ACTA ARITHMETICA
146.4 (2011)

Second moments of holomorphic Hilbert modular forms and subconvexity

by

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We have two results in this note. First, we generalize the result of Sarnak [Sa] to holomorphic Hilbert cusp forms (not necessarily newforms) over a totally real number field of degree n by applying the technique of Titchmarsh [Ti] and obtain the average version of the second moments. Second, by applying the technique of [PSa], we obtain the subconvexity bound in t-aspect.

We recall some facts on Hilbert cusp forms from [G, §1.9]: Let F be a totally real number field. Let $[F:\mathbb{Q}]=n$. Let $\mathfrak{o}$ be the ring of integers and $\mathfrak{n}$ be an ideal. Let

$$\Gamma=\Gamma(\mathfrak{n})=\{\gamma\in \mathrm{GL}^+(2,\mathfrak{o}):\gamma\equiv 1_2 \bmod \mathfrak{n}\}.$$

Let f be a Hilbert cusp form with respect to Γ of weight $k=(k,\ldots,k)$, where k is a positive integer. Let $z=(z_1,\ldots,z_n)\in\mathcal{H}^n$. Let $\Lambda=\big\{u\in F:\left(\begin{smallmatrix}1&u\\0&1\end{smallmatrix}\right)\in\Gamma\big\}$. Then f has the Fourier expansion

$$f(z)=\sum_{\xi\in\Lambda^*}a(\xi)N(\xi)^{(k-1)/2}e^{2\pi i\,\mathrm{Tr}(\xi z)},$$

where Tr is the $\mathbb{C}$-linear extension to $\mathbb{C}^n\to\mathbb{C}$ of the Galois trace $F\to\mathbb{Q}$, and $\Lambda^*=\{u\in F:\mathrm{Tr}(u\Lambda)\subset\mathfrak{o}\}$.

Let $T=\mathbb{R}_+^n$, and $\chi:T\to\mathbb{C}^\times$ be a continuous group homomorphism which is trivial on the two subgroups

$$\Delta=\{(y,\ldots,y)\in\mathbb{R}_+^n:y>0\},\quad U=\{\eta\in T:\eta\in\mathfrak{o}^\times,\,\eta\equiv 1 \bmod \mathfrak{n}\}.$$

We write

$$T/U\simeq\{(y_1,\ldots,y_n):y_1\cdots y_n=1\}/U\times\{(r^{1/n},\ldots,r^{1/n}):r>0\}.$$

Then by the units theorem, the first factor is compact. Choose a compact set X in T of representatives of the first factor, and identify $(r^{1/n},\ldots,r^{1/n})$ with

2010 *Mathematics Subject Classification*: Primary 11F41; Secondary 11M41.
Key words and phrases: Hilbert cusp forms, second moments, subconvexity.

DOI: 10.4064/aa146-4-2

$r^{1/n}$. Then we can write any element $(y_1,\dots,y_n)\in T/U$ as $(y_1,\dots,y_n) = xr^{1/n}$ for some $x\in X$.

Here χ is a character of X, and we can write

$$\chi(y)=\chi(y_1,\dots,y_n)=\prod_j y_j^{i\nu_j},$$

where $\nu_j\in\mathbb{R}$ and $\nu_1+\dots+\nu_n=0$.

For simplicity, we assume that $\mathfrak{n}=\mathfrak{o}$. Then $\Lambda=\mathfrak{o}$ and $\Lambda^*=\mathfrak{d}^{-1}$, where $\mathfrak{d}$ is the different of F. In this case, we can write down ν_j's explicitly in terms of fundamental units: Let $u_1,\dots,u_{n-1}$ be fundamental units. Since U is the image of the map $\mathfrak{o}^\times\to T$ given by $u\mapsto(u^{(1)},\dots,u^{(n)})$, $|u_j^{(1)}|^{i\nu_1}\cdots|u_j^{(n)}|^{i\nu_n}=1$ for each $j=1,\dots,n-1$, namely, for $m_1,\dots,m_{n-1}\in\mathbb{Z}$,

$$(\nu_1,\dots,\nu_n)\begin{pmatrix}1 & \log|u_1^{(1)}| & \cdots & \log|u_{n-1}^{(1)}|\\ \vdots & \vdots & \cdots & \vdots\\ 1 & \log|u_1^{(n)}| & \cdots & \log|u_{n-1}^{(n)}|\end{pmatrix}=(0,2\pi m_1,\dots,2\pi m_{n-1}).$$

Hence for $\xi\in\mathfrak{d}^*$, $\chi(\xi)=\prod_{j=1}^n\left|\frac{\xi^{(j)}}{N(\xi)^{1/n}}\right|^{i\nu_j}$.

Define the L-function

$$L(s,f,\chi)=\sum_{\xi \bmod U}a(\xi)\chi(\xi)N(\xi)^{-s}.$$

Then we have the following integral representation:

$$\begin{aligned}\Lambda(s,f,\chi)&=L(s,f,\chi)\prod_{j=1}^n(2\pi)^{-(s+(k-1)/2+i\nu_j)}\Gamma\left(s+\frac{k-1}{2}+i\nu_j\right)\\&=\int_{T/U}f(iy)\bar{\chi}(y)y^{s+(k-1)/2}\,d^\times y,\end{aligned}$$

where $d^\times y=\frac{dy_1\cdots dy_n}{y_1\cdots y_n}$. If f is an eigenfunction with eigenvalue $\lambda\in\{\pm1,\pm i\}$ for the map $f\mapsto f^\sharp=f|_kJ$, where $J=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, then we have the functional equation

$$\Lambda(s,f,\chi)=\lambda i^{nk}\Lambda(1-s,f,\bar{\chi}).$$

1. Average of second moments. We write

$$\Lambda(s,f,\chi)=\int_0^\infty\int_X f(ir^{1/n}x)\bar{\chi}(x)r^{s+(k-1)/2}\,d^\times x\,\frac{dr}{r}.$$

By Mellin inversion, we have

$$\int_X f(ir^{1/n}x)\bar{\chi}(x)\,d^\times x=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Lambda(s,f,\chi)r^{-s-(k-1)/2}\,ds.$$

The above equation is valid by substituting r with z with $\mathrm{Re}(z^{1/n}) > 0$, i.e.,

$$\int_X f(iz^{1/n}x)\bar{\chi}(x)\,d^\times x = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \Lambda(s,f,\chi) z^{-s-(k-1)/2}\,ds.$$

Since $L(s,f,\chi)$ is entire, we can move the contour to $\mathrm{Re}(s) = \sigma$, $0 < \sigma < 1$. We will set $z^{1/n} = r^{1/n}e^{i(\pi/2-\delta)}$. Then

$$\int_X f(iz^{1/n}x)\bar{\chi}(x)\,d^\times x = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Lambda(s,f,\chi)e^{-i(s+(k-1)/2)n(\pi/2-\delta)}r^{-s-(k-1)/2}\,ds.$$

Hence $r^{(k-1)/2}\int_X f(iz^{1/n}x)\bar{\chi}(x)\,d^\times x$ and $\Lambda(s,f,\chi)e^{-i(s+(k-1)/2)n(\pi/2-\delta)}$ are Mellin transforms and by Parseval's formula,

$$\int_0^\infty \Big|\int_X f(iz^{1/n}x)\bar{\chi}(x)\,d^\times x\Big|^2 r^{k+2\sigma-2}\,dr = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\Lambda(\sigma+it,f,\chi)|^2 e^{tn\pi-2\delta t}\,dt.$$

Now, $c(\chi) = \int_X f(iz^{1/n}x)\bar{\chi}(x)\,d^\times x$ is the Fourier coefficient of $f(iz^{1/n}x) = \sum_\chi c(\chi)\chi(x)$. By Parseval's formula,

$$\sum_\chi \Big|\int_X f(iz^{1/n}x)\bar{\chi}(x)\,d^\times x\Big|^2 = \int_X |f(iz^{1/n}x)|^2\,d^\times x.$$

Therefore, we have

$$(1.1)\qquad \sum_\chi \frac{1}{2\pi}\int_{-\infty}^{\infty} |\Lambda(\sigma+it,f,\chi)|^2 e^{tn\pi-2\delta t}\,dt = \int_0^\infty\int_X |f(iz^{1/n}x)|^2 r^{k+2\sigma-2}\,d^\times x\,dr.$$

Set $\sigma = 1/2$.

We first analyze the RHS of (1.1). We write $\int_0^\infty = \int_0^1 + \int_1^\infty$. By the functional equation, we see that $\int_0^1 = \int_1^\infty$. We write

$$\int_1^\infty = \int_1^{(\sin\delta)^{-n}} + \int_{(\sin\delta)^{-n}}^\infty.$$

If $r > (\sin\delta)^{-n}$, then $r^{1/n}\sin\delta > 1$, and $|f(iz^{1/n}x)| \ll e^{-c(x)r^{1/n}\sin\delta}$ for a constant $c(x)$ depending only on x. Then

$$\int_{(\sin\delta)^{-n}}^\infty \ll \int_{(\sin\delta)^{-n}}^\infty r^{k-1}e^{-2c(x)r^{1/n}\sin\delta}\,dr = O((\sin\delta)^{-nk}).$$

For $\int_1^{(\sin\delta)^{-n}}$, we use the fact that $y^k|f(z)|^2 < C$ for some constant C [G, p. 24]. Then $|f(iz^{1/n}x)|^2 \ll c(x,k)r^{-k}(\sin\delta)^{-nk}$ for some constant $c(x,k)$, depending only on x,k. Hence

$$\int_1^{(\sin\delta)^{-n}} \ll (\sin\delta)^{-nk}\int_1^{(\sin\delta)^{-n}} r^{-1}\,dr = O\bigg((\sin\delta)^{-nk}\log\frac{1}{\sin\delta}\bigg).$$

Therefore,

$$\text{RHS of (1.1)} \ll (\sin\delta)^{-nk}\log\frac{1}{\sin\delta}.$$

Next we analyze the LHS of (1.1). By a change of variables,

$$\int_{-\infty}^{0} |\Lambda(1/2+it,f,\chi)|^2 e^{tn\pi-2\delta t}\,dt = \int_0^{\infty} |\Lambda(1/2-it,f,\chi)|^2 e^{-tn\pi+2\delta t}\,dt.$$

By Stirling's formula, if $k>1$,

$$|\Gamma(k/2-it+i\nu_j)|^2 = 2\pi|t-\nu_j|^{k-1}e^{-\pi|t-\nu_j|}(1+O(|t-\nu_j|^{-1})).$$

If $k=1$,

$$|\Gamma(1/2-it+i\nu_j)|^2 = 2\pi e^{-\pi|t-\nu_j|}+O(e^{-3\pi|t-\nu_j|}).$$

Let $\|\chi\| = \max|\nu_j|$ and $\|m\| = \max|m_j|$. Then clearly $\|\chi\| \ll \|m\|$ and $\|m\| \ll \|\chi\|$. By the convexity bound, $|L(1/2-it,f,\chi)| \ll \prod_{j=1}^n |t-\nu_j|^{k/2+\epsilon}$ for any $\epsilon>0$.

Let $R = \|\chi\|$. Then since $|t-\nu_j| \le t+R$ for $t\ge 0$,

$$\begin{aligned}\int_R^{\infty} |\Lambda(1/2-it,f,\chi)|^2 e^{-tn\pi+2\delta t}\,dt &\ll \int_R^{\infty}(t+R)^{2nk-n+2n\epsilon}e^{-2t(\pi n-\delta)}\,dt\\ &\ll R^{2nk-n+2n\epsilon}e^{-2R(n-\delta)}.\end{aligned}$$

If $t\le R$, then since $|t-\nu_j| \ge |\nu_j|-t$, we have

$$\begin{aligned}\int_0^{R} |\Lambda(1/2-it,f,\chi)|^2 e^{-tn\pi+2\delta t}\,dt &\ll \int_0^{R}(t+R)^{2nk-n+2n\epsilon}e^{2\delta t}e^{-\pi(|\nu_1|+\cdots+|\nu_n|)}\,dt\\ &\ll R^{2nk-n+1+2n\epsilon}e^{2\delta R}e^{-\pi(|\nu_1|+\cdots+|\nu_n|)}.\end{aligned}$$

Here $|\nu_1|+\cdots+|\nu_n|-2\delta R/\pi \ge cR/\pi$ for some constant $c>0$ if we take δ very small. Therefore

$$\int_0^{\infty} |\Lambda(1/2-it,f,\chi)|^2 e^{-tn\pi+2\delta t}\,dt \ll R^{2nk-n+1+2n\epsilon}e^{-c'R}$$

for some constant $c'>0$. For each positive integer l, let $N(l)$ be the number of χ's such that $l-1 \le \|\chi\| < l$. Then $N(l) \ll l^{n-1}$. So

$$\sum_\chi \int_0^\infty |\Lambda(1/2-it,f,\chi)|^2 e^{-tn\pi+2\delta t}\,dt \ll \sum_{l=1}^\infty l^{2nk+2n\epsilon} e^{-c'l} = O(1).$$

Hence on the left hand side of (1.1), the sum of the integrals $\int_{-\infty}^0$ is $O(1)$. So as $\delta \to 0+$,

$$\sum_\chi \int_0^\infty |\Lambda(1/2+it,f,\chi)|^2 e^{tn\pi-2\delta t}\,dt = O\bigg(\delta^{-nk}\log\frac{1}{\delta}\bigg).$$

By [Ti, p. 157], this is equivalent to:

$$\sum_\chi \int_0^T |\Lambda(1/2+it,f,\chi)|^2 e^{\pi nt}\,dt = O(T^{nk}\log T)$$

as $T\to\infty$. By integration by parts, we have

$$\sum_\chi \int_0^T |\Lambda(\sigma+it,f,\chi)|^2 t^{-nk+n} e^{\pi nt}\,dt = O(T^n\log T).$$

Letting $M(\chi,t) = t^{-nk+n}e^{\pi tn}\prod_{j=1}^n |\Gamma(k/2+it+i\nu_j)|^2$, we have proved

Theorem 1.1. *As $T\to\infty$,*

$$\sum_\chi \int_0^T |L(1/2+it,f,\chi)|^2 M(\chi,t)\,dt = O(T^n\log T).$$

When $\chi=1$, $M(1,t)\sim 1/(2\pi)^n$, and so

Corollary 1.2. *As $T\to\infty$,*

$$\int_0^T |L(1/2+it,f)|^2\,dt = O(T^n\log T).$$

Now we can prove a result analogous to [Sa].

Theorem 1.3. *As $T\to\infty$, for any constant $\alpha<1/2$,*

$$\sum_{\|\chi\|\le\alpha T} \int_{T/2}^T |L(1/2+it,f,\chi)|^2\,dt = O(T^n\log T). \tag{1.2}$$

Proof. By Stirling's formula,

$$M(\chi,t) = \prod_{j=1}^n e^{\pi(t-|t+\nu_j|)}\bigg(\frac{|t+\nu_j|}{t}\bigg)^{k-1}(1+O(|t+\nu_j|^{-1})).$$

If $\|\chi\| \le \alpha T$ and $t \ge T/2$, then $|t+\nu_j| = t+\nu_j \ge (1-2\alpha)t$. Hence, $M(\chi,t) \gg 1$. Therefore,

$$\sum_{\|\chi\|\le\alpha T} \int_{T/2}^{T} |L(1/2+it,f,\chi)|^2\,dt \ll \sum_{\chi} \int_{0}^{T} |L(1/2+it,f,\chi)|^2 M(\chi,t)\,dt.$$

Our result follows. ∎

REMARK 1.4. In [D, p. 214], it is claimed that the above estimate would imply the estimate

$$\sum_{\|\chi\|\le T} \int_{0}^{T} |L(1/2+it,f,\chi)|^2\,dt = O(T^n \log T).$$

However, we do not see how it is possible.

2. Subconvexity at the critical line. As the referee pointed out, the L-function of an arbitrary holomorphic Hilbert cusp form is a finite linear combination of L-functions of holomorphic newforms with coefficients being bounded on the critical line (cf. [BH, p. 11]; any holomorphic Hilbert cusp form f is a finite linear combination of $R_{\mathfrak{t}}h$, where $R_{\mathfrak{t}}$ is the shift operator with an ideal $\mathfrak{t}$, and h is a newform; now $L(s,R_{\mathfrak{t}}h) = N(\mathfrak{t})^s L(s,h)$). So for our purpose of obtaining a subconvexity bound in t-aspect, we can assume that f is a newform, i.e., an eigenform of all Hecke operators. In this case, f is attached to a cuspidal representation of $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)$, and we can use the result in [H].

In equation (1.2), by taking one term, we have $\int_0^T |L(1/2+it,f,\chi_0)|^2\,dt = O_{\chi_0}(T^n\log T)$ for a fixed χ_0. This implies $L(1/2+it,f,\chi_0) = O_{\chi_0}(|t|^{n/2+\epsilon})$. This is the convexity bound. We want to prove

THEOREM 2.1. *For a fixed χ_0,*

$$L(1/2+it,f,\chi_0) = O_{\chi_0}(|t|^{n/2-7/216+\epsilon}).$$

By considering $f\otimes\chi_0$ instead of f, we assume that $\chi_0 = 1$. We follow [PSa] closely. Recall the definition of analytic conductor due to [IS]:

$$C = C(t) = \frac{1}{(2\pi)^{2n}} \prod_{j=1}^{n} |(k/2+it)(k/2+1+it)|.$$

We use the uniform approximate functional equation due to Harcos: Theorem 2.5 of [H] implies, for any $\epsilon > 0$,

$$L(1/2+it,f) = \sum_{\xi} \frac{a(\xi)}{N(\xi)^{1/2+it}} V\Big(\frac{N(\xi)}{\sqrt{C}}\Big) + i^{nk}\lambda \sum_{\xi} \frac{\overline{a(\xi)}}{N(\xi)^{1/2-it}} V\Big(\frac{N(\xi)}{\sqrt{C}}\Big) + O_{\epsilon,V}(\eta^{-1}C^{1/4+\epsilon}).$$

Here λ is a complex number of absolute value 1, and $V : (0,\infty) \to \mathbb{C}$ is a smooth function, independent of t, with the functional equation $V(x) + V(1/x) = 1$ and derivatives decaying faster than any negative power of x as $x \to \infty$, and

$$\eta = \min_{j=1,\dots,n} \{|k/2 + it|, |k/2 + 1 + it|\}.$$

Now for any χ, we define a "fake" L-value (this idea is due to the referee):

$$\tilde{L}(1/2+it,f,\chi) = \sum_{\xi} \frac{a(\xi)\chi(\xi)}{N(\xi)^{1/2+it}} V\left(\frac{N(\xi)}{\sqrt{C}}\right) + i^{nk}\lambda \sum_{\xi} \frac{\overline{a(\xi)}\chi^{-1}(\xi)}{N(\xi)^{1/2-it}} V\left(\frac{N(\xi)}{\sqrt{C}}\right). \tag{2.1}$$

We reduce the size of averaging in (1.2): namely, we show, for $T^{101/108} \le H \le T$ and $\epsilon > 0$,

$$\int \sum |\tilde{L}(1/2+it,f,\chi)|^2\,dt \ll (T^{n-1}H)^{1+\epsilon},$$

where the integral and sum are over the domain $T - H \le |\nu_j + it| \le T + H$ for $j = 1,\dots,n$, and χ is given by $\nu_1,\dots,\nu_n$. Let $H = T^{101/108}$, and take one term corresponding to $\chi = 1$. Here $\tilde{L}(1/2+it,f)$ and $L(1/2+it,f)$ differ by the error term $O_{\epsilon,V}(\eta^{-1}C^{1/4+\epsilon})$, and it gives rise to $O(T^{n-2+\epsilon})$. Hence we have

$$\int_{T-\log^2 T}^{T+\log^2 T} |L(1/2+it,f)|^2\,dt \ll T^{n-7/108+\epsilon}. \tag{2.2}$$

By a standard argument (for example, see [Go, p. 294] or [Iv, (7.2)]), this implies Theorem 2.1. At the end of the paper, we give an outline of how the mean-value estimate (2.2) implies the pointwise estimate in Theorem 2.1.

As in [CPSS], we introduce a smooth dyadic partition of the identity on $(0,\infty)$ by $1 = \sum_{\alpha=-\infty}^{\infty} g(x/2^{\alpha/2})$ with $g(x)$ a smooth function with support in $[1,2]$. Let $X_\alpha = 2^{\alpha/2}$. Then the first term on the right hand side of (2.1) can be written as

$$\sum_{\xi} \sum_{\alpha=-1}^{\infty} \frac{a(\xi)\chi(\xi)}{N(\xi)^{1/2+it}} V\left(\frac{N(\xi)}{\sqrt{C}}\right) g\left(\frac{N(\xi)}{X_\alpha}\right).$$

If we set $W_X(x) = \sqrt{X/x}\, x^{-it} V(x/\sqrt{C}) g(x/X)$, then the above becomes

$$\sum_{\alpha=-1}^{\infty} \frac{1}{\sqrt{X_\alpha}} S_{X_\alpha}(t,\chi),$$

where $S_X(t,\chi)=\sum_\xi a(\xi)\chi(\xi)W_X(N(\xi))$. Then (2.1) can be written as

$$\tilde{L}(1/2+it,f,\chi)=\sum_{\alpha=-1}^{\infty}\frac{S_{X_\alpha}(t,\chi)}{\sqrt{X_\alpha}}+i^{nk}\lambda\sum_{\alpha=-1}^{\infty}\frac{\overline{S_{X_\alpha}(t,\chi)}}{\sqrt{X_\alpha}}.$$

If we take r so that $X_r\le C^{1/2+\epsilon}<X_{r+1}$, then

$$\tilde{L}(1/2+it,f,\chi)=\sum_{\alpha=-1}^{r}\frac{S_{X_\alpha}(t,\chi)}{\sqrt{X_\alpha}}+i^{nk}\lambda\sum_{\alpha=-1}^{r}\frac{\overline{S_{X_\alpha}(t,\chi)}}{\sqrt{X_\alpha}}+O(C^{-M})$$

for some positive constant M. Note that the length of the sum is $r+2$, and $r\ll\log C\ll r+1$. So it is enough to show that

$$\int\sum|S_X(t,\chi)|^2\,dt\ll X(T^{n-1}H)^{1+\epsilon} \tag{2.3}$$

for $T^{101/108}\le H\le T$, where the sum and integral are over the domain $T-H\le|\nu_j+it|\le T+H$ for $j=1,\dots,n$, and $X\le C^{1/2+\epsilon}\le T^{n+\epsilon}$.

In order to apply the Poisson summation formula, recall the map

$$u\mapsto(\log|u^{(1)}|,\dots,\log|u^{(n)}|)\quad\text{for } u\in F^\times.$$

The image of $\mathfrak{o}_+^\times$ is a lattice Γ in $P=\{(x_1,\dots,x_n):x_1+\dots+x_n=0\}\simeq\mathbb{R}^{n-1}$. We identify χ with $\nu=(\nu_1,\dots,\nu_n)$. Then $\chi(\xi)=e^{2\pi i(\log\xi,\nu)}$, where $\log\xi=(\log\xi^{(1)},\dots,\log\xi^{(n)})$, and $(\log\xi,\nu)=\sum_{j=1}^n\nu_j\log\xi^{(j)}$. Then by the definition of χ, the set of χ's is the dual lattice Γ' of Γ.

Let $\psi(x_1,\dots,x_n)$ be a non-negative function on P such that $\psi(I_1)=1$ and the support of ψ is in I_2; here $I_1=\{(x_1,\dots,x_n)\in P:|x_i|\le1\}$, $I_2=\{(x_1,\dots,x_n)\in P:|x_i|\le2\}$. Then the left hand side of (2.3) is less than

$$A=\sum_\chi\int_{-\infty}^{\infty}\psi\biggl(\frac{|\nu_1+it|-T}{H},\dots,\frac{|\nu_n+it|-T}{H}\biggr)|S_X(t,\chi)|^2\,dt.$$

Hence we need to show that

$$A\ll X(T^{n-1}H)^{1+\epsilon}\quad\text{for } T^{\frac{101}{18}n/(7n-1)}\le H\le T.$$

We write

$$\begin{aligned}A=X\sum_{\xi,\eta}&\frac{a(\xi)\overline{a(\eta)}}{(N(\xi)N(\eta))^{1/2}}g\biggl(\frac{N(\xi)}{X}\biggr)g\biggl(\frac{N(\eta)}{X}\biggr)\\&\times\int_{-\infty}^{\infty}\biggl(\frac{N(\xi)}{N(\eta)}\biggr)^{it}V\biggl(\frac{N(\xi)}{\sqrt{C}}\biggr)\overline{V\biggl(\frac{N(\eta)}{\sqrt{C}}\biggr)}\\&\times\biggl(\sum_\chi\psi\biggl(\frac{|\nu_1+it|-T}{H},\dots,\frac{|\nu_n+it|-T}{H}\biggr)\chi(\xi)\overline{\chi(\eta)}\biggr)\,dt.\end{aligned}\tag{2.4}$$

We apply the Poisson summation formula in χ:

$$\sum_{\chi}\psi\Big(\frac{|\nu_1+it|-T}{H},\dots,\frac{|\nu_n+it|-T}{H}\Big)\chi(\xi)\overline{\chi(\eta)} \tag{2.5}$$
$$=\sum_{\gamma\in\Gamma}\int_P\psi\Big(\frac{|x_1+it|-T}{H},\dots,\frac{|x_n+it|-T}{H}\Big)$$
$$\times e^{2\pi i\sum_{i=1}^n x_i(\log\xi^{(i)}-\log\eta^{(i)})-2\pi i(\gamma,x)}\,dx.$$

Since $x_1+\dots+x_n=0$ in P, we write the integral as

$$\int_P\psi\Big(\frac{|x_1+it|-T}{H},\dots,\frac{|x_n+it|-T}{H}\Big)e^{2\pi i\sum_{i=1}^n x_i(\log\xi^{(i)}-\log\eta^{(i)})-2\pi i(\gamma,x)}\,dx$$
$$=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\psi\Big(\frac{|x_1+it|-T}{H},\dots,\frac{|x_n+it|-T}{H}\Big)$$
$$\times e^{2\pi i(x,\log\xi-\log\eta-\gamma-(\log\xi^{(n)}-\log\eta^{(n)}-\gamma^{(n)}))}\,dx_1\cdots dx_{n-1},$$

where $\log\xi-\log\eta-\gamma-(\log\xi^{(n)}-\log\eta^{(n)}-\gamma^{(n)})=(\log\xi^{(1)}-\log\eta^{(1)}-\gamma^{(1)}-(\log\xi^{(n)}-\log\eta^{(n)}-\gamma^{(n)}),\dots,\log\xi^{(n-1)}-\log\eta^{(n-1)}-\gamma^{(n-1)}-(\log\xi^{(n)}-\log\eta^{(n)}-\gamma^{(n)}))$.

By the change of variables, the integral becomes

$$H^{n-1}\hat{\psi}_{T,H,t}(H(\log\xi-\log\eta-\gamma-(\log\xi^{(n)}-\log\eta^{(n)}-\gamma^{(n)}))),$$

where $\psi_{T,H,t}(y_1,\dots,y_{n-1})=\psi(|it/H+y_1|-T/H,\dots,|it/H+y_n|-T/H)$, and $y_n=-(y_1+\dots+y_{n-1})$. By integration by parts,

$$\hat{\psi}_{T,H,t}(y_1,\dots,y_{n-1})\ll(\|y\|+1)^{-N}$$

for any $N\ge 1$, where $\|y\|=\min\{|y_1|,\dots,|y_{n-1}|\}$. Since $\xi,\eta\in\mathfrak{d}/\mathfrak{o}_+^\times$, we can choose ξ,η so that $\log\xi-\log\eta$ is in the fundamental domain of Γ in P. Hence in (2.5), only the term $\gamma=0$ is significant. That is,

$$\sum_{\chi}\psi\Big(\frac{|\nu_1+it|-T}{H},\dots,\frac{|\nu_n+it|-T}{H}\Big)\chi(\xi)\overline{\chi(\eta)}$$
$$=\int_P\psi\Big(\frac{|x_1+it|-T}{H},\dots,\frac{|x_n+it|-T}{H}\Big)e^{2\pi i\sum_{i=1}^n x_i(\log\xi^{(i)}-\log\eta^{(i)})}\,dx$$
$$+O(H^{-N}).$$

Plugging this into (2.4), we have

$$A=X\sum_{\xi,\eta}\frac{a(\xi)\overline{a(\eta)}}{(N(\xi)N(\eta))^{1/2}}g\Big(\frac{N(\xi)}{X}\Big)g\Big(\frac{N(\eta)}{X}\Big) \tag{2.6}$$
$$\times\int_{-\infty}^{\infty}\Big(\frac{N(\xi)}{N(\eta)}\Big)^{it}V\Big(\frac{N(\xi)}{\sqrt{C}}\Big)\overline{V\Big(\frac{N(\eta)}{\sqrt{C}}\Big)}$$

$$\times \int_P \psi\Big(\frac{|x_1+it|-T}{H},\dots,\frac{|x_n+it|-T}{H}\Big)e^{2\pi i\sum_{i=1}^n x_i(\log\xi^{(i)}-\log\eta^{(i)})}\,dx\,dt$$
$$+\text{ small error.}$$

Note that $(N(\xi)/N(\eta))^{it}=e^{it\sum_{i=1}^n(\log\xi^{(i)}-\log\eta^{(i)})}$. So the above integral is

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\psi\Big(\frac{|x_1+it|-T}{H},\dots,\frac{|x_n+it|-T}{H}\Big)V\Big(\frac{N(\xi)}{\sqrt{C}}\Big)\overline{V\Big(\frac{N(\eta)}{\sqrt{C}}\Big)}$$
$$\times e^{2\pi i(x,\log\xi-\log\eta-\gamma-(\log\xi^{(n)}-\log\eta^{(n)}-\gamma^{(n)}))+it\sum_{i=1}^n(\log\xi^{(i)}-\log\eta^{(i)})}dx_1\cdots dx_{n-1}\,dt.$$

Set $t/H=t', x_i/H=y_i$. Then the above integral is $H^n\hat{\phi}_{T,H}$, where

$$\hat{\phi}_{T,H}=\hat{\phi}_{T,H}\Big(H(\log\xi^{(1)}-\log\eta^{(1)}-\log\xi^{(n)}+\log\eta^{(n)}),\dots,$$
$$H(\log\xi^{(n-1)}-\log\eta^{(n-1)}-\log\xi^{(n)}+\log\eta^{(n)}),\frac{H}{2\pi}(\log N(\xi)-\log N(\eta))\Big),$$

and $\phi_{T,H}(y_1,\dots,y_{n-1},t')=\psi(|y_1+it'|-T/H,\dots,|y_n+it'|-T/H)\times V(N(\xi)/\sqrt{C})\overline{V(N(\eta)/\sqrt{C})}$. Repeated integration by parts shows that

$$\hat{\phi}_{T,H}(u_1,\dots,u_{n-1},u_n)\ll(T/H)^{n-1}(1+\|u\|)^{-N}$$

for any $N\ge 1$, where $\|u\|=\min\{|u_1|,\dots,|u_{n-1}|,|u_n|\}$. Hence if $\delta>0$ is arbitrarily small, the contribution to (2.6) of the terms with

$$\min\{|\log\xi^{(i)}-\log\eta^{(i)}-\log\xi^{(n)}+\log\eta^{(n)}|\ (i=1,\dots,n-1),$$
$$|\log N(\xi)-\log N(\eta)|\}\gg H^{\delta-1}$$

is negligible. Also $N(\xi),N(\eta)$ are of size X. Hence $|N(\xi)-N(\eta)|\ll XH^{\delta-1}$. Also

$$|\log\xi^{(i)}-\log\eta^{(i)}-\log\xi^{(n)}+\log\eta^{(n)}|\ll H^{\delta-1},$$
$$|\log N(\xi)-\log N(\eta)|\ll H^{\delta-1}$$

implies that $|\log\xi^{(i)}-\log\eta^{(i)}|\ll H^{\delta-1}$ for each $i=1,\dots,n$. So $|\xi^{(i)}-\eta^{(i)}|\ll H^{\delta-1}|\eta^{(i)}|$ for each i. Therefore $\prod_{i=1}^n|\xi^{(i)}-\eta^{(i)}|\ll XH^{n\delta-n}$. Hence

$$A=XH^n\sum_{N(\xi-\eta)\ll XH^{n\delta-n}}\frac{a(\xi)\overline{a(\eta)}}{(N(\xi)N(\eta))^{1/2}}g\Big(\frac{N(\xi)}{X}\Big)g\Big(\frac{N(\eta)}{X}\Big)\hat{\phi}_{T,H}\tag{2.7}$$

with small error. The contribution to (2.6) of the diagonal $\xi=\eta$ is

$$XH^n\sum_{\xi}\frac{|a(\xi)|^2}{N(\xi)}g\Big(\frac{N(\xi)}{X}\Big)^2\hat{\phi}_{T,H}(0).$$

Here

$$\hat{\phi}_{T,H}(0)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\psi(|y_1+it'|-T/H,\dots,|y_n+it'|-T/H)$$
$$\times V\bigg(\frac{N(\xi)}{\sqrt{C}}\bigg)\overline{V\bigg(\frac{N(\eta)}{\sqrt{C}}\bigg)}\,dy_1\cdots dy_{n-1}\,dt'\ll(T/H)^{n-1}.$$

Also by Rankin–Selberg convolution,

$$\sum_{N(\xi)\le X^{1+\epsilon}}\frac{|a(\xi)|^2}{N(\xi)}=O(X^{\epsilon}).$$

Therefore, the diagonal contribution to (2.7) is

$$X(T^{n-1}H)^{1+\epsilon}.$$

For the off-diagonal terms, let $\xi-\eta=h$. Then $N(h)\ll XH^{n\delta-n}$. We estimate the sum for each h: Let

$$S(h)=\sum_{\eta}\frac{a(\eta+h)\overline{a(\eta)}}{(N(\eta+h)N(\eta))^{1/2}}g\bigg(\frac{N(\eta+h)}{X}\bigg)g\bigg(\frac{N(\eta)}{X}\bigg)\hat{\phi}_{T,H}.$$

Now we have $N(\eta+h)=N(\eta)+O(XH^{\delta-1})$, and $\log\xi^{(i)}-\log\eta^{(i)}=\log(1+h^{(i)}/\eta^{(i)})=h^{(i)}/\eta^{(i)}+O(H^{2\delta-2})$. We can see easily that

$$\frac{\partial^{i_1+\dots+i_n}}{\partial u_1^{i_1}\cdots\partial u_n^{i_n}}\hat{\phi}_{T,H}(u_1,\dots,u_{n-1},u_n)\ll(T/H)^{n-1+i_1+\dots+i_n}. \tag{2.8}$$

Hence

$$\hat{\phi}_{T,H}=\hat{\phi}_{T,H}\bigg(H\frac{h}{\eta}\bigg)+O((T/H)^nH^{2\delta-1}),$$

where

$$\hat{\phi}_{T,H}\bigg(H\frac{h}{\eta}\bigg)=\hat{\phi}_{T,H}\bigg(H\bigg(\frac{h^{(1)}}{\eta^{(1)}}-\frac{h^{(n)}}{\eta^{(n)}}\bigg),\dots,$$
$$H\bigg(\frac{h^{(n-1)}}{\eta^{(n-1)}}-\frac{h^{(n)}}{\eta^{(n)}}\bigg),\frac{H}{2\pi}\bigg(\frac{h^{(1)}}{\eta^{(1)}}+\dots+\frac{h^{(n)}}{\eta^{(n)}}\bigg)\bigg).$$

Therefore,

$$S(h)=\sum_{\eta}\frac{a(\eta+h)\overline{a(\eta)}}{N(\eta)}g\bigg(\frac{N(\eta)}{X}\bigg)^2\hat{\phi}_{T,H}\bigg(H\frac{h}{\eta}\bigg)(1+O((T/H)^nH^{2\delta-1})). \tag{2.9}$$

Let $s=(s_1,\dots,s_n)$ and use the notation $y^s=y_1^{s_1}\cdots y_n^{s_n}$ for $y=(y_1,\dots,y_n)$. Also for each $i=1,\dots,n$, let $\eta^{(i)}=X^{1/n}y_i$. Let

$$B_{h,T,X}(s)=\int_0^{\infty}\cdots\int_0^{\infty}g(y_1\cdots y_n)^2\hat{\phi}_{T,H}\bigg(H\frac{h}{X^{1/n}y}\bigg)y^s\,\frac{dy}{y},$$

$$D_f(s,h)=\sum_\eta \frac{a(\eta+h)\overline{a(\eta)}}{\eta^s}.$$

For $-1\le\sigma_j\le 2$, we integrate by parts N times, where $N=i_1+\cdots+i_n$, and using (2.8), we obtain

$$B_{h,T,X}(\sigma_j+it)\ll (T/H)^{N+n-1+\epsilon}\prod_{j=1}^n(1+|t_j|)^{-i_j}.$$

Recall the following.

THEOREM 2.2 ([CPSS]). *$D_f(s,h)$ has an analytic continuation to* $\mathrm{Re}(s_j)>11/18$, *and for $s_j=\sigma_j+it_j$,*

$$D_f(s,h)\ll N(h)^{1/9+\epsilon}\prod_{j=1}^n|h^{(j)}|^{1/2-\sigma_j}(1+|t_j|)^{3+\epsilon}.$$

Proof. In [CPSS, Theorem 1.3], it is proved that the Dirichlet series

$$D(s,\alpha_1,\alpha_2,h)=\sum_{\alpha_1,\alpha_2,\,\alpha_1-\alpha_2=h}\frac{a(\alpha_1)\overline{a(\alpha_2)}}{(\alpha_1+\alpha_2)^s}\left(\frac{(\alpha_1\alpha_2)^{1/2}}{\alpha_1+\alpha_2}\right)^{k-1}$$

extends analytically as a function of several variables $s=(s_1,\dots,s_n)$, $s_j=\sigma_j+it_j$ to the region $\sigma_j>1/2+1/9$, and in this region

$$D(s,\alpha_1,\alpha_2,h)\ll N(h)^{1/9+\epsilon}\prod_{j=1}^n|h^{(j)}|^{1/2-\sigma_j}(1+|t_j|)^{3+\epsilon}.$$

It is easy to see that this implies our result. ∎

By multi-variable inverse Mellin transform, we have

$$g(y_1\cdots y_n)^2\hat\phi_{T,H}\left(H\frac{h}{X^{1/n}y}\right)=\frac{1}{(2\pi i)^n}\int\limits_{\mathrm{Re}(s_1)=2}\cdots\int\limits_{\mathrm{Re}(s_n)=2}B_{h,H,X}(s)y^{-s}\,ds.$$

Hence we can write the main term of (2.9) as follows:

$$\sum_\eta\frac{a(\eta+h)\overline{a(\eta)}}{N(\eta)}g\left(\frac{N(\eta)}{X}\right)^2\hat\phi_{T,H}\left(H\frac{h}{\eta}\right)$$
$$=\frac{1}{(2\pi i)^n}\int\limits_{\mathrm{Re}(s_1)=2}\cdots\int\limits_{\mathrm{Re}(s_n)=2}D_f(s+1,h)(X^{1/n})^{s_1+\cdots+s_n}B_{h,H,X}(s)\,ds.$$

Now we move the contour to $\mathrm{Re}(s_j)=-7/18+\epsilon_1$, where ϵ_1 is arbitrarily small. Then

$$S(h)\ll$$
$$\int\limits_{-\infty}^{\infty}\cdots\int\limits_{-\infty}^{\infty}X^{-7/18}(T/H)^{n-1+i_1+\cdots+i_n+\epsilon}N(h)^\epsilon\prod_{j=1}^n(1+|t_j|)^{-i_j+3+\epsilon}\,dt_1\cdots dt_n.$$

Take $i_j = 5$ for each $j = 1, \ldots, n$. Then

$$S(h) \ll X^{-7/18}(T/H)^{6n-1+\epsilon}N(h)^{\epsilon}.$$

Now sum over h in (2.7). Then the off-diagonal contribution to A is

$$\ll XH^n(XH^{n\delta-n})^{1+\epsilon}X^{-7/18}(T/H)^{6n-1+\epsilon} \ll X^{29/18+\epsilon}T^{6n-1+\epsilon}H^{-6n+1}.$$

Since $X \le T^{n+\epsilon}$, it satisfies the desired bound $O(X(T^{n-1}H)^{1+\epsilon})$ as long as

$$H \ge T^{101/108}.$$

This concludes the proof of (2.3).

We give an outline of how the mean-value estimate (2.2) implies the pointwise estimate in Theorem 2.1. We do it for general L-functions. We merely imitate the argument for the Riemann zeta function in [Iv, (7.2)]: Let $L(s)$ be a Dirichlet series which converges absolutely for $\mathrm{Re}(s) \gg 0$, and has a meromorphic continuation to all of $\mathbb{C}$ with pole only at $s = 1$, and satisfies the functional equation

$$\Lambda(s) = L(s)Q^s \prod_{j=1}^{m} \Gamma(a_j s + b_j), \quad \Lambda(s) = \omega\overline{\Lambda(1-\bar{s})},$$

where Q, a_j are positive real numbers and ω, b_j are complex numbers with $\mathrm{Re}(b_j) \ge 0$ and $|\omega| = 1$. Then we prove, for k a fixed positive integer and $T/2 \le t \le 2T$,

$$|L(1/2+it)|^k \ll (\log T)\Big(1 + \int_{-\log^2 T}^{\log^2 T} |L(1/2+i(t+v))|^k e^{-|v|}\, dv\Big), \tag{2.10}$$

where the implied constant depends only on k, Λ. Let $L(s)^k = \sum_{n=1}^{\infty} a(n)n^{-s}$, and $c = 1/\log T$. By using the fact that

$$e^{-x} = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \Gamma(s)x^{-s}\, ds,$$

we have

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \Gamma(w)L(1/2+c+it+w)^k\, dw = \sum_{n=1}^{\infty} a(n)e^{-n}n^{-1/2-c-it} \ll 1.$$

Moving the contour to $\mathrm{Re}(w) = -c$ and using Stirling's formula $\Gamma(\pm c \pm iv) \ll e^{-|v|}(c+|v|)^{-1}$, we have, for $T/3 \le t \le 3T$,

$$L(1/2+c+it)^k \ll 1 + \int_{-\infty}^{\infty} |L(1/2+i(t+v))|^k e^{-|v|}(c+|v|)^{-1}\, dv.$$

By the functional equation,

$$|L(1/2-c+it)| \ll |L(1/2+c+it)| \prod_{j=1}^{m} |t|^{2a_j c}$$
$$\ll (T^c)^{2\sum_{j=1}^m a_j} |L(1/2+c+it)| \ll |L(1/2+c+it)|,$$

since $T^c = e$. On the other hand, by the residue theorem,

$$L(1/2+it)^k = \frac{1}{2\pi i} \int_C L(1/2+it+z)^k \Gamma(z)\, dz,$$

where C is the rectangle with vertices $\pm c \pm i \log^2 T$. By Stirling's formula, the integrals over horizontal sides of C are $o(1)$ as $T \to \infty$. By using the above estimate,

$$|L(1/2+it)|^k \ll 1 + \int_{-\log^2 T}^{\log^2 T} e^{-|u|}(c+|u|)^{-1}$$
$$\times \Big(1 + \int_{-\infty}^{\infty} |L(1/2+it+i(u+v))|^k (c+|v|)^{-1} e^{-|v|}\, dv\Big)\, du.$$

By using the estimate $\int_{-\log^2 T}^{\log^2 T} e^{-|u|}(c+|u|)^{-1}\, du \ll \log T$, and making the substitution $x = u+v$, we have

$$|L(1/2+it)|^k \ll \log T + \int_{-\infty}^{\infty} |L(1/2+it+ix)|^k$$
$$\times \Big(\int_{-\infty}^{\infty} e^{-|u|-|x-u|}(c+|u|)^{-1}(c+|x-u|)^{-1}\, du\Big)\, dx.$$

Ivić [Iv, p. 173] showed that

$$\int_{-\infty}^{\infty} e^{-|u|-|x-u|}(c+|u|)^{-1}(c+|x-u|)^{-1}\, du \ll e^{-|x|}\log T.$$

Using convexity bound, one can show easily

$$\int_{\log^2 T}^{\infty} |L(1/2+it+ix)|^k e^{-|x|}\, dx = o(1),$$
$$\int_{-\infty}^{-\log^2 T} |L(1/2+it+ix)|^k e^{-|x|}\, dx = o(1).$$

This proves (2.10).

Remark 2.3. Diaconu and Garrett [DG] have more general results over arbitrary number fields. In our special case, we give a very short proof by using the technique of [Ti] and [PSa].

Acknowledgments. I thank P. Sarnak who suggested to use [PSa] for the subconvexity bound and provided the crucial Theorem 2.2 which is contained in the preprint [CPSS]. Thanks are due to the referee who read the paper very carefully and pointed out many oversights. Without his/her help, this paper could not have been finished. In particular, the idea of using "fake" L-values in the proof of Theorem 2.1 is due to the referee.

This research was partially supported by an NSERC grant.

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Received on 25.3.2009
and in revised form on 26.5.2010 (5981)