-1}$ (see Hall and Tenenbaum [5], Chapter 0, for example). Therefore

$$\mathcal{Y}(N) \ll X^2\mathcal{L}^{5-A/2}.$$

This proves Theorem 1.

It remains to establish the asymptotic formula (3.17). Consider, for example, the sum $\mathcal{H}^+(n)$. We have

$$\mathcal{H}^+(n) = \mathcal{H}' + \mathcal{H}'', \tag{3.21}$$

where in $\mathcal{H}'$ we sum over $\delta_1, \delta_2, \delta_3$ such that $(\delta_1, \delta_2), (\delta_1, \delta_3), (\delta_2, \delta_3) \le \mathcal{L}^{6A}$. The sum $\mathcal{H}''$ is the contribution from the other summands. Using (2.10)–(2.14) we may easily estimate the product from the formula for $\mathcal{H}^+(n)$ to get

$$\mathcal{H}'' \ll \mathcal{L} \sum_{\substack{\delta_1,\delta_2,\delta_3 \mid \mathcal{P}_0\\ (\delta_1,\delta_2)>\mathcal{L}^{6A}}} \frac{\mu^2(\delta_1)\mu^2(\delta_2)\mu^2(\delta_3)}{\varphi(\delta_1)\varphi(\delta_2)\varphi(\delta_3)}\tau^4(\delta_1)\tau^4(\delta_2)\tau^4(\delta_3)(\delta_1,\delta_2,\delta_3).$$

After some standard calculations, which we leave to the reader, we find that

$$\mathcal{H}'' \ll \mathcal{L}^{-2A}. \tag{3.22}$$

Consider now $\mathcal{H}'$. We have $\prod_{Q_0\le p<Q}(1+t(p)) = \Pi_0\Pi_1\Pi_2\Pi_3$, where Π_ν denotes the product of the primes dividing exactly ν of the integers $\delta_1, \delta_2, \delta_3$. It is clear that Π_2 and Π_3 are actually functions of $(\delta_2, \delta_3), (\delta_1, \delta_3), (\delta_1, \delta_2)$. Consider Π_0 and Π_1. For $h_0(p)$ defined by (2.10), we have $1 + h_0(p) > 0$ for any prime $p \ge Q_0$. The product $\mathcal{P}_0$ defined by (3.2) does not contain prime factors $p > 2$ such that $p \mid n-4$ and $p \equiv 3 \pmod 4$. Hence for any $p \mid \mathcal{P}_0$ we also have $1 + h_1(p) > 0$. We use the inclusion-exclusion principle and find that

$$\Pi_0\Pi_1 = \xi(n)\prod_{p\mid\delta_1}\frac{1+h_1(p)}{1+h_0(p)}\prod_{p\mid\delta_2}\frac{1+h_1(p)}{1+h_0(p)}\prod_{p\mid\delta_3}\frac{1+h_1(p)}{1+h_0(p)}\Pi',$$

where

$$\xi(n) = \prod_{Q_0\le p<Q}(1+h_0(p)), \tag{3.23}$$

and where Π' is actually a function of $(\delta_2, \delta_3), (\delta_1, \delta_3), (\delta_1, \delta_2)$. So we may write

$$\mathcal{H}' = \xi(n) \sum_{\substack{\delta_1,\delta_2,\delta_3 \mid \mathcal{P}_0 \\ (\delta_i,\delta_j) \le \mathcal{L}^{6A} \\ 1 \le i < j \le 3}} \kappa((\delta_2, \delta_3), (\delta_1, \delta_3), (\delta_1, \delta_2)) \prod_{\nu=1}^{3} \left(\lambda_0^+(\delta_\nu) \frac{\omega(\delta_\nu)}{\delta_\nu} \right), \tag{3.24}$$

where κ arises from Π', Π_2, Π_3 and where

$$\omega(k) = \begin{cases} \dfrac{k}{\varphi(k)} \displaystyle\prod_{p \mid k} \frac{1 + h_1(p)}{1 + h_0(p)} & \text{if } (k, 2\mathcal{B}_0) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.25}$$

We may easily find an explicit formula for $\kappa(l_1, l_2, l_3)$. Then we use (2.10)–(2.13) to find that

$$\kappa(l_1, l_2, l_3) \ll (l_1 l_2 l_3)^{10}. \tag{3.26}$$

In fact, a much sharper estimate is available. We leave the calculations to the reader.

We use (3.24) to represent $\mathcal{H}'$ as follows:

$$\begin{aligned} \mathcal{H}' &= \xi(n) \sum_{\substack{l_1,l_2,l_3 \mid \mathcal{P}_0 \\ l_1,l_2,l_3 \le \mathcal{L}^{6A}}} \kappa(l_1, l_2, l_3) \\ &\quad \times \sum_{\substack{\delta_1,\delta_2,\delta_3 \mid \mathcal{P}_0 \\ (\delta_2,\delta_3)=l_1,\, (\delta_1,\delta_3)=l_2 \\ (\delta_1,\delta_2)=l_3}} \prod_{\nu=1}^{3} \left(\lambda_0^+(\delta_\nu) \frac{\omega(\delta_\nu)}{\delta_\nu} \right) \\ &= \xi(n) \sum_{\substack{l_1,l_2,l_3 \mid \mathcal{P}_0 \\ l_1,l_2,l_3 \le \mathcal{L}^{6A}}} \kappa(l_1, l_2, l_3) \\ &\quad \times \sum_{\substack{\delta_1,\delta_2,\delta_3 \mid \mathcal{P}_0 \\ \delta_1 \equiv 0\, ([l_2,l_3]),\, \delta_2 \equiv 0\, ([l_1,l_3]) \\ \delta_3 \equiv 0\, ([l_1,l_2])}} \prod_{\nu=1}^{3} \left(\lambda_0^+(\delta_\nu) \frac{\omega(\delta_\nu)}{\delta_\nu} \right) \\ &\quad \times \sum_{h_1 \mid (\delta_2/l_1, \delta_3/l_1)} \mu(h_1) \sum_{h_2 \mid (\delta_1/l_2, \delta_3/l_2)} \mu(h_2) \sum_{h_3 \mid (\delta_1/l_3, \delta_2/l_3)} \mu(h_3) \\ &= \xi(n) \sum_{\substack{l_1,l_2,l_3 \mid \mathcal{P}_0 \\ l_1,l_2,l_3 \le \mathcal{L}^{6A}}} \kappa(l_1, l_2, l_3) \sum_{\substack{h_i \mid \mathcal{P}_0/l_i \\ i=1,2,3}} \mu(h_1)\mu(h_2)\mu(h_3) \mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3, \end{aligned} \tag{3.27}$$

where

$$\mathcal{D}_i = \sum_{\substack{\delta | \mathcal{P}_0 \\ \delta \equiv 0\, (\varrho_i)}} \lambda_0^+(\delta) \frac{\omega(\delta)}{\delta}, \qquad i = 1, 2, 3$$

and

$$\varrho_1 = [l_2 h_2, l_3 h_3], \qquad \varrho_2 = [l_1 h_1, l_3 h_3], \qquad \varrho_3 = [l_1 h_1, l_2 h_2]. \tag{3.28}$$

It is not difficult to see that the function $\omega(k)$ defined by (3.25) satisfies

$$\prod_{w_1 \le p < w_2} \left(1 - \frac{\omega(p)}{p} \right)^{-1} \le \frac{\log w_2}{\log w_1} \left(1 + \frac{c}{\log w_1} \right),$$

for some constant $c > 0$ and for arbitrary $2 \le w_1 < w_2$. Only at this point do we use the fact that the integers n satisfy $\nu(n-4) \le A \log \mathcal{L}$. We note that $\log D_0 / \log Q \ge \sqrt{\mathcal{L}}$. Therefore we may use Lemma 11 of [1] to get

$$\mathcal{D}_i = \mathcal{E}_i + \mathcal{O}(\tau(\varrho_i) \exp(-\sqrt{\mathcal{L}})), \qquad i = 1, 2, 3, \tag{3.29}$$

where

$$\mathcal{E}_i = \sum_{\substack{\delta | \mathcal{P}_0 \\ \delta \equiv 0\, (\varrho_i)}} \mu(\delta) \frac{\omega(\delta)}{\delta}, \qquad i = 1, 2, 3.$$

It is also easy to see that the sums $\mathcal{D}_i$ and $\mathcal{E}_i$ defined above satisfy

$$|\mathcal{D}_i|, |\mathcal{E}_i| \ll \mu^2(\varrho_i) \tau^3(\varrho_i) \varrho_i^{-1} \mathcal{L}. \tag{3.30}$$

We replace the product $\mathcal{D}_1 \mathcal{D}_2 \mathcal{D}_3$ from (3.27) by $\mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3$ and denote the new sum by $\mathcal{H}^*$. Proceeding as in Section 7 of [23] and using (3.23), (3.26)–(3.30) we get

$$\mathcal{H}' = \mathcal{H}^* + \mathcal{O}(\mathcal{L}^{-2A}). \tag{3.31}$$

To study $\mathcal{H}^*$ we apply the procedures above in reverse order and we obtain

$$\mathcal{H}^* = \mathcal{H}_0(n) + \mathcal{O}(\mathcal{L}^{-2A}). \tag{3.32}$$

Formula (3.17) for $\mathcal{H}^+(n)$ is a consequence of (3.21), (3.22), (3.31) and (3.32).

4. Proof of the Proposition—minor arcs. The object of this section is to prove the inequality (2.23) for $\mathcal{U}_2$. We substitute the expression for I_2, given by (2.6), in formula (2.21) and change the order of summation and integration to obtain

$$\mathcal{U}_2 = \sum_{n \le N} \Big| \int_{E_2} \mathcal{K}_1(\alpha) \mathcal{K}_2(\alpha) \mathcal{K}_3(\alpha) e(-n\alpha)\, d\alpha \Big|,$$

where

$$\mathcal{K}_i(\alpha)=\sum_{k\le K_i}\beta_i(k)S_k(\alpha),\qquad i=1,2,3. \tag{4.1}$$

We apply the Cauchy and Bessel inequalities to get

$$\begin{aligned}\mathcal{U}_2^2&\ll N\sum_{n\le N}\Big|\int_{E_2}\mathcal{K}_1(\alpha)\mathcal{K}_2(\alpha)\mathcal{K}_3(\alpha)e(-n\alpha)\,d\alpha\Big|^2\\&\ll N\int_{E_2}|\mathcal{K}_1(\alpha)\mathcal{K}_2(\alpha)\mathcal{K}_3(\alpha)|^2\,d\alpha\\&\ll N(\max_{\alpha\in E_2}|\mathcal{K}_3(\alpha)|)^2\int_0^1|\mathcal{K}_1(\alpha)\mathcal{K}_2(\alpha)|^2\,d\alpha\\&\ll N(\max_{\alpha\in E_2}|\mathcal{K}_3(\alpha)|)^2\Big(\int_0^1|\mathcal{K}_1(\alpha)|^4\,d\alpha+\int_0^1|\mathcal{K}_2(\alpha)|^4\,d\alpha\Big).\end{aligned} \tag{4.2}$$

To estimate the last expression we prove the inequalities

$$\int_0^1|\mathcal{K}_i(\alpha)|^4\,d\alpha\ll X^2\mathcal{L}^{1027},\qquad i=1,2, \tag{4.3}$$

and

$$\max_{\alpha\in E_2}|\mathcal{K}_3(\alpha)|\ll X\mathcal{L}^{-2A}. \tag{4.4}$$

Formula (2.23) for $\mathcal{U}_2$ is a consequence of (4.2)–(4.4).

First we prove (4.3). Denote the integral on the left-hand side of (4.3) by $\mathfrak{I}$. We use (2.2), (2.18) and (4.1) to get

$$\begin{aligned}\mathfrak{I}&=\int_0^1\sum_{k_1,\dots,k_4\le K_i}\beta_i(k_1)\beta_i(k_2)\overline{\beta_i(k_3)\beta_i(k_4)}S_{k_1}(\alpha)S_{k_2}(\alpha)S_{k_3}(-\alpha)S_{k_4}(-\alpha)\,d\alpha\\&=\sum_{k_1,\dots,k_4\le K_i}\beta_i(k_1)\beta_i(k_2)\overline{\beta_i(k_3)\beta_i(k_4)}\\&\qquad\times\int_0^1\sum_{\substack{p_1,\dots,p_4\le X\\ p_j+2\equiv 0\,(k_j),\,1\le j\le 4}}(\log p_1)\dots(\log p_4)e(\alpha(p_1^2+p_2^2-p_3^2-p_4^2))\,d\alpha\\&=\sum_{k_1,\dots,k_4\le K_i}\beta_i(k_1)\beta_i(k_2)\overline{\beta_i(k_3)\beta_i(k_4)}\\&\qquad\times\sum_{\substack{p_1,\dots,p_4\le X\\ p_j+2\equiv 0\,(k_j),\,1\le j\le 4\\ p_1^2+p_2^2=p_3^2+p_4^2}}(\log p_1)\dots(\log p_4)\end{aligned}$$

$$\ll \mathcal{L}^4 \sum_{k_1,\dots,k_4 \le K_i} \tau(k_1)\dots\tau(k_4) \sum_{\substack{n_1,\dots,n_4 \le X \\ n_j+2 \equiv 0\,(k_j),\, 1\le j\le 4 \\ n_1^2+n_2^2=n_3^2+n_4^2}} 1$$

$$= \mathcal{L}^4 \sum_{\substack{n_1,\dots,n_4 \le X \\ n_1^2+n_2^2=n_3^2+n_4^2}} \Big(\sum_{\substack{k_1 \le K_i \\ k_1 \mid n_1+2}} \tau(k_1)\Big) \dots \Big(\sum_{\substack{k_4 \le K_i \\ k_4 \mid n_4+2}} \tau(k_4)\Big)$$

$$\ll \mathcal{L}^4 \sum_{\substack{n_1,\dots,n_4 \le X \\ n_1^2+n_2^2=n_3^2+n_4^2}} \tau^2(n_1+2)\tau^2(n_2+2)\tau^2(n_3+2)\tau^2(n_4+2).$$

To estimate the last sum we apply the inequality $xyzt \le x^4+y^4+z^4+t^4$. Then we split the new sum into two parts to obtain

$$\mathfrak{I} \ll \mathcal{L}^4 \sum_{\substack{n_1,\dots,n_4 \le X \\ n_1^2+n_2^2=n_3^2+n_4^2}} \tau^8(n_1+2) \ll X^2\mathcal{L}^{259} + \mathcal{L}^4 U_0, \tag{4.5}$$

where

$$U_0 = \sum_{\substack{n_1,\dots,n_4 \le X \\ (n_1-n_3)(n_1+n_3)=(n_4-n_2)(n_4+n_2) \\ n_1\ne n_3,\, n_1\ne n_4,\, n_2\ne n_3,\, n_2 \ne n_4}} \tau^8(n_1+2). \tag{4.6}$$

We divide U_0 into two subsums:

$$U_0 = U_1 + U_2. \tag{4.7}$$

In the domain of summation of U_1 the condition $n_1 \ne n_3$ is replaced by $n_1 > n_3$, in U_2 it is replaced by $n_1 < n_3$.

Consider U_1. We have

$$U_1 = \sum_{\substack{h_1,\dots,h_4 \le 2X \\ h_1h_3=h_2h_4 \\ h_1\equiv h_3\,(2),\, h_2\equiv h_4\,(2)}} \sum_{\substack{n_1,\dots,n_4\le X \\ n_1-n_3=h_1,\, n_1+n_3=h_3 \\ n_4-n_2=h_2,\, n_4+n_2=h_4}} \tau^8(n_1+2)$$

$$\ll \sum_{\substack{h_1,\dots,h_4 \le 2X \\ h_1h_3=h_2h_4}} \tau^8(h_1+h_3+4)$$

$$= \sum_{k,l\le 2X} \sum_{\substack{h_1,\dots,h_4 \le 2X \\ h_1h_3=h_2h_4 \\ (h_1,h_2)=k,\, (h_3,h_4)=l}} \tau^8(h_1+h_3+4)$$

$$\ll \sum_{k,l\le 2X} \sum_{\substack{h_1,h_2\le (2X)/k;\ h_3,h_4 \le (2X)/l \\ h_1h_3=h_2h_4 \\ (h_1,h_2)=(h_3,h_4)=1}} \tau^8(h_1k+h_3l+4).$$

The conditions $(h_1,h_2)=(h_3,h_4)=1$, $h_1h_3=h_2h_4$ imply $h_1=h_4$, $h_2=h_3$. Hence

$$\begin{aligned} U_1 &\ll \sum_{k,l\le 2X} \sum_{h_1,h_2\le \min((2X)/k,(2X)/l)} \tau^8(h_1k+h_2l+4) \\ &= \sum_{m_1,m_2\le 2X} \tau^8(m_1+m_2+4) \sum_{\substack{k,l\le 2X \\ h_1,h_2\le\min((2X)/k,(2X)/l) \\ h_1k=m_1,\, h_2l=m_2}} 1 \\ &\ll \sum_{m_1,m_2\le 2X} \tau^8(m_1+m_2+4)\tau(m_1)\tau(m_2). \end{aligned}$$

Now we apply the inequality $x^8yz\le x^{10}+y^{10}+z^{10}$ to get

$$\begin{aligned} U_1 &\ll \sum_{m_1,m_2\le 2X} \tau^{10}(m_1+m_2+4) + \sum_{m_1,m_2\le 2X} \tau^{10}(m_1) \\ &\ll \sum_{l\le 4X+4} \tau^{10}(l) \sum_{\substack{m_1,m_2\le 2X \\ m_1+m_2+4=l}} 1 + X^2\mathcal{L}^{2^{10}-1} \ll X^2\mathcal{L}^{2^{10}-1}. \end{aligned} \tag{4.8}$$

We treat U_2 similarly to obtain

$$U_2 \ll X^2\mathcal{L}^{2^{10}-1}. \tag{4.9}$$

The inequality (4.3) follows from (4.5), (4.7)–(4.9).

Let us now prove (4.4). For simplicity we write K and $\beta(k)$ instead of K_3 and $\beta_3(k)$, respectively. We decompose $\mathcal{K}_3(\alpha)$ into $\mathcal{O}(\mathcal{L})$ sums of the form

$$\mathcal{K}(\alpha,Y) = \sum_{k\le K} \beta(k) \sum_{\substack{Y<p\le 2Y \\ p+2\equiv 0\,(k)}} \log p\, e(\alpha p).$$

We may assume that $X\mathcal{L}^{-2A-4}<Y\le X/2$, for otherwise we can use the trivial estimate for $\mathcal{K}(\alpha,Y)$. We have

$$\mathcal{K}(\alpha,Y) = W(Y,K,\alpha) + \mathcal{O}(X^{2/3}), \tag{4.10}$$

where

$$W(Y,K,\alpha) = \sum_{Y<n\le 2Y} \Lambda(n)e(\alpha n^2) \sum_{\substack{k\le K \\ k\mid n+2}} \beta(k).$$

We apply Heath-Brown's identity [7] to decompose $W(Y,K,\alpha)$ into $\mathcal{O}(\mathcal{L}^6)$ sums of two types.

Type I sums are

$$W_1 = \sum_{\substack{M<m\le M_1 \\ Y<ml\le 2Y}} \sum_{L<l\le L_1} a_m e(\alpha m^2 l^2) \sum_{\substack{k\le K \\ k\mid ml+2}} \beta(k)$$

and

$$W_1' = \sum_{\substack{M<m\le M_1 \\ Y<ml\le 2Y}} \sum_{L<l\le L_1} a_m (\log l) e(\alpha m^2 l^2) \sum_{\substack{k\le K \\ k|ml+2}} \beta(k),$$

where

(4.11) $M_1 \le 2M, \quad L_1 \le 2L, \quad ML \asymp Y, \quad L \ge Y^{0.498}, \quad |a_m| \ll \tau_5(m)\mathcal{L}.$

Type II sums are

$$W_2 = \sum_{\substack{M<m\le M_1 \\ Y<ml\le 2Y}} \sum_{L<l\le L_1} a_m b_l e(\alpha m^2 l^2) \sum_{\substack{k\le K \\ k|ml+2}} \beta(k),$$

where

$$\begin{gathered} M_1 \le 2M, \quad L_1 \le 2L, \quad ML \asymp Y, \quad Y^{0.001} \le L \le 2^{30} Y^{1/3}, \\ |a_m| \ll \tau_5(m)\mathcal{L}, \quad |b_l| \ll \tau_5(l)\mathcal{L}. \end{gathered} \tag{4.12}$$

Consider type II sums. We have

$$|W_2| \ll \mathcal{L} \sum_{M<m\le M_1} \tau_5(m) \Big| \sum_{\substack{L<l\le L_1 \\ Y<ml\le 2Y \\ ml+2\equiv 0\,(k)}} \sum_{k\le K} b_l \beta(k) e(\alpha m^2 l^2) \Big|.$$

An application of Cauchy's inequality gives

$$\begin{aligned} |W_2|^2 &\ll M\mathcal{L}^{26} \sum_{M<m\le M_1} \Big| \sum_{\substack{L<l\le L_1 \\ Y<ml\le 2Y \\ ml+2\equiv 0\,(k)}} \sum_{k\le K} b_l \beta(k) e(\alpha m^2 l^2) \Big|^2 \\ &= M\mathcal{L}^{26} \sum_{M<m\le M_1} \sum_{\substack{L<l_1,l_2\le L_1 \\ Y<l_1 m, l_2 m\le 2Y \\ l_i m+2\equiv 0\,(k_i),\, i=1,2}} \sum_{k_1,k_2\le K} b_{l_1} \overline{b}_{l_2} \\ &\quad \times \beta(k_1)\overline{\beta(k_2)} e(\alpha m^2 (l_1^2 - l_2^2)). \end{aligned}$$

Therefore, by (2.18) and (4.12),

$$|W_2|^2 \ll M\mathcal{L}^{28} \sum_{k_1,k_2\le K} \sum_{\substack{L<l_1,l_2\le L_1 \\ (k_1 k_2, 2)=(l_1,k_1)=(l_2,k_2)=1 \\ l_1\equiv l_2\,((k_1,k_2))}} \tau(k_1)\tau(k_2)\tau_5(l_1)\tau_5(l_2)|V|, \tag{4.13}$$

where

$$V = \sum_{\substack{M'<m\le M_1' \\ l_i m+2\equiv 0\,(k_i),\, i=1,2}} e(\alpha m^2 (l_1^2 - l_2^2)),$$

(4.14) $M' = \max(Y/l_1, Y/l_2, M), \quad M_1' = \min(2Y/l_1, 2Y/l_2, M_1).$

Note that if $l_1 \not\equiv l_2 \ ((k_1,k_2))$ then the system of congruences $l_i m + 2 \equiv 0 \ (k_i)$, $i = 1, 2$, is not solvable and, therefore, $V = 0$. Using only the basic properties of the congruences we easily find that if the conditions imposed on l_i, k_i in (4.13) hold, then there exists some integer $h_0 = h_0(l_1, l_2, k_1, k_2)$ satisfying $1 \le h_0 \le [k_1, k_2]$ and such that the system $l_i m + 2 \equiv 0\ (k_i)$, $i = 1, 2$, is equivalent to the congruence $m \equiv h_0\ ([k_1, k_2])$. In this case we have

$$\begin{aligned}|V| &= \Big| \sum_{\substack{M' < m \le M_1' \\ m \equiv h_0\ ([k_1,k_2])}} e(\alpha m^2 (l_1^2 - l_2^2)) \Big| \\ &= \Big| \sum_{H < r \le H_1} e(\alpha (h_0 + r[k_1, k_2])^2 (l_1^2 - l_2^2)) \Big| \\ &= \Big| \sum_{H < r \le H_1} e(\alpha (r^2 [k_1, k_2]^2 + 2h_0 r [k_1, k_2])(l_1^2 - l_2^2)) \Big|,\end{aligned}$$

where

$$H = \frac{M' - h_0}{[k_1, k_2]}, \qquad H_1 = \frac{M_1' - h_0}{[k_1, k_2]}. \tag{4.15}$$

The trivial estimate for the sum V is

$$|V| \ll \frac{M}{[k_1, k_2]}.$$

Note that, according to (2.17), (4.12) and our assumption $Y > X\mathcal{L}^{-2A-4}$, we have $[k_1, k_2] \ll M\mathcal{L}^{-200A}$. If the upper bound for K given by (4.3) were greater, for example $X^{1/3+\varepsilon}$ for some $\varepsilon > 0$, then our method would not work. Indeed, in this case the trivial estimate for V would be $|V| \ll 1$ for some k_1, k_2 and it would be difficult to find a non-trivial estimate for the sum W_2.

We easily see that the contribution of the summands with $l_1 = l_2$ in the expression on the right-hand side of (4.13) is

$$\ll M^2 \mathcal{L}^{28} \sum_{k_1, k_2 \le K} \frac{\tau(k_1)\tau(k_2)}{[k_1, k_2]} \sum_{L < l \le L_1} \tau_5^2(l) \ll M^2 L \mathcal{L}^{100}.$$

By the last observation, Cauchy's inequality and the estimate (4.13) we get

$$\begin{aligned}|W_2|^4 &\ll M^4 L^2 \mathcal{L}^{200} \\ &\quad + M^2 \mathcal{L}^{60} \Big(\sum_{k_1, k_2 \le K} \frac{\tau^2(k_1)\tau^2(k_2)}{[k_1, k_2]} \sum_{L < l_1, l_2 \le L_1} \tau_5^2(l_1)\tau_5^2(l_2) \Big) \\ &\quad \times \Big(\sum_{\substack{k_1, k_2 \le K \\ (k_1 k_2, 2) = 1}} [k_1, k_2] \sum_{\substack{L < l_1, l_2 \le L_1,\, l_1 \ne l_2 \\ (k_1, l_1) = (k_2, l_2) = 1 \\ l_1 \equiv l_2\ ((k_1, k_2))}} |V|^2 \Big) \\ &\ll M^4 L^2 \mathcal{L}^{200} + M^2 L^2 \mathcal{L}^{200} \Sigma_0,\end{aligned} \tag{4.16}$$

where

$$\Sigma_0 = \sum_{\substack{k_1,k_2\le K\\ (k_1k_2,2)=1}} [k_1,k_2] \sum_{\substack{L<l_1,l_2\le L_1,\, l_1\ne l_2\\ (k_1,l_1)=(k_2,l_2)=1\\ l_1\equiv l_2\ ((k_1,k_2))}} \times \sum_{H<r_1,r_2\le H_1} e(\alpha((r_1^2-r_2^2)[k_1,k_2]^2+2h_0[k_1,k_2](r_1-r_2))(l_1^2-l_2^2)).$$

We have

$$\begin{aligned}(4.17)\quad \Sigma_0 &= \sum_{\substack{k_1,k_2\le K\\ (k_1k_2,2)=1}} [k_1,k_2] \sum_{\substack{L<l_1,l_2\le L_1,\, l_1\ne l_2\\ (k_1,l_1)=(k_2,l_2)=1\\ l_1\equiv l_2\ ((k_1,k_2))}} \\ &\quad\times \sum_{s_1,s_2} e(\alpha(s_1s_2[k_1,k_2]^2+2h_0s_1[k_1,k_2])(l_1^2-l_2^2)) \sum_{\substack{H<r_1,r_2\le H_1\\ r_1-r_2=s_1\\ r_1+r_2=s_2}} 1 \\ &= \sum_{\substack{k_1,k_2\le K\\ (k_1k_2,2)=1}} [k_1,k_2] \sum_{\substack{L<l_1,l_2\le L_1,\, l_1\ne l_2\\ (k_1,l_1)=(k_2,l_2)=1\\ l_1\equiv l_2\ ((k_1,k_2))}} \\ &\quad\times \sum_{\substack{s_1,s_2\colon s_1\equiv s_2\ (2)\\ 2H<s_2+s_1\le 2H_1\\ 2H<s_2-s_1\le 2H_1}} e(\alpha(s_1s_2[k_1,k_2]^2+2h_0s_1[k_1,k_2])(l_1^2-l_2^2)) \\ &= \sum_{\substack{k_1,k_2\le K\\ (k_1k_2,2)=1}} [k_1,k_2] \sum_{\substack{L<l_1,l_2\le L_1,\, l_1\ne l_2\\ (k_1,l_1)=(k_2,l_2)=1\\ l_1\equiv l_2\ ((k_1,k_2))}} \\ &\quad\times \sum_{|s_1|\le 2H_1-2H} e(2\alpha h_0s_1[k_1,k_2](l_1^2-l_2^2)) \\ &\quad\times \sum_{\substack{s_2\colon s_2\equiv s_1\ (2)\\ 2H-s_1<s_2\le 2H_1-s_1\\ 2H+s_1<s_2\le 2H_1+s_1}} e(\alpha s_1s_2[k_1,k_2]^2(l_1^2-l_2^2)).\end{aligned}$$

Define

$$(4.18)\qquad K_0=\mathcal{L}^{50A}.$$

We divide the sum Σ_0 into two parts:

$$(4.19)\qquad \Sigma_0=\Sigma_1+\Sigma_2.$$

In Σ_1 the restriction $[k_1,k_2]\le K_0$ is imposed on the domain of summation over k_1,k_2, whilst in Σ_2 we sum over k_1,k_2 satisfying the condition $[k_1,k_2]>K_0$. According to (4.17) and the definitions above, we put $s_2=s_1+2t$ and obtain

$$\Sigma_1 \le \sum_{\substack{k_1,k_2\le K_0\\(k_1k_2,2)=1}} [k_1,k_2] \sum_{\substack{L<l_1,l_2\le L_1,\, l_1\ne l_2\\(k_1,l_1)=(k_2,l_2)=1\\ l_1\equiv l_2\ ((k_1,k_2))}} \times \sum_{|s_1|\le 2H_1-2H} \Big| \sum_{H'<t\le H_1'} e(2\alpha s_1[k_1,k_2]^2(l_1^2-l_2^2)t)\Big|, \tag{4.20}$$

$$\Sigma_2 \le \sum_{\substack{k_1,k_2\le K\\(k_1k_2,2)=1\\ [k_1,k_2]>K_0}} [k_1,k_2] \sum_{\substack{L<l_1,l_2\le L_1,\, l_1\ne l_2\\(k_1,l_1)=(k_2,l_2)=1\\ l_1\equiv l_2\ ((k_1,k_2))}} \times \sum_{|s_1|\le 2H_1-2H} \Big| \sum_{H'<t\le H_1'} e(2\alpha s_1[k_1,k_2]^2(l_1^2-l_2^2)t)\Big|, \tag{4.21}$$

where

$$H' = \max(H-s_1,H), \qquad H_1' = \min(H_1-s_1,H_1). \tag{4.22}$$

Consider first Σ_1. We have

$$\Sigma_1 = \Sigma_1^{(1)} + \Sigma_1^{(2)}, \tag{4.23}$$

where $\Sigma_1^{(1)}$ and $\Sigma_1^{(2)}$ denote the respective contributions of the summands with $s_1\ne 0$ and $s_1=0$ on the right-hand side of (4.20). Obviously

$$\Sigma_1^{(2)} \ll ML^2K_0^2. \tag{4.24}$$

Using the well known estimate for the linear exponential sums and (4.12), (4.14), (4.15), (4.20), (4.22) we get

$$\begin{aligned}
\Sigma_1^{(1)} &\ll \sum_{k_1,k_2\le K_0} [k_1,k_2] \sum_{\substack{L<l_1,l_2\le L_1\\ l_1\ne l_2}} \\
&\quad\times \sum_{0<|s|\le 2M/[k_1,k_2]} \min\left(\frac{M}{[k_1,k_2]}, \frac{1}{\|2\alpha(l_1^2-l_2^2)[k_1,k_2]^2 s\|}\right)\\
&\ll K_0^3 \sum_{h\le K_0^2} \sum_{\substack{L<l_1,l_2\le L_1\\ l_1\ne l_2}} \sum_{0<|s|\le 2M} \min\left(M, \frac{1}{\|2\alpha(l_1^2-l_2^2)h^2s\|}\right)\\
&= K_0^3 \sum_{h\le K_0^2} \sum_{t_1,t_2} \Bigg(\sum_{\substack{L<l_1,l_2\le L_1\\ l_1-l_2=t_1,\, l_1+l_2=t_2\\ l_1\ne l_2}} 1\Bigg) \sum_{0<|s|\le 2M} \min\left(M, \frac{1}{\|2\alpha t_1t_2h^2s\|}\right)\\
&\ll K_0^3 \sum_{h\le K_0^2} \sum_{\substack{0<|t_1|\le L\\ 1\le t_2\le 4L}} \sum_{0<|s|\le 2M} \min\left(M, \frac{1}{\|2\alpha t_1t_2h^2s\|}\right)
\end{aligned}$$

$$\ll K_0^3 \sum_{h\le K_0^2} \sum_{1\le t_1,t_2\le 4L} \sum_{1\le s\le 2M} \min\left(M, \frac{1}{\|2\alpha t_1 t_2 h^2 s\|}\right)$$

$$\ll K_0^3 \sum_{1\le m\le 64K_0^4L^2M} \tau_5(m) \min\left(M, \frac{1}{\|\alpha m\|}\right).$$

To get rid of $\tau_5(m)$ weights we apply Cauchy's inequality. Then we use Lemma 2.2 of Vaughan [27] and (2.3), (2.4), (4.12), (4.18) to obtain $\Sigma_1^{(1)} \ll M^2L^2\mathcal{L}^{-140A}$. We leave the calculations to the reader. The last estimate and (4.23), (4.24) give

$$\Sigma_1 \ll M^2L^2\mathcal{L}^{-140A}. \tag{4.25}$$

Consider now the sum Σ_2. According to (4.21) we have

$$\Sigma_2 \ll \mathcal{L} \max_{K_0\le T\le K^2} (T\Sigma_2^{(1)}), \tag{4.26}$$

where

$$\Sigma_2^{(1)} = \Sigma_2^{(1)}(T) = \sum_{\substack{k_1,k_2\le K\\ (k_1k_2,2)=1\\ T\le[k_1,k_2]\le 2T}} \sum_{\substack{L<l_1,l_2\le L_1,\, l_1\ne l_2\\ (k_1,l_1)=(k_2,l_2)=1\\ l_1\equiv l_2\ ((k_1,k_2))}} \times \sum_{|s_1|\le 2H_1-2H} \Big| \sum_{H'<t\le H_1'} e(2\alpha s_1[k_1,k_2]^2(l_1^2-l_2^2)t)\Big|.$$

The interval of summation over t in the sum above depends on the other variables, which is not convenient. To get rid of this dependence, we apply Lemma 2.2 of Bombieri and Iwaniec [2] and estimate $\Sigma_2^{(1)}$ by means of the mean value of a similar sum, in which the interval of summation over t does not depend on k_i, l_i, s_1. In the new sum we may already extend the domain of summation over k_i, l_i, s_1. After that the quantity under consideration does not decrease. More precisely, using (4.14), (4.15), (4.22) and the lemma mentioned above, we obtain

$$\begin{aligned} \Sigma_2^{(1)} &\le \sum_{\substack{k_1,k_2\le K\\ T\le[k_1,k_2]\le 2T}} \sum_{\substack{L<l_1,l_2\le L_1\\ l_1\ne l_2}} \sum_{|s|\le 2M/T} \\ &\quad\times \int_{-\infty}^{\infty} \mathcal{K}(\theta)\Big| \sum_{M/(4T)<t\le 4M/T} e(\theta t)e(2\alpha s[k_1,k_2]^2(l_1^2-l_2^2)t)\Big|\, d\theta \\ &= \int_{-\infty}^{\infty} \mathcal{K}(\theta)\Sigma_2^{(2)}(\theta,T)\, d\theta, \end{aligned} \tag{4.27}$$

where

$$\mathcal{K}(\theta) = \min(15M/(4T)+1, (\pi|\theta|)^{-1}, (\pi\theta)^{-2}) \tag{4.28}$$

and

$$\Sigma_2^{(2)} = \Sigma_2^{(2)}(\theta, T) = \sum_{\substack{k_1,k_2 \le K \\ T<[k_1,k_2]\le 2T}} \sum_{\substack{L<l_1,l_2\le L_1 \\ l_1\ne l_2}} \sum_{|s|\le 2M/T} \times \Big| \sum_{M/(4T)<t\le 4M/T} e(2\alpha s[k_1,k_2]^2(l_1^2-l_2^2)t+\theta t)\Big|.$$

From (4.27), (4.28) we get

(4.29) $$\Sigma_2^{(1)} \ll \mathcal{L} \max_{0\le\theta\le 1} \Sigma_2^{(2)}.$$

Consider $\Sigma_2^{(2)}$. We have

(4.30)
$$\begin{aligned}
\Sigma_2^{(2)} &= \sum_{T<h\le 2T} \Big(\sum_{\substack{k_1,k_2\le K \\ [k_1,k_2]=h}} 1\Big) \sum_{\substack{L<l_1,l_2\le L_1 \\ l_1\ne l_2}} \\
&\quad \times \sum_{|s|\le 2M/T} \Big| \sum_{M/(4T)<t\le 4M/T} e(2\alpha sh^2(l_1^2-l_2^2)t+\theta t)\Big| \\
&\ll \sum_{T<h\le 2T} \tau^2(h) \sum_{\substack{L<l_1,l_2\le L_1 \\ l_1\ne l_2}} \\
&\quad \times \sum_{|s|\le 2M/T} \Big| \sum_{M/(4T)<t\le 4M/T} e(2\alpha sh^2(l_1^2-l_2^2)t+\theta t)\Big| \\
&= \sum_{T<h\le 2T} \tau^2(h) \sum_{t_1,t_2} \Big(\sum_{\substack{L<l_1,l_2\le L_1 \\ l_1-l_2=t_1,\, l_1+l_2=t_2 \\ l_1\ne l_2}} 1\Big) \\
&\quad \times \sum_{|s|\le 2M/T} \Big| \sum_{M/(4T)<t\le 4M/T} e(2\alpha sh^2 t_1 t_2 t+\theta t)\Big| \\
&\ll \sum_{T<h\le 2T} \tau^2(h) \sum_{0<|t_1|,|t_2|\le 4L} \\
&\quad \times \sum_{|s|\le 2M/T} \Big| \sum_{M/(4T)<t\le 4M/T} e(2\alpha sh^2 t_1 t_2 t+\theta t)\Big| \\
&\ll \sum_{T<h\le 2T} \tau^2(h) \sum_{0<|t_1|,|t_2|\le 4L} \\
&\quad \times \sum_{0<|s|\le 2M/T} \Big| \sum_{M/(4T)<t\le 4M/T} e(2\alpha sh^2 t_1 t_2 t+\theta t)\Big| + ML^2\mathcal{L}^3 \\
&\ll ML^2\mathcal{L}^3 + \Sigma_2^{(3)},
\end{aligned}$$

where

$$\Sigma_2^{(3)} = \sum_{T<h\le 2T} \tau^2(h) \times \sum_{0<|m|\le 32ML^2/T} \tau^3(|m|) \Bigl| \sum_{M/(4T)<t\le 4M/T} e(2\alpha h^2 mt + \theta t) \Bigr|.$$

We use the Cauchy inequality to get

$$(\Sigma_2^{(3)})^2 \le \Bigl(\sum_{T<h\le 2T} \tau^4(h) \sum_{0<|m|\le 32ML^2/T} \tau^6(|m|) \Bigr) \Sigma_2^{(4)} \ll ML^2 \mathcal{L}^{100} \Sigma_2^{(4)}, \tag{4.31}$$

where

$$\Sigma_2^{(4)} = \sum_{T<h\le 2T} \sum_{0<|m|\le 32ML^2/T} \Bigl| \sum_{M/(4T)<t\le 4M/T} e(2\alpha h^2 mt + \theta t) \Bigr|^2.$$

For the last sum we have

$$\begin{aligned} \Sigma_2^{(4)} &= \sum_{T<h\le 2T} \sum_{0<|m|\le 32ML^2/T} \times \sum_{M/(4T)<t_1,t_2\le 4M/T} e((2\alpha m h^2 + \theta)(t_1 - t_2)) \\ &\ll \sum_{0<|m|\le 32ML^2/T} \sum_{M/(4T)<t_1,t_2\le 4M/T} \times \Bigl| \sum_{T<h\le 2T} e((2\alpha m h^2)(t_1 - t_2)) \Bigr| \\ &\ll \frac{M^2L^2}{T} + \frac{M}{T} \sum_{0<|m|\le 32ML^2/T} \sum_{0<|l|\le 4M/T} \Bigl| \sum_{T<h\le 2T} e(2\alpha m h^2 l) \Bigr| \\ &\ll \frac{M^2L^2}{T} + \frac{M}{T} \Sigma_2^{(5)}, \end{aligned} \tag{4.32}$$

where

$$\Sigma_2^{(5)} = \sum_{0<|s|\le 256M^2L^2/T^2} \tau(|s|) \Bigl| \sum_{T<h\le 2T} e(\alpha s h^2) \Bigr|.$$

By Cauchy's inequality we obtain

$$(\Sigma_2^{(5)})^2 \ll \Bigl(\sum_{1\le s\le 256M^2L^2/T^2} \tau^2(s) \Bigr) \times \Bigl(\sum_{1\le s\le 256M^2L^2/T^2} \Bigl| \sum_{T<h\le 2T} e(\alpha s h^2) \Bigr|^2 \Bigr) \tag{4.33}$$

$$\ll \frac{M^2L^2}{T^2}\mathcal{L}^3 \sum_{1\le s\le 256M^2L^2/T^2} \sum_{T<h_1,h_2\le 2T} e(\alpha s(h_1^2-h_2^2))$$
$$\ll \frac{M^4L^4}{T^3}\mathcal{L}^3 + \frac{M^2L^2}{T^2}\mathcal{L}^3|\Sigma_2^{(6)}|,$$

where

$$\Sigma_2^{(6)} = \sum_{1\le s\le 256M^2L^2/T^2} \sum_{\substack{T<h_1,h_2\le 2T\\ h_1\ne h_2}} e(\alpha s(h_1^2-h_2^2)).$$

Applying the estimate for the linear sums again we get

$$|\Sigma_2^{(6)}| = \Big|\sum_{m_1,m_2}\Big(\sum_{\substack{T<h_1,h_2\le 2T\\ h_1-h_2=m_1,\, h_1+h_2=m_2\\ h_1\ne h_2}} 1\Big)\sum_{1\le s\le 256M^2L^2/T^2} e(\alpha s m_1 m_2)\Big|$$
$$\ll \sum_{0<|m_1|,|m_2|\le 4T}\Big|\sum_{1\le s\le 256M^2L^2/T^2} e(\alpha s m_1 m_2)\Big|$$
$$\ll \sum_{1\le m_1,m_2\le 4T} \min\left(\frac{M^2L^2}{T^2}, \frac{1}{\|\alpha m_1 m_2\|}\right)$$
$$\ll \sum_{1\le m\le 16T^2} \tau(m)\min\left(\frac{M^2L^2}{T^2}, \frac{1}{\|\alpha m\|}\right).$$

Now we proceed as in the estimation of $\Sigma_1^{(1)}$ to get

(4.34) $$\Sigma_2^{(6)} \ll M^2L^2\mathcal{L}^{2-50A}.$$

The inequalities (2.17), (4.12), (4.18), (4.26), (4.29)–(4.34) imply

(4.35) $$\Sigma_2 \ll M^2L^2\mathcal{L}^{-12A}.$$

Taking into account (4.12), (4.16), (4.19), (4.25) and (4.35), we find that

(4.36) $$|W_2| \ll X\mathcal{L}^{50-3A}.$$

Let us now estimate type I sums. Consider, for example, the sum W_1. According to (2.18) and (4.11) we have

(4.37) $$|W_1| \ll \mathcal{L}^2 \max_{1/2\le T\le K} \Sigma_3,$$

where

$$\Sigma_3 = \Sigma_3(T) = \sum_{\substack{T<k\le 2T\\ (k,2)=1}} \tau(k) \sum_{\substack{M<m\le M_1\\ (m,k)=1}} \tau_5(m)\Big|\sum_{\substack{L'<l\le L_1'\\ ml+2\equiv 0\ (k)}} e(\alpha m^2 l^2)\Big|,$$

and

(4.38) $$L' = \max(L, Y/m), \qquad L_1' = \min(L_1, 2Y/m).$$

For any m coprime to k we define $\overline{m}$ by $m\overline{m} \equiv 1\ (k)$, $0 \le \overline{m} < k$. Let

$$R = (L' + 2\overline{m})/k, \qquad R_1 = (L_1' + 2\overline{m})/k. \tag{4.39}$$

By Cauchy's inequality we get

$$\begin{aligned}(\Sigma_3)^2 &\ll \Big(\sum_{T<k\le 2T} \tau^2(k) \sum_{M<m\le M_1} \tau_5^2(m) \Big) \\ &\quad \times \Big(\sum_{\substack{T<k\le 2T \\ (k,2)=1}} \sum_{\substack{M<m\le M_1 \\ (m,k)=1}} \Big| \sum_{R<r\le R_1} e(\alpha m^2(-2\overline{m} + rk)^2) \Big|^2 \Big) \\ &\ll MT\mathcal{L}^{100} \sum_{\substack{T<k\le 2T \\ (k,2)=1}} \sum_{\substack{M<m\le M_1 \\ (m,k)=1}} \Big| \sum_{R<r\le R_1} e(\alpha m^2(r^2k^2 - 4\overline{m}rk)) \Big|^2 \\ &= MT\mathcal{L}^{100} \sum_{\substack{T<k\le 2T \\ (k,2)=1}} \sum_{\substack{M<m\le M_1 \\ (m,k)=1}} \\ &\quad \times \sum_{R<r_1,r_2\le R_1} e(\alpha m^2(k^2(r_1^2 - r_2^2) - 4\overline{m}k(r_1 - r_2))) \\ &\ll M^2LT\mathcal{L}^{100} + MT\mathcal{L}^{100}|\Sigma_3^{(1)}|,\end{aligned} \tag{4.40}$$

where

$$\begin{aligned}\Sigma_3^{(1)} = &\sum_{\substack{T<k\le 2T \\ (k,2)=1}} \sum_{\substack{M<m\le M_1 \\ (m,k)=1}} \\ &\times \sum_{\substack{R<r_1,r_2\le R_1 \\ r_1\ne r_2}} e(\alpha m^2(k^2(r_1^2 - r_2^2) - 4\overline{m}k(r_1 - r_2))).\end{aligned}$$

We have

$$\begin{aligned}|\Sigma_3^{(1)}| &= \Big| \sum_{\substack{T<k\le 2T \\ (k,2)=1}} \sum_{\substack{M<m\le M_1 \\ (m,k)=1}} \\ &\quad \times \sum_{\substack{s_1,s_2 \\ s_1\ne 0}} e(\alpha m^2(k^2 s_1 s_2 - 4\overline{m}ks_1)) \sum_{\substack{R<r_1,r_2\le R_1 \\ r_1-r_2=s_1 \\ r_1+r_2=s_2}} 1 \Big| \\ &= \Big| \sum_{\substack{T<k\le 2T \\ (k,2)=1}} \sum_{\substack{M<m\le M_1 \\ (m,k)=1}} \\ &\quad \times \sum_{\substack{s_1,s_2:\, s_1\ne 0 \\ 2R<s_1+s_2,\, s_2-s_1\le 2R_1 \\ s_1\equiv s_2\ (2)}} e(\alpha m^2(k^2 s_1 s_2 - 4\overline{m}ks_1)) \Big|\end{aligned} \tag{4.41}$$

$$\ll \sum_{\substack{T<k\le 2T\\ (k,2)=1}} \sum_{\substack{M<m\le M_1\\ (m,k)=1}} \sum_{0<|s_1|\le 10L/T} \Big| \sum_{\substack{s_2\colon s_2\equiv s_1\,(2)\\ 2R-s_1<s_2\le 2R_1-s_1\\ 2R+s_1\le s_2\le 2R_1+s_1}} e(\alpha m^2k^2s_1s_2)\Big|$$

$$\ll \sum_{\substack{T<k\le 2T\\ (k,2)=1}} \sum_{\substack{M<m\le M_1\\ (m,k)=1}} \sum_{1\le s_1\le 10L/T} \Big| \sum_{\substack{R<t\le R_1\\ R-s_1<t\le R_1-s_1}} e(2\alpha m^2k^2s_1t)\Big|.$$

First we consider the case

$$MT\le K_0, \tag{4.42}$$

where K_0 is defined by (4.18). We apply Cauchy's inequality, Lemma 2.2 of Vaughan [27] and also (2.3), (2.4), (4.11), (4.18), (4.42) to get

$$\begin{aligned}|\Sigma_3^{(1)}| &\ll \sum_{T<k\le 2T} \sum_{M<m\le M_1} \sum_{1\le s\le 20L} \min\left(L, \frac{1}{\|2\alpha m^2k^2s\|}\right)\\ &\ll \sum_{1\le n\le 640K_0^2L} \tau^3(n)\min\left(L,\frac{1}{\|\alpha n\|}\right) \ll X^2\mathcal{L}^{-300A}.\end{aligned} \tag{4.43}$$

Hence, by (4.11), (4.40), (4.42), (4.43) we find that

$$\Sigma_3 \ll X\mathcal{L}^{-100A} \quad \text{if } MT\le K_0. \tag{4.44}$$

Consider now the case

$$MT > K_0. \tag{4.45}$$

Using (4.38), (4.39), (4.41) and Lemma 2.2 of Bombieri and Iwaniec [2] we obtain

$$|\Sigma_3^{(1)}| \ll \mathcal{L} \max_{0\le\theta\le 1} \Sigma_3^{(2)}, \tag{4.46}$$

where

$$\begin{aligned}\Sigma_3^{(2)} &= \Sigma_3^{(2)}(\theta,T)\\ &= \sum_{T<k\le 2T} \sum_{M<m\le M_1} \sum_{1\le s\le 10L/T} \Big| \sum_{L/(4T)<t\le 4L/T} e(2\alpha k^2m^2st+\theta t)\Big|.\end{aligned}$$

It is clear that

$$\Sigma_3^{(2)} \ll \sum_{MT<h\le 4MT} \tau(h) \sum_{1\le s\le 10L/T} \Big| \sum_{L/(4T)<t\le 4L/T} e(2\alpha h^2st+\theta t)\Big|.$$

Hence an application of Cauchy's inequality gives

$$(4.47)\qquad (\Sigma_3^{(2)})^2 \ll \Big(\sum_{MT<h\le 4MT} \tau^2(h) \sum_{1\le s\le 10L/T} 1\Big)$$

$$\times\Big(\sum_{MT<h\le 4MT} \sum_{1\le s\le 10L/T} \Big| \sum_{L/(4T)<t\le 4L/T} e(2\alpha h^2 st+\theta t)\Big|^2\Big)$$

$$\ll ML\mathcal{L}^3 \sum_{MT<h\le 4MT} \sum_{1\le s\le 10L/T} \sum_{L/(4T)<t_1,t_2\le 4L/T} e((2\alpha h^2 s+\theta)(t_1-t_2))$$

$$\ll ML\mathcal{L}^3 \sum_{1\le s\le 10L/T} \sum_{L/(4T)<t_1,t_2\le 4L/T} \Big| \sum_{MT<h\le 4MT} e(2\alpha h^2 s(t_1-t_2))\Big|$$

$$\ll \frac{M^2L^3}{T}\mathcal{L}^3 + ML\mathcal{L}^3\Sigma_3^{(3)},$$

where

$$\Sigma_3^{(3)} = \sum_{1\le s\le 10L/T} \sum_{\substack{L/(4T)<t_1,t_2\le 4L/T \\ t_1\ne t_2}} \Big| \sum_{MT<h\le 4MT} e(2\alpha h^2 s(t_1-t_2))\Big|.$$

For the last sum we have

$$\Sigma_3^{(3)} = \sum_{1\le s\le 10L/T} \sum_{\substack{0<|t|\le 4L/T \\ 1\le u\le 8L/T}} \Big(\sum_{\substack{L/(4T)<t_1,t_2\le 4L/T \\ t_1-t_2=t \\ t_1+t_2=u}} 1\Big) \Big| \sum_{MT<h\le 4MT} e(2\alpha h^2 st)\Big|$$

$$\ll \frac{L}{T} \sum_{1\le s\le 10L/T} \sum_{1\le t\le 4L/T} \Big| \sum_{MT<h\le 4MT} e(2\alpha h^2 st)\Big|$$

$$\ll \frac{L}{T} \sum_{1\le m\le 80L^2/T^2} \tau(m) \Big| \sum_{MT<h\le 4MT} e(\alpha h^2 m)\Big|.$$

Hence

$$(4.48)\qquad (\Sigma_3^{(3)})^2 \ll \frac{L^2}{T^2}\Big(\sum_{1\le m\le 80L^2/T^2} \tau^2(m)\Big)$$

$$\times\Big(\sum_{1\le m\le 80L^2/T^2} \Big| \sum_{MT<h\le 4MT} e(\alpha h^2 m)\Big|^2\Big)$$

$$\ll \frac{L^4}{T^4}\mathcal{L}^3 \sum_{1\le m\le 80L^2/T^2} \sum_{MT<h_1,h_2\le 4MT} e(\alpha(h_1^2-h_2^2)m)$$

$$\ll \frac{L^6M}{T^5}\mathcal{L}^3 + \frac{L^4}{T^4}\mathcal{L}^3\Sigma_3^{(4)},$$

where

$$\Sigma_3^{(4)} = \sum_{\substack{MT<h_1,h_2\le 4MT\\ h_1\ne h_2}} \Big| \sum_{1\le m\le 80L^2/T^2} e(\alpha(h_1^2-h_2^2)m)\Big|.$$

So we get as before

$$(4.49)\quad \Sigma_3^{(4)} = \sum_{0<|s_1|,|s_2|\le 8MT} \Big(\sum_{\substack{TM<h_1,h_2\le 4MT\\ h_1-h_2=s_1\\ h_1+h_2=s_2}} 1\Big) \times \Big| \sum_{1\le m\le 80L^2/T^2} e(\alpha s_1 s_2 m)\Big|$$

$$\ll \sum_{1\le s_1,s_2\le 8MT} \Big| \sum_{1\le m\le 80L^2/T^2} e(\alpha s_1 s_2 m)\Big|$$

$$\ll \sum_{1\le s_1,s_2\le 8MT} \min\Big(\frac{L^2}{T^2}, \frac{1}{\|\alpha s_1 s_2\|}\Big)$$

$$\ll \sum_{1\le s\le 64M^2T^2} \tau(s)\min\Big(\frac{L^2}{T^2}, \frac{1}{\|\alpha s\|}\Big) \ll M^2L^2\mathcal{L}^{2-50A}.$$

Using (2.17), (4.11), (4.18), (4.40), (4.45)–(4.49) we find that

$$(4.50)\qquad \Sigma_3 \ll X\mathcal{L}^{-6A} \quad \text{if } MT > K_0.$$

Hence by (4.37), (4.44) and (4.50) we obtain the estimate

$$(4.51)\qquad |W_1| \ll X\mathcal{L}^{2-6A}.$$

We treat type I sums W_1' in the same way and we find that

$$(4.52)\qquad |W_1'| \ll X\mathcal{L}^{2-6A}.$$

The estimate (4.4) follows from (4.10), (4.36), (4.51) and (4.52). Now the proof of the estimate (2.23) for $\mathcal{U}_2$ is complete.

5. Proof of the Proposition—major arcs. In this section we prove that for the sum $\mathcal{U}_1$, defined by (2.20), the estimate (2.23) holds. However, now we do not need such a restrictive upper bound for K_3, as in Section 4. Now we assume that

$$(5.1)\qquad K_i \le X^{1/2}\mathcal{L}^{-20000A}, \quad i=1,2,3.$$

According to (2.4) and (2.6) we have

$$(5.2)\qquad I_1 = \sum_{q<Q} \sum_{\substack{0\le a\le q-1\\ (a,q)=1}} H(a,q),$$

where

$$H(a,q)=\int\limits_{-1/(q\tau)}^{1/(q\tau)} S_{k_1}\left(\frac{a}{q}+\alpha\right)S_{k_2}\left(\frac{a}{q}+\alpha\right) \times S_{k_3}\left(\frac{a}{q}+\alpha\right)e\left(-n\left(\frac{a}{q}+\alpha\right)\right)d\alpha \tag{5.3}$$

and where $S_k(\alpha)$ is defined by (2.2). Denote

$$M(\alpha)=\sum_{m\le N}\frac{1}{2\sqrt{m}}e(\alpha m),\qquad \varDelta(y,h)=\max_{z\le y}\max_{(l,h)=1}\Bigg|\sum_{\substack{p\le z\\ p\equiv l\,(h)}}\log p-\frac{z}{\varphi(h)}\Bigg|$$

and let $s_k(a,q)$ be defined by (2.7). We write

$$S_k\left(\frac{a}{q}+\alpha\right)=\frac{s_k(a,q)}{\varphi(k)}M(\alpha)+\mathcal{G}(\alpha;k,q,a). \tag{5.4}$$

For α,a,q satisfying

$$|\alpha|\le(q\tau)^{-1},\qquad 0\le a<q<Q,\qquad (a,q)=1 \tag{5.5}$$

and for $k\le X^{1/2}\mathcal{L}^{-20000A}$ we have

$$\mathcal{G}(\alpha;k,q,a)\ll(1+\varDelta(X,[k,q]))\frac{X^2}{\tau}. \tag{5.6}$$

The calculations are similar to those in Section 4.1 of [21], so we do not present them here. We define

$$\varGamma_i(\alpha,q,a)=\sum_{k\le K_i}\beta_i(k)\mathcal{G}(\alpha;k,q,a),\qquad i=1,2,3. \tag{5.7}$$

By (5.6) we get

$$\begin{aligned}\max_{\substack{\alpha,q,a\\(5.5)}}|\varGamma_i(\alpha,q,a)| &\ll \frac{X^2}{\tau}\sum_{q\le Q}\sum_{k\le K_i}\tau(k)(1+\varDelta(X,[k,q]))\\ &\ll \frac{X^2}{\tau}\sum_{h\le K_iQ}\tau^3(h)(1+\varDelta(X,h)).\end{aligned}$$

Applying Cauchy's inequality and Bombieri–Vinogradov's theorem (Chapter 28 of Davenport [4]) and using (2.3), (5.1) we get

$$\max_{\substack{\alpha,q,a\\(5.5)}}|\varGamma_i(\alpha,q,a)|\ll X\mathcal{L}^{-7000A},\qquad i=1,2,3. \tag{5.8}$$

Define

$$\mathcal{S}_i=S_{k_i}\left(\frac{a}{q}+\alpha\right),\qquad \mathcal{M}_i=\frac{s_{k_i}(a,q)}{\varphi(k_i)}M(\alpha),\qquad \mathcal{G}_i=\mathcal{S}_i-\mathcal{M}_i. \tag{5.9}$$

We use (5.2)–(5.4), (5.9) and the identity

$$\mathcal{S}_1\mathcal{S}_2\mathcal{S}_3 = \mathcal{M}_1\mathcal{M}_2\mathcal{M}_3 + \mathcal{S}_1\mathcal{S}_2\mathcal{G}_3 + \mathcal{S}_1\mathcal{G}_2\mathcal{M}_3 + \mathcal{G}_1\mathcal{M}_2\mathcal{M}_3$$

to get

$$I_1 = J' + J_1 + J_2 + J_3, \tag{5.10}$$

where

$$J' = \sum_{q<Q} \sum_{\substack{0\le a\le q-1\\(a,q)=1}} \int\limits_{-1/(q\tau)}^{1/(q\tau)} \mathcal{M}_1\mathcal{M}_2\mathcal{M}_3 e\bigg(-n\bigg(\frac{a}{q}+\alpha\bigg)\bigg)\,d\alpha, \tag{5.11}$$

and where J_1, J_2 and J_3 are the contributions of the other summands.

Consequently,

$$\mathcal{U}_1 \ll \mathcal{U}' + \mathcal{Z}_1 + \mathcal{Z}_2 + \mathcal{Z}_3, \tag{5.12}$$

where

$$\mathcal{U}' = \sum_{n\le N}{}^{*} \bigg| \sum_{\substack{k_i\le K_i\\ i=1,2,3}} \beta_1(k_1)\beta_2(k_2)\beta_3(k_3)\bigg(J' - \frac{\pi}{4}\sqrt{n}\,\frac{\mathfrak{S}(n;Q;k_1,k_2,k_3)}{\varphi(k_1)\varphi(k_2)\varphi(k_3)}\bigg)\bigg|,$$

$$\mathcal{Z}_1 = \sum_{n\le N} \bigg| \sum_{\substack{k_i\le K_i\\ i=1,2,3}} \beta_1(k_1)\beta_2(k_2)\beta_3(k_3) \times \sum_{q<Q} \sum_{\substack{0\le a\le q-1\\(a,q)=1}} \int\limits_{-1/(q\tau)}^{1/(q\tau)} \mathcal{S}_1\mathcal{S}_2\mathcal{G}_3 e\bigg(-n\bigg(\frac{a}{q}+\alpha\bigg)\bigg)\,d\alpha\bigg|,$$

the definitions of $\mathcal{Z}_2$ and $\mathcal{Z}_3$ are clear. First we show that

$$\mathcal{Z}_i \ll X^3\mathcal{L}^{-A}, \quad i=1,2,3. \tag{5.13}$$

Consider, for example, $\mathcal{Z}_1$. We have

$$\mathcal{Z}_1 \ll \sum_{q<Q} \sum_{\substack{0\le a\le q-1\\(a,q)=1}} \sum_{n\le N} \bigg| \int\limits_{-1/(q\tau)}^{1/(q\tau)} \mathcal{K}_1\bigg(\frac{a}{q}+\alpha\bigg) \times \mathcal{K}_2\bigg(\frac{a}{q}+\alpha\bigg)\varGamma_3(\alpha,q,a) e\bigg(-n\bigg(\frac{a}{q}+\alpha\bigg)\bigg)\,d\alpha\bigg|,$$

where $\mathcal{K}_i(\alpha)$ are defined by (4.1) and $\varGamma_3(\alpha,q,a)$ by (5.7). We apply the Cauchy and Bessel inequalities to get

$$\mathcal{Z}_1^2 \ll Q^2X^2 \sum_{q<Q} \sum_{\substack{0\le a\le q-1\\(a,q)=1}} \sum_{n\le N} \bigg| \int\limits_{-1/(q\tau)}^{1/(q\tau)} \mathcal{K}_1\Big(\frac{a}{q}+\alpha\Big) \times \mathcal{K}_2\Big(\frac{a}{q}+\alpha\Big)\Gamma_3(\alpha,q,a)e\Big(-n\Big(\frac{a}{q}+\alpha\Big)\Big)\,d\alpha\bigg|^2$$

$$\ll Q^2X^2 \sum_{q<Q} \sum_{\substack{0\le a\le q-1\\(a,q)=1}} \int\limits_{-1/(q\tau)}^{1/(q\tau)} \bigg|\mathcal{K}_1\Big(\frac{a}{q}+\alpha\Big)\mathcal{K}_2\Big(\frac{a}{q}+\alpha\Big)\Gamma_3(\alpha,q,a)\bigg|^2\,d\alpha$$

$$\ll Q^2X^2 \max_{\substack{\alpha,q,a\\(5.5)}} |\Gamma_3(\alpha,q,a)|^2 \int\limits_0^1 |\mathcal{K}_1(\alpha)\mathcal{K}_2(\alpha)|^2\,d\alpha$$

$$\ll Q^2X^2 \max_{\substack{\alpha,q,a\\(5.5)}} |\Gamma_3(\alpha,q,a)|^2 \int\limits_0^1 (|\mathcal{K}_1(\alpha)|^4 + |\mathcal{K}_2(\alpha)|^4)\,d\alpha.$$

We use (4.3), (5.8) and the estimate (5.13) for $\mathcal{Z}_1$ follows. To treat $\mathcal{Z}_2$ and $\mathcal{Z}_3$ we also need the inequality

$$\int\limits_0^1 \bigg|\sum_{k\le K_i} \beta_i(k)\mathcal{M}_i\bigg|^4\,d\alpha \ll X^2\mathcal{L}^{10}\tau^4(q),$$

whose proof is easy. We leave it to the reader to verify that the estimate (5.13) holds also for $\mathcal{Z}_2$ and $\mathcal{Z}_3$.

Consider the quantity J' defined by (5.11). Using (2.8) and (5.9) we get

$$J' = \frac{1}{\varphi(k_1)\varphi(k_2)\varphi(k_3)} \sum_{q<Q} t(q) \int\limits_{-1/(q\tau)}^{1/(q\tau)} M^3(\alpha)e(-n\alpha)\,d\alpha.$$

It follows from (2.9)–(2.14) that for squarefree odd integers k_1, k_2, k_3 we have

$$t(q) \ll \tau^3(q)q^{-1}(k_1,q)(k_2,q)(k_3,q). \tag{5.14}$$

We also apply the well known formula

$$\int\limits_{-1/(q\tau)}^{1/(q\tau)} M^3(\alpha)e(-n\alpha)\,d\alpha = \frac{\pi}{4}\sqrt{n} + \mathcal{O}((q\tau)^{1/2}), \tag{5.15}$$

whose proof is available in Vaughan [27], Chapter 2, for example.

We use (2.3), (5.14) to estimate the contribution to $\mathcal{U}'$ arising from the error term in (5.15). We leave this computation to the reader. We find

$$\mathcal{U}' \ll \mathcal{U}'' + X^2\mathcal{L}^{-A}, \tag{5.16}$$

where

$$\mathcal{U}'' = \sum_{n\le N}{}^{*}\sqrt{n}\,\Big|\sum_{\substack{k_i\le K_i\\ i=1,2,3}}\frac{\beta_1(k_1)\beta_2(k_2)\beta_3(k_3)}{\varphi(k_1)\varphi(k_2)\varphi(k_3)}\Big(\sum_{q<Q}t(q)-\mathfrak{S}(n;Q;k_1,k_2,k_3)\Big)\Big|.$$

To estimate $\mathcal{U}''$ we apply some arguments of Mikawa [16]. Consider the function

$$\Psi(k)=\begin{cases}0 & \text{if } k \text{ has a prime divisor} \ge Q,\\ 1 & \text{otherwise.}\end{cases}$$

Let

$$M = X^2Q^{-1} \quad \text{and} \quad T = 8\prod_{p<Q}p. \tag{5.17}$$

By the definition (2.15) of $\mathfrak{S}$ we get

$$\mathfrak{S}-\sum_{q<Q}t(q)=\sum_{Q\le q\le M}t(q)\Psi(q)+\sum_{M<q\le T}t(q)\Psi(q).$$

Therefore

$$\mathcal{U}''\ll\mathcal{U}^{*}+\mathcal{U}^{**}, \tag{5.18}$$

where

$$\mathcal{U}^{*}=X\sum_{n\le N}{}^{*}\sum_{\substack{k_i\le K_i\\ i=1,2,3}}{}'\frac{\tau(k_1)\tau(k_2)\tau(k_3)}{\varphi(k_1)\varphi(k_2)\varphi(k_3)}\Big|\sum_{Q\le q\le M}t(q)\Psi(q)\Big|,$$

$$\mathcal{U}^{**}=X\sum_{n\le N}{}^{*}\sum_{\substack{k_i\le K_i\\ i=1,2,3}}{}'\frac{\tau(k_1)\tau(k_2)\tau(k_3)}{\varphi(k_1)\varphi(k_2)\varphi(k_3)}\sum_{M<q\le T}|t(q)|\Psi(q).$$

Here and later $\sum'_{k\le K_i}$ means that we sum over squarefree odd integers k only.

Consider $\mathcal{U}^{*}$. Using Cauchy's inequality we get

$$\begin{aligned}\mathcal{U}^{*2}&\ll X^4\mathcal{L}^{14}\sum_{\substack{k_i\le K_i\\ i=1,2,3}}{}'\frac{1}{k_1k_2k_3}\sum_{n\le N}\Big|\sum_{Q\le q\le M}t(q)\Psi(q)\Big|^2\\ &=X^4\mathcal{L}^{14}\sum_{\substack{k_i\le K_i\\ i=1,2,3}}{}'\frac{1}{k_1k_2k_3}\mathfrak{F},\end{aligned} \tag{5.19}$$

say. We use the definition (2.8) of $t(q)$ to represent the sum $\mathfrak{F}$ as

$$\mathfrak{F}=\sum_{n\le N}\Big|\sum_{r\in\mathfrak{X}}\eta(r)e(-nr)\Big|^2,$$

where

$$\mathfrak{X}=\{a/q: Q\le q\le M,\ 1\le a\le q-1,\ (a,q)=1\}$$

and

$$\eta(a/q) = s_{k_1}(a,q)s_{k_2}(a,q)s_{k_3}(a,q)\Psi(q).$$

For any $r \in \mathfrak{X}$ we set $\delta_r = \min\{\|r - r'\| : r' \in \mathfrak{X},\ r' \neq r\}$, so if $r = a/q$ then $\delta_r \geq (qM)^{-1}$. We apply the dual form of the large sieve inequality (see Montgomery [17], Montgomery–Vaughan [18]) to get

$$\mathfrak{F} \ll \sum_{r\in\mathfrak{X}} (N + \delta_r^{-1})|\eta(r)|^2 \ll \sum_{Q\leq q\leq M} (N + qM)\Psi(q)\varpi(q), \tag{5.20}$$

where

$$\varpi(q) = \varpi(q;k_1,k_2,k_3) = \sum_{\substack{0\leq a\leq q-1\\(a,q)=1}} |s_{k_1}(a,q)s_{k_2}(a,q)s_{k_3}(a,q)|^2.$$

This function is multiplicative with respect to q and we may easily compute $\varpi(p^l)$ for prime p. So we establish that if k_1, k_2, k_3 are squarefree odd integers then $\varpi(q) \ll q^{-2}\tau^6(q)(k_1,q)(k_2,q)(k_3,q)$. Now we use (5.17), (5.19), (5.20) and after some straightforward calculations we get

$$\mathcal{U}^* \ll X^2\mathcal{L}^{-A}. \tag{5.21}$$

Consider $\mathcal{U}^{**}$. We apply the estimate (5.14) to get

$$\mathcal{U}^{**} \ll X^3\mathcal{L} \sum_{M<q\leq T} \Psi(q)\frac{\tau^3(q)}{q}\bigg(\sum_{k\leq X}\frac{\tau(k)(k,q)}{k}\bigg)^3 \ll X^3\mathcal{L}^7\mathfrak{T}, \tag{5.22}$$

where

$$\begin{aligned}\mathfrak{T} = \sum_{M<q\leq T} \Psi(q)\frac{\tau^9(q)}{q} &\ll \sum_{M<q_1\ldots q_9\leq T} \frac{\Psi(q_1\ldots q_9)}{q_1\ldots q_9}\\ &\ll \sum_{\substack{M<q_1\ldots q_9\leq T\\ q_1\leq q_2\leq\ldots\leq q_9}} \frac{\Psi(q_1)\ldots\Psi(q_9)}{q_1\ldots q_9}\\ &\ll \bigg(\sum_{q\leq T}\frac{\Psi(q)}{q}\bigg)^8 \sum_{M^{1/9}<q\leq T}\frac{\Psi(q)}{q}.\end{aligned} \tag{5.23}$$

As in Mikawa's paper [16] we find that

$$\sum_{M^{1/9}<q\leq T} \Psi(q)/q \ll \exp(-\sqrt{\mathcal{L}})$$

and obviously $\sum_{q\leq T}\Psi(q)/q \ll \mathcal{L}$. Hence using (5.22), (5.23) we get

$$\mathcal{U}^{**} \ll X^3\mathcal{L}^{-A}. \tag{5.24}$$

The estimate (2.23) for $\mathcal{U}_1$ is a consequence of (5.12), (5.13), (5.16), (5.18), (5.21) and (5.24).

Now the proof of Theorem 1 is complete.

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Added in proof (September 2000). After the present paper was submitted for publication Professor H. Mikawa sent to the author the manuscript *On exponential sums over primes in arithmetic progressions*. In this article he establishes non-trivial estimates for the sums

$$\sum_{(d,c)=1} \lambda(d) \sum_{n\le x,\, n\equiv c\,(d)} \Lambda(n)e(\alpha n),$$

where α belongs to the set of minor arcs, $c \neq 0$ is a fixed integer, λ is any well-factorable function of level $x^{4/9}(\log x)^{-B}$ and $B > 0$. This result implies a slight improvement of Theorem 2 and Corollaries 2 and 3. The method can be used to improve also Theorem 1. However the calculations will be quite difficult.

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