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Chains of metric invariants over a local field

by

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1. Introduction. By a *local field* we mean a field K which is complete with respect to a discrete non-archimedean absolute value $|\cdot|$. The main example we have in mind is that of a p-adic field, that is, a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. If $\mathbb{Q}_p \subseteq K$ we normalize the absolute value $|\cdot|$ on K so that $|p| = 1/p$. We associate two finite chains of metric invariants to each element α separable over K, and investigate the connection between these chains. In what follows by a *finite chain* we mean a matrix

$$A = \begin{pmatrix} t_1 & \dots & t_l \\ D_1 & \dots & D_l \end{pmatrix}$$

such that $t_1 > \dots > t_l = 0$ are real numbers and $1 = D_1 < \dots < D_l$ are integers with the property that D_j divides D_{j+1} for any $j \in \{1, \dots, l-1\}$. We call l the *length* of the chain A. By an *infinite chain* we mean a pair of sequences $((t_n)_{n\ge 1}, (D_n)_{n\ge 1})$ such that $(t_n)_{n\ge 1}$ is a strictly decreasing sequence of real numbers with $\lim_{n\to\infty} t_n = 0$ and $(D_n)_{n\ge 1}$ is a strictly increasing sequence of integers such that $D_1 = 1$ and each D_j divides D_{j+1}. We will associate a pair of infinite chains to each element $T \in \Omega$ which is transcendental over K, where Ω denotes the completion of a fixed separable closure $\overline{K}$ of K with respect to the unique absolute value induced by $|\cdot|$ on $\overline{K}$. In case K is a p-adic field we denote Ω as usual by $\mathbb{C}_p$. The chains we are going to construct consist of some of the most basic metric invariants which can be associated to α and respectively T. In this paper our goal is to show that although the two chains associated to the same element are defined in completely different ways, they are very much related to each other.

The first chain associated to α is given in terms of the distances between α and the elements of $\overline{K}$ of smaller degree over K. In order to define this chain we first prove the following theorem, which reveals a nice structure for the set of degrees over K of elements from any (open or closed) ball in

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$\overline{K}$. For any $\beta \in \overline{K}$ we denote by $\deg_K \beta$ the degree of β over K, that is, $\deg_K \beta = [K(\beta) : K]$.

THEOREM 1. *For any ball B in $\overline{K}$ there is a positive integer $D = D(K, B)$ such that*

$$\{\deg_K \beta : \beta \in B\} = D\mathbb{N}^* = \{D, 2D, 3D, \ldots\}.$$

Let now $\alpha \in \overline{K}$ and consider the map from $(0, \infty)$ to $\mathbb{N}^*$ given by $t \mapsto D(K, B[\alpha, t])$, where $B[\alpha, t]$ denotes the closed ball in $\overline{K}$ of radius t centered at α. This is a decreasing step function which is bounded by $\deg_K \alpha$ since $\alpha \in B[\alpha, t]$ for any t, thus its image consists of finitely many positive integers $D_1 < \ldots < D_l$. From Theorem 1 it follows that D_j divides D_{j+1} for any $j \in \{1, \ldots, l-1\}$. Note that for t large enough $B[\alpha, t]$ will contain elements from K, hence $D_1 = 1$. Also, from Krasner's Lemma we know that if $t < \min\{|\alpha - \sigma(\alpha)| : \sigma \in \mathrm{Gal}(\overline{K}/K),\ \sigma(\alpha) \neq \alpha\}$ then for any $\beta \in B[\alpha, t]$ one has $K(\alpha) \subseteq K(\beta)$, and so $\deg_K \alpha$ divides $\deg_K \beta$. This shows that for t small enough we have

$$D(K, B[\alpha, t]) = D_l = \deg_K \alpha. \tag{1.1}$$

For any $j \in \{1, \ldots, l\}$ we set

$$t_j = \inf\{t > 0 : D(K, B[\alpha, t]) = D_j\}. \tag{1.2}$$

Thus $t_1 > \ldots > t_l = 0$ and we have a chain

$$\mathcal{D}_K(\alpha) = \begin{pmatrix} t_1 & \ldots & t_l \\ D_1 & \ldots & D_l \end{pmatrix}. \tag{1.3}$$

Note that t_1 equals the distance from α to K

$$t_1 = d(\alpha, K) := \inf\{|\alpha - z| : z \in K\}. \tag{1.4}$$

The length l of the chain $\mathcal{D}_K(\alpha)$ in (1.3) will be denoted by $l_K(\alpha)$. These chains $\mathcal{D}_K(\alpha)$ play an essential role in the description of the structure of irreducible polynomials over K. We mention in this context that they can also be defined in terms of the so-called saturated distinguished chains introduced in [6] and later studied also in [1] and [5]. In this paper we try to keep the presentation short and as self-contained as possible. In particular we chose to define the chains $\mathcal{D}_K(\alpha)$ via Theorem 1 rather than to recall the terminology from the above mentioned papers. If one replaces α by an element $T \in \Omega \setminus \overline{K}$, the construction is similar. We intersect the balls centered at T with $\overline{K}$, so we work with the function $t \mapsto D(K, B[T, t] \cap \overline{K})$. This is a decreasing step function whose image consists of infinitely many integer numbers $1 = D_1 < D_2 < \ldots$ Then we define for any D_j a real number $t_j > 0$ as in (1.2) and obtain an infinite chain

$$\mathcal{D}_K(T) = \begin{pmatrix} t_1 & t_2 & \ldots \\ D_1 & D_2 & \ldots \end{pmatrix}. \tag{1.5}$$

Clearly any finite part

$$\begin{pmatrix} t_1 & \dots & t_r \\ D_1 & \dots & D_r \end{pmatrix}$$

of $\mathcal{D}_K(T)$ coincides with the corresponding part in a chain $\mathcal{D}_K(\alpha)$ as in (1.3), provided α is close enough to T.

In this connection we point out that the distinguished sequences $(\alpha_n)_{n\in\mathbb{N}}$ associated to T (see [1]) provide in some sense best possible approximations to T, and some of their properties are reminiscent of those of continued fractions associated to real numbers. In particular they have the property that for each such α_n the entire chain $\mathcal{D}_K(\alpha_n)$ (with the obvious exception of $t_{l_K(\alpha_n)} = 0$) is contained in $\mathcal{D}_K(T)$. This is analogous to the property of convergents α_n to the continued fraction of an irrational real number T to have the same continued fraction as the corresponding part of the continued fraction of T. We remark that even when K is a p-adic field our chains have nothing to do with continued fractions in $\mathbb{Q}_p$ which approximate the elements of $\mathbb{Q}_p$ with rational numbers and which are the true analogs of the continued fractions in $\mathbb{R}$. The chains $\mathcal{D}_{\mathbb{Q}_p}(T)$ are produced by algebraic elements of higher and higher degree over $\mathbb{Q}_p$ and this has no analogue in the archimedean case where the only algebraic extensions of $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{C}$.

We now proceed to construct a second chain for each element $T \in \Omega$. This chain is defined in terms of the distances between T and its conjugates over K, and plays an important role in various questions concerned with the action of the Galois group $G_K = \mathrm{Gal}(\overline{K}/K) \cong \mathrm{Gal}_{\mathrm{cont}}(\Omega/K)$ on Ω, such as the problem of extending the definition of the trace over K to elements from Ω which are not algebraic over K (see [2]). In order to define this chain, we consider for each $\varepsilon > 0$ all the closed balls of radius ε in Ω and intersect them with the orbit $C_K(T) := \{\sigma(T) : \sigma \in G_K\}$. The set $C_K(T)$ is finite if $T \in \overline{K}$, and infinite, but still compact, if $T \in \Omega \setminus \overline{K}$. Since Ω is an ultrametric space, any two balls of radius ε either coincide or they are disjoint. Denote by $N(K,T,\varepsilon)$ the number of disjoint closed balls of radius ε which cover $C_K(T)$. The map from $(0,\infty)$ to $\mathbb{N}^*$ given by $\varepsilon \mapsto N(K,T,\varepsilon)$ is a decreasing step function which is bounded precisely if $T \in \overline{K}$. Thus its image consists of a sequence of positive integers $1 = N_1 < N_2 < \dots$, which is finite or infinite according as $T \in \overline{K}$ or $T \in \Omega \setminus \overline{K}$. For any N_j we set

$$\varepsilon_j = \inf\{\varepsilon > 0 : N(K,T,\varepsilon) = N_j\}. \tag{1.6}$$

Since the Galois group G_K acts transitively on $C_K(T)$ and each automorphism $\sigma \in G_K$ is an isometry one easily sees that for any j, each of the N_j closed balls of radius ε_j which cover $C_K(T)$ will produce the same number of closed balls of radius ε_{j+1} which intersect $C_K(T)$. Thus each N_j

divides N_{j+1}. If $T \in \Omega \setminus \overline{K}$ we get an infinite chain

$$\mathcal{N}_K(T) = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \dots \\ N_1 & N_2 & \dots \end{pmatrix}. \tag{1.7}$$

In case of an $\alpha \in \overline{K}$ this construction produces a finite chain, say of length $l'_K(\alpha)$,

$$\mathcal{N}_K(\alpha) = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_{l'_K(\alpha)} \\ N_1 & \dots & N_{l'_K(\alpha)} \end{pmatrix}. \tag{1.8}$$

Here $\varepsilon_{l'_K(\alpha)} = 0$ and $N_{l'_K(\alpha)} = \deg_K \alpha$. Note that for any $T \in \Omega$, ε_1 equals the diameter of $C_K(T)$:

$$\varepsilon_1 = \operatorname{diam} C_K(T) = \sup \{|T - \sigma(T)| : \sigma \in G_K\}. \tag{1.9}$$

Several precise relations involving the chains $\mathcal{D}_K(\alpha)$ and $\mathcal{N}_K(\alpha)$ have been established by Ota in [5]. For instance, Theorem 3.2 and Proposition 3.4 of [5] imply that the chains $\mathcal{D}_K(\alpha)$ and $\mathcal{N}_K(\alpha)$ coincide if the characteristic of the residue field of K does not divide $\deg_K \alpha$ or if $K(\alpha)/K$ is unramified. There are however elements $\alpha \in \overline{K}$ for which the chains $\mathcal{D}_K(\alpha)$ and $\mathcal{N}_K(\alpha)$ do not coincide. To see this, use (1.4) and (1.9) which give $t_1 = d(\alpha, K)$ and $\varepsilon_1 = \operatorname{diam} C_K(\alpha)$. One always has $\operatorname{diam} C_K(\alpha) \le d(\alpha, K)$, but there are elements $\alpha \in \overline{K}$ for which this inequality is strict. One does have an inequality of the form

$$d(\alpha, K) \le c_p \operatorname{diam} C_K(\alpha)$$

valid for any $\alpha \in \overline{K}$, where c_p is a constant (the so-called Sen–Ax constant) which depends on the characteristic p (or 0) of the residue field of K only, and this inequality was the key ingredient in the construction of the Galois theory in Ω (see Tate [8], Sen [7] and Ax [4]). Thus, for any $\alpha \in \overline{K}$, the first elements t_1 and ε_1 from $\mathcal{D}_K(\alpha)$ and respectively $\mathcal{N}_K(\alpha)$, do not differ too much.

We would like to have a result which gives some control over the other elements of the chains $\mathcal{D}_K(\alpha)$ and $\mathcal{N}_K(\alpha)$ as well. In Section 4 we will prove such a result, which is valid for all elements of Ω. Here we state a simpler version of it, for elements $T \in \Omega$ which are transcendental over K. If

$$\mathcal{A} = \begin{pmatrix} a_1 & \dots & a_n & \dots \\ A_1 & \dots & A_n & \dots \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} b_1 & \dots & b_m & \dots \\ B_1 & \dots & B_m & \dots \end{pmatrix}$$

are two infinite chains, we say that $\mathcal{A}$ is *dominated* by $\mathcal{B}$ if there exists a real number $c > 0$ such that for any pair $\binom{a_n}{A_n}$ from $\mathcal{A}$ there is a pair $\binom{b_m}{B_m}$ in $\mathcal{B}$ such that

$$b_m \le c a_n \tag{1.10}$$

and

$$B_m \text{ divides } A_n. \tag{1.11}$$

If $\mathcal{A}$ is dominated by $\mathcal{B}$ and $\mathcal{B}$ is dominated by $\mathcal{A}$ we say that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent*. One checks that this is indeed an equivalence relation on the set of all infinite chains. The reader might wonder at this point whether our definition of equivalence is strong enough to enable us to transfer relevant information from an infinite chain to an equivalent one. Lemma 3 from Section 2 below provides a quantitative version of equivalence where one can conclude that two chains actually coincide. Also, Lemma 5 shows an effective way of exploiting a natural descent procedure in moving back and forth between two equivalent chains. We will prove the following

THEOREM 2. *For any $T \in \Omega$ which is transcendental over K, the chains $\mathcal{D}_K(T)$ and $\mathcal{N}_K(T)$ are equivalent.*

In many cases one has more information on $\mathcal{D}_K(T)$ than on $\mathcal{N}_K(T)$ (see for example the constructive approach from [1] and [6] which gives all the elements T of Ω, where the chain $\mathcal{D}_K(T)$ is built into the construction). So from this point of view one may interpret Theorem 2 as giving information on the chain $\mathcal{N}_K(T)$ in terms of the "known" chain $\mathcal{D}_K(T)$. As an application of Theorem 2, we will see in Section 2 how one can express a metric obstruction appearing in the definition of the trace over K for elements $T \in \Omega \setminus \overline{K}$, in terms of the chain $\mathcal{D}_K(T)$.

2. Chains. We start with a discussion on chains, which is self-contained and independent of local fields. We use the definition of finite and infinite chains given in the introduction. Let $c > 0$ be a real number and let

$$\mathcal{A} = \begin{pmatrix} a_1 & a_2 & \dots \\ A_1 & A_2 & \dots \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} b_1 & b_2 & \dots \\ B_1 & B_2 & \dots \end{pmatrix}$$

be two finite or infinite chains. We say that $\mathcal{A}$ is c-*dominated* by $\mathcal{B}$ and write $\mathcal{A} \overset{c}{\leq} \mathcal{B}$, provided for any pair $\binom{a_n}{A_n}$ from $\mathcal{A}$ there is a pair $\binom{b_m}{B_m}$ in $\mathcal{B}$ such that (1.10) and (1.11) hold true. Note that if $\mathcal{A} \overset{c}{\leq} \mathcal{B}$ and $\mathcal{B} \overset{c'}{\leq} \mathcal{A}$ for some $c, c' > 0$ then the chains $\mathcal{A}$ and $\mathcal{B}$ are either both finite or both infinite. Indeed, $\mathcal{A}$ is finite if and only if it contains a pair $\binom{a_n}{A_n}$ with $a_n = 0$, and similarly for $\mathcal{B}$, from which the above statement follows immediately.

LEMMA 3. *If $\mathcal{A}$ and $\mathcal{B}$ are finite or infinite chains such that $\mathcal{A} \overset{1}{\leq} \mathcal{B}$ and $\mathcal{B} \overset{1}{\leq} \mathcal{A}$, then $\mathcal{A}$ and $\mathcal{B}$ coincide.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be as in the statement of the lemma. Since $\mathcal{A} \overset{1}{\leq} \mathcal{B}$, for any pair $\binom{a_n}{A_n}$ from $\mathcal{A}$ there is a pair $\binom{b_m}{B_m}$ in $\mathcal{B}$ such that

$$b_m \leq a_n \tag{2.1}$$

and

$$B_m \text{ divides } A_n. \tag{2.2}$$

From $\mathcal{B} \overset{1}{\leq} \mathcal{A}$ it follows that there exists a pair $\binom{a_{n'}}{A_{n'}}$ in $\mathcal{A}$ such that

$$a_{n'} \leq b_m \tag{2.3}$$

and

$$A_{n'} \text{ divides } B_m. \tag{2.4}$$

Relations (2.1) and (2.3) imply $a_{n'} \leq a_n$ so $n' \geq n$, while from (2.2) and (2.4) it follows that $A_{n'}$ divides A_n, which implies $n' \leq n$. Hence $n = n'$, and from (2.1)–(2.4) we see that $a_n = b_m$ and $A_n = B_m$. Therefore each pair $\binom{a_n}{A_n}$ from $\mathcal{A}$ belongs to $\mathcal{B}$. Similarly each pair from $\mathcal{B}$ belongs to $\mathcal{A}$, thus $\mathcal{A}$ and $\mathcal{B}$ consist of the same pairs. Clearly the order of pairs will also be the same, thus the chains $\mathcal{A}$ and $\mathcal{B}$ coincide and the lemma is proved. ■

Let $\mathcal{A} = ((a_n)_{n\geq 1}, (A_n)_{n\geq 1})$ be an infinite chain. We say that $\mathcal{A}$ has the *property* $(\diamondsuit)$ provided

$$\lim_{n\to\infty} \frac{a_n}{|A_n|} = 0 \tag{2.5}$$

where $|\cdot|$ denotes the absolute value on $\overline{K}$. The motivation for introducing this property comes from the definition of the trace over $\mathbb{Q}_p$, or over a finite extension K of $\mathbb{Q}_p$, for elements $T \in \mathbb{C}_p \setminus \overline{K}$, as an integral with respect to the p-adic Haar measure π_T on the orbit $C_K(T)$ (see [2])

$$\operatorname{Tr}_{\mathbb{C}_p/K}(T) := \int_{C_K(T)} z \, d\pi_T(z). \tag{2.6}$$

In general π_T is an unbounded measure and the integral on the right hand side of (2.6) may or may not be defined. By breaking $C_K(T)$ into balls of radius ε and reasoning in terms of the corresponding Riemann sums, a sufficient condition in order for this integral to be defined has been given in [2], in terms of the chain $\mathcal{N}_K(T)$. Similarly, one finds that a necessary condition in terms of $\mathcal{N}_K(T)$ for the integral to be defined is to have (in the notation from the introduction)

$$\lim_{n\to\infty} \frac{\varepsilon_n}{|N_n|} = 0,$$

in other words the condition is $\mathcal{N}_K(T)$ to have property $(\diamondsuit)$. We would like to have this condition expressed if possible in terms of the chain $\mathcal{D}_K(T)$. Keeping in mind the fact that the chains $\mathcal{D}_K(T)$ and $\mathcal{N}_K(T)$ are equivalent by Theorem 2, we ask whether any infinite chain equivalent to one which has the property $(\diamondsuit)$, will also satisfy $(\diamondsuit)$. The answer is “no”. In order to see this, we construct two equivalent chains $\mathcal{A} = ((a_n)_{n\geq 1}, (A_n)_{n\geq 1})$ and $\mathcal{B} = ((b_m)_{m\geq 1}, (B_m)_{m\geq 1})$ in the p-adic case such that $\mathcal{A}$ satisfies $(\diamondsuit)$ but $\mathcal{B}$ does not satisfy $(\diamondsuit)$. We first construct the sequences $(A_n)_{n\geq 1}$ and $(B_m)_{m\geq 1}$.

We set $A_1 = B_1 = 1,\ B_2 = p$. Next, we let $A_2 = B_3$ and $B_4 = p^2 A_2$. Then we let $A_3 = B_5$ and $B_6 = p^3 A_3$, and so on. Here there is no connection between the numbers $A_1, A_2, \dots, A_n, \dots$, except we have to arrange that each A_n divides A_{n+1} and each B_n divides B_{n+1}. Assuming the sequences $(A_n)_{n\ge 1}$ and $(B_m)_{m\ge 1}$ are already constructed, we choose a strictly decreasing sequence $(a_n)_{n\ge 1}$ such that

$$\frac{a_n}{|A_n|} \to 0 \quad \text{and} \quad \frac{a_n}{|p^n A_n|} \to \infty.$$

Next, we set $b_1 = a_1,\ b_3 = a_2,\ b_5 = a_3, \dots$, and then we choose the numbers $b_2, b_4, \dots$ so as to have, for any k,

$$b_{2k-1} > b_{2k} > \max\{b_{2k-1}/2, b_{2k+1}\}.$$

In this way we obtain two equivalent chains $\mathcal{A} = ((a_n)_{n\ge 1}, (A_n)_{n\ge 1})$ and $\mathcal{B} = ((b_m)_{m\ge 1}, (B_m)_{m\ge 1})$ such that $\mathcal{A}$ satisfies $(\diamondsuit)$ while $\mathcal{B}$ does not satisfy $(\diamondsuit)$.

Let now $\mathcal{A} = ((a_n)_{n\ge 1}, (A_n)_{n\ge 1})$ and $\mathcal{B} = ((b_m)_{m\ge 1}, (B_m)_{m\ge 1})$ be arbitrary infinite chains. We say that *$\mathcal{A}$ and $\mathcal{B}$ have finite intersection* if the intersection of the sets $\{A_1, A_2, \dots, A_n, \dots\}$ and $\{B_1, B_2, \dots, B_m, \dots\}$ is finite. Note that the chains from the above example do not have finite intersection. For pairs of chains $(\mathcal{A}, \mathcal{B})$ which have finite intersection we are able to prove a result of the desired shape:

THEOREM 4. *Let $\mathcal{A}$ and $\mathcal{B}$ be two equivalent infinite chains which have finite intersection. If $\mathcal{A}$ has the property $(\diamondsuit)$ then so does $\mathcal{B}$.*

Let $\mathcal{A} = ((a_n)_{n\ge 1}, (A_n)_{n\ge 1})$ and $\mathcal{B} = ((b_m)_{m\ge 1}, (B_m)_{m\ge 1})$ be two equivalent infinite chains which have finite intersection. For each positive integer m we consider the set

$$\mathcal{M}_m = \{n \in \mathbb{N}^* : B_m \text{ divides } A_n\}.$$

It is easy to see that $\mathcal{M}_m$ is not empty. Denote by $n(m)$ the smallest element of $\mathcal{M}_m$. The proof of Theorem 4 is based on the following

LEMMA 5. *Let $\mathcal{A} = ((a_n)_{n\ge 1}, (A_n)_{n\ge 1})$ and $\mathcal{B} = ((b_m)_{m\ge 1}, (B_m)_{m\ge 1})$ be two equivalent chains having finite intersection. Then the sequence*

$$\left(\frac{b_m}{a_{n(m)}}\right)_{m\ge 1}$$

is bounded.

Proof of Theorem 4. Assuming Lemma 5, for any m we write

$$\frac{b_m}{|B_m|} = \frac{b_m}{a_{n(m)}} \cdot \frac{a_{n(m)}}{|A_{n(m)}|} \cdot \frac{|A_{n(m)}|}{|B_m|}. \tag{2.7}$$

On the right hand side of (2.7), the first quotient is bounded as $m \to \infty$ by Lemma 5. The last quotient is ≤ 1 since $A_{n(m)}/B_m$ is an integer by the definition of $n(m)$. Finally, the middle quotient converges to zero as $m \to \infty$ from the assumption that $\mathcal{A}$ has the property $(\diamondsuit)$ together with the fact that $n(m) \to \infty$ as $m \to \infty$. Hence $\lim_{m\to\infty} b_m/|B_m| = 0$ and Theorem 4 is proved.

Proof of Lemma 5. Let $\mathcal{A}$ and $\mathcal{B}$ be as in the statement of the lemma, and let $c > 0$ be such that $\mathcal{A} \overset{c}{\leq} \mathcal{B}$ and $\mathcal{B} \overset{c}{\leq} \mathcal{A}$. For any $m \geq 1$ we consider the pair $\binom{a_{n(m)}}{A_{n(m)}}$ from $\mathcal{A}$. Since $\mathcal{A} \overset{c}{\leq} \mathcal{B}$, there is a pair $\binom{b_{m_1}}{B_{m_1}}$ in $\mathcal{B}$ such that

$$b_{m_1} \leq c a_{n(m)} \tag{2.8}$$

and

$$B_{m_1} \text{ divides } A_{n(m)}. \tag{2.9}$$

From $\mathcal{B} \overset{c}{\leq} \mathcal{A}$ it follows that there is a pair $\binom{a_{n_1}}{A_{n_1}}$ in $\mathcal{A}$ for which

$$a_{n_1} \leq c b_{m_1} \tag{2.10}$$

and

$$A_{n_1} \text{ divides } B_{m_1}. \tag{2.11}$$

By applying again the relation $\mathcal{A} \overset{c}{\leq} \mathcal{B}$ we get a pair $\binom{b_{m_2}}{B_{m_2}}$ in $\mathcal{B}$ such that

$$b_{m_2} \leq c a_{n_1} \tag{2.12}$$

and

$$B_{m_2} \text{ divides } A_{n_1}. \tag{2.13}$$

We stop this “descent” here and, as we shall see in what follows, this is the key point in the proof of Lemma 5. From (2.9) and (2.11) it follows that A_{n_1} divides $A_{n(m)}$, hence $n_1 \leq n(m)$.

We claim that for m large enough the inequality is strict: $n_1 < n(m)$. Indeed, if $n_1 = n(m)$ then one has $A_{n_1} = B_{m_1} = A_{n(m)}$. But the chains $\mathcal{A}$ and $\mathcal{B}$ have finite intersection, so there are positive integers m_0 and n_0 such that the sets $\{A_{n_0}, A_{n_0+1}, \ldots\}$ and $\{B_{m_0}, B_{m_0+1}, \ldots\}$ are disjoint. By the definition of $n(m)$ we see that $n(m) \to \infty$ as $m \to \infty$. Then by (2.8) it follows also that $m_1 \to \infty$ as $m \to \infty$. Thus for m large enough the numbers $A_{n(m)}$ and B_{m_1} are distinct, which proves the claim.

Next, we claim that $m_2 < m$. Indeed, if $m_2 \geq m$ then B_m divides B_{m_2}, and from (2.13) it will follow that B_m divides A_{n_1}. But $n_1 < n(m)$ and $n(m)$ is the smallest integer n for which B_m divides A_n. Therefore B_m does not divide A_{n_1} and so $m_2 < m$ as claimed. As a consequence one has

$$b_m < b_{m_2}. \tag{2.14}$$

On combining the inequalities (2.8), (2.10), (2.12) and (2.14) we obtain

$$b_m < b_{m_2} \leq c a_{n_1} \leq c^2 b_{m_1} \leq c^3 a_{n(m)}.$$

In conclusion, for m large enough one has $b_m/a_{n(m)} < c^3$, which completes the proof of the lemma.

3. Reduction to $\overline{K}$. In this section we show how statements concerning the chains $\mathcal{D}_K(T)$ and $\mathcal{N}_K(T)$ for a general element $T \in \Omega$ can be reduced to the case when $T \in \overline{K}$. For this we need the following comparison lemma. First we introduce one more piece of terminology. Let $\delta > 0$ and let

$$\mathcal{A} = \begin{pmatrix} t_1 & t_2 & \dots \\ D_1 & D_2 & \dots \end{pmatrix}$$

be a finite or infinite chain. Let r be the largest integer for which $t_r \geq \delta$. Then the matrix

$$\begin{pmatrix} t_1 & \dots & t_r \\ D_1 & \dots & D_r \end{pmatrix}$$

will be called the δ-*part* of the chain $\mathcal{A}$.

LEMMA 6. *Let $\delta > 0$ and let T, U be elements of Ω with $|T - U| < \delta$. Then the δ-parts of $\mathcal{D}_K(T)$ and $\mathcal{D}_K(U)$ coincide, and so do the δ-parts of $\mathcal{N}_K(T)$ and $\mathcal{N}_K(U)$.*

Proof. Let $t \in [\delta, \infty)$. Then the closed balls $B[T,t]$ and $B[U,t]$ coincide. It follows that the functions $t \mapsto D(K, B[T,t] \cap \overline{K})$ and $t \mapsto D(K, B[U,t] \cap \overline{K})$ coincide on the interval $[\delta, \infty)$. In terms of the chains $\mathcal{D}_K(T)$ and $\mathcal{D}_K(U)$ this translates exactly into their δ-parts being the same. A similar argument works for $\mathcal{N}_K(T)$ and $\mathcal{N}_K(U)$. More precisely, let $\varepsilon \geq \delta$ and decompose the entire space Ω into a union of disjoint closed balls of radius ε. Then $N(K,T,\varepsilon)$ equals the number of such disjoint balls which have a non-empty intersection with $C_K(T)$, and similarly for $N(K,U,\varepsilon)$. Now any closed ball B of radius ε which intersects $C_K(T)$ will also intersect $C_K(U)$. Indeed, if $\sigma(T) \in B$ for some $\sigma \in G_K$, then since

$$|\sigma(T) - \sigma(U)| = |T - U| < \varepsilon$$

it follows that $\sigma(U) \in B$. Thus a closed ball of radius ε intersects $C_K(T)$ if and only if it intersects $C_K(U)$, hence $N(K,U,\varepsilon) = N(K,T,\varepsilon)$. The maps $\varepsilon \mapsto N(K,T,\varepsilon)$ and $\varepsilon \mapsto N(K,U,\varepsilon)$ being the same on the interval $[\delta, \infty)$, the δ-parts of $\mathcal{N}_K(T)$ and $\mathcal{N}_K(U)$ will coincide, and the lemma is proved. ∎

Using this lemma we now reduce our statements concerning general elements $T \in \Omega$ to the corresponding statements for elements $\alpha \in \overline{K}$. First of all let us see that for each $T \in \Omega \setminus \overline{K}$, $\mathcal{D}_K(T)$ and $\mathcal{N}_K(T)$ are indeed infinite chains. For any $\delta > 0$ we choose an $\alpha \in \overline{K}$ such that $|\alpha - T| < \delta$. Then by Lemma 6 we know that the δ-parts of $\mathcal{D}_K(\alpha)$ and $\mathcal{D}_K(T)$ coincide, and similarly for $\mathcal{N}_K(\alpha)$ and $\mathcal{N}_K(T)$. Since $\mathcal{D}_K(\alpha)$ and $\mathcal{N}_K(\alpha)$ are finite chains it follows that the δ-parts

$$\begin{pmatrix} t_1 & \dots & t_r \\ D_1 & \dots & D_r \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_{r'} \\ N_1 & \dots & N_{r'} \end{pmatrix}$$

of $\mathcal{D}_K(T)$ and $\mathcal{N}_K(T)$ respectively have the required properties $t_1 > \ldots > t_r > 0$, $\varepsilon_1 > \ldots > \varepsilon_{r'} > 0$, $1 = D_1 < \ldots < D_r$, $1 = N_1 < \ldots < N_{r'}$ and also D_j divides D_{j+1}, $1 \le j < r$, and N_i divides N_{i+1}, $1 \le i < r'$. Since this holds for any $\delta > 0$, we see that $\mathcal{D}_K(T)$ and $\mathcal{N}_K(T)$ are infinite chains. Actually there is one more thing we need to show, namely that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \varepsilon_n = 0$. Assume one of these fails, say $\lim_{n\to\infty} t_n = \eta > 0$. Then we take an $\alpha \in \overline{K}$ for which $|\alpha - T| < \eta$ and from the above comparison lemma we get a contradiction since the η-part of $\mathcal{D}_K(\alpha)$ has finite length while the η-part of $\mathcal{D}_K(T)$ has infinite length under the assumption $\lim_{n\to\infty} t_n = \eta$. Thus $\lim_{n\to\infty} t_n = 0$ and similarly $\lim_{n\to\infty} \varepsilon_n = 0$. In conclusion $\mathcal{D}_K(T)$ and $\mathcal{N}_K(T)$ are infinite chains.

Theorem 2 is a consequence of the following more precise result:

THEOREM 7. *Let K be a locally compact field. For any $T \in \Omega$ one has $\mathcal{D}_K(T) \overset{1}{\le} \mathcal{N}_K(T)$ and $\mathcal{N}_K(T) \overset{c_p}{\le} \mathcal{D}_K(T)$, where c_p is the Sen–Ax constant.*

The case $T \in \Omega \setminus \overline{K}$ in Theorem 7 follows easily from the corresponding statement for elements $\alpha \in \overline{K}$. For instance, in order to prove that $\mathcal{D}_K(T) \overset{1}{\le} \mathcal{N}_K(T)$, fix a pair $\left(\begin{smallmatrix} t_n \\ D_n \end{smallmatrix}\right)$ from $\mathcal{D}_K(T)$ (notations are as in (1.5) and (1.7)). We need to show that there is a pair $\left(\begin{smallmatrix} \varepsilon_m \\ N_m \end{smallmatrix}\right)$ in $\mathcal{N}_K(T)$ such that

$$\varepsilon_m \le t_n \tag{3.1}$$

and

$$N_m \text{ divides } D_n. \tag{3.2}$$

Let r be the smallest integer for which $N_r > D_n$ and $\varepsilon_r < t_n$. Choose $\alpha \in \overline{K}$ such that $|\alpha - T| < \varepsilon_r$. From Lemma 6 we know that the ε_r-part of $\mathcal{D}_K(T)$ coincides with that of $\mathcal{D}_K(\alpha)$ and the ε_r-part of $\mathcal{N}_K(T)$ coincides with that of $\mathcal{N}_K(\alpha)$. Note that the pair $\left(\begin{smallmatrix} t_n \\ D_n \end{smallmatrix}\right)$ belongs to the ε_r-part of $\mathcal{D}_K(T)$, so it will also belong to $\mathcal{D}_K(\alpha)$. Assuming that $\mathcal{D}_K(\alpha) \overset{1}{\le} \mathcal{N}_K(\alpha)$, there will be a pair $\left(\begin{smallmatrix} \varepsilon_s(\alpha) \\ N_s(\alpha) \end{smallmatrix}\right)$ in $\mathcal{N}_K(\alpha)$ such that

$$\varepsilon_s(\alpha) \le t_n \tag{3.3}$$

and

$$N_s(\alpha) \text{ divides } D_n. \tag{3.4}$$

Since $\left(\begin{smallmatrix} \varepsilon_r \\ N_r \end{smallmatrix}\right)$ belongs to the ε_r-part of $\mathcal{N}_K(T)$, it will also belong to $\mathcal{N}_K(\alpha)$. One has $N_s(\alpha) \le D_n < N_r$, so the pairs $\left(\begin{smallmatrix} \varepsilon_s(\alpha) \\ N_s(\alpha) \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \varepsilon_r \\ N_r \end{smallmatrix}\right)$ appear in this order in $\mathcal{N}_K(\alpha)$. Thus the former belongs to the ε_r-part of $\mathcal{N}_K(\alpha)$.

Therefore it also belongs to $\mathcal{N}_K(T)$, that is, $\varepsilon_s(\alpha) = \varepsilon_m$ and $N_s(\alpha) = N_m$ for some m, and (3.1) and (3.2) follow from (3.3) and (3.4). The relation $\mathcal{N}_K(T) \overset{c_p}{\leq} \mathcal{D}_K(T)$ can be obtained in a similar way.

4. Proof of Theorems 1 and 7. We first prove Theorem 1. Let B be a ball in $\overline{K}$ and set

$$D = \min\{\deg_K \beta : \beta \in B\}.$$

We need to show that

$$\{\deg_K \beta : \beta \in B\} = \{D, 2D, 3D, \ldots\}. \tag{4.1}$$

Choose $\alpha \in B$ with $\deg_K \alpha = D$. In order to prove the inclusion "$\supseteq$" in (4.1), let m be a positive integer. Choose an element $\theta \in \overline{K}$ such that $\deg_{K(\alpha)} \theta = m$. Thus $[K(\alpha, \theta) : K] = mD$. For any positive integer n set $\beta_n = \alpha + \pi^n \theta$, where π is a uniformizing element of K. One has $\beta_n \in B$ for n large enough. For large n one also has

$$|\beta_n - \alpha| < \min\{|\sigma(\alpha) - \alpha| : \sigma \in G_K,\ \sigma(\alpha) \neq \alpha\}.$$

From Krasner's Lemma (see [3, Ch. 2, Section 6, Theorem 8]) it follows that $K(\alpha) \subseteq K(\beta_n)$. Then $\theta = (\beta_n - \alpha)/\pi^n \in K(\beta_n)$ so $K(\beta_n) = K(\alpha, \theta)$ and one has $\deg_K \beta_n = [K(\alpha, \theta) : K] = mD$. This gives the inclusion "$\supseteq$" in (4.1). For the other inclusion we use the so-called fundamental principle from [6, p. 109]:

Let $\alpha, \beta \in \overline{K}$ be such that

$$|\beta - \alpha| < |\gamma - \alpha| \tag{4.2}$$

for any $\gamma \in \overline{K}$ with $\deg_K \gamma < \deg_K \alpha$. Then the residue field of $K(\alpha)$ is contained in the residue field of $K(\beta)$ and the ramification index of $K(\alpha)/K$ divides the ramification index of $K(\beta)/K$. As a consequence $\deg_K \alpha$ divides $\deg_K \beta$.

With our choice of α, any $\gamma \in \overline{K}$ with $\deg_K \gamma < \deg_K \alpha$ will lie outside the ball B. Therefore (4.2) holds for any $\beta \in B$. It follows that $\deg_K \alpha$ divides $\deg_K \beta$ for any $\beta \in B$, which completes the proof of Theorem 1.

Next we turn to Theorem 7, the remaining case: $\alpha \in \overline{K}$. First we show that $\mathcal{D}_K(\alpha) \overset{1}{\leq} \mathcal{N}_K(\alpha)$. Notations are as in (1.3) and (1.8). Let $\left(\begin{smallmatrix} t_n \\ D_n \end{smallmatrix}\right)$ be a pair in $\mathcal{D}_K(\alpha)$. We need to find $\left(\begin{smallmatrix} \varepsilon_m \\ N_m \end{smallmatrix}\right)$ in $\mathcal{N}_K(\alpha)$ such that

$$\varepsilon_m \leq t_n \tag{4.3}$$

and

$$N_m \text{ divides } D_n. \tag{4.4}$$

We choose m to be the smallest integer for which (4.3) holds. It remains to show that (4.4) also holds. From the definition of t_n, t_{n-1}, ε_m and ε_{m-1} we know that on the open interval (t_n, t_{n-1}) the function $t \mapsto D(K, B[\alpha, t])$ is constantly D_n, while on $(\varepsilon_m, \varepsilon_{m-1})$ the function $\varepsilon \mapsto N(K, \alpha, \varepsilon)$ is constantly N_m. The intervals (t_n, t_{n-1}) and $(\varepsilon_m, \varepsilon_{m-1})$ are not disjoint by our choice of m. Fix a point ϱ in their intersection. Then $D(K, B[\alpha, \varrho]) = D_n$ and $N(K, \alpha, \varrho) = N_m$. Choose an element $\beta \in B[\alpha, \varrho]$ of smallest degree over K, thus $\deg_K \beta = D_n$. Recall that $C_K(\alpha)$ is covered by $N(K, \alpha, \varrho) = N_m$ disjoint closed balls of radius ϱ, denote them by $B_1 = B[\alpha, \varrho], B_2, \ldots, B_{N_m}$.

We claim that $C_K(\beta)$ is covered by the balls $B_1, \ldots, B_{N_m}$ and each of them contains the same number of elements from $C_K(\beta)$. The first statement follows from the fact that each $\sigma(\beta) \in C_K(\beta)$, $\sigma \in G_K$, satisfies $|\sigma(\beta) - \sigma(\alpha)| = |\beta - \alpha| \leq \varrho$, so if B_j is the ball which contains $\sigma(\alpha)$ then it will also contain $\sigma(\beta)$. As for the second statement, let $i, j \in \{1, \ldots, N_m\}$ and choose $\sigma, \tau \in G_K$ such that $\sigma(\alpha) \in B_i$ and $\tau(\alpha) \in B_j$. Then the automorphisms $\tau\sigma^{-1}$ and $\sigma\tau^{-1}$ will send $\tau(\alpha)$ and $\sigma(\alpha)$ to each other, and since they are isometries they will also send the entire balls B_i and B_j to each other. Therefore they will send the sets $B_i \cap C_K(\beta)$ and $B_j \cap C_K(\beta)$ to each other, so $\#(B_i \cap C_K(\beta)) = \#(B_j \cap C_K(\beta))$, as claimed. It follows that N_m divides $\#C_K(\beta) = \deg_K \beta = D_n$, proving (4.4). In conclusion $\mathcal{D}_K(\alpha) \overset{1}{\leq} \mathcal{N}_K(\alpha)$.

Next we show that $\mathcal{N}_K(\alpha) \overset{c_p}{\leq} \mathcal{D}_K(\alpha)$. Let $\binom{\varepsilon_m}{N_m}$ be a pair in $\mathcal{N}_K(\alpha)$. We look for $\binom{t_n}{D_n}$ in $\mathcal{D}_K(\alpha)$ such that

$$t_n \leq c_p \varepsilon_m \tag{4.5}$$

and

$$D_n \text{ divides } N_m. \tag{4.6}$$

Fix $\varrho \in (\varepsilon_m, \varepsilon_{m-1})$. Then $N(K, \alpha, \varrho) = N_m$. Denote as before by $B_1 = B[\alpha, \varrho], B_2, \ldots, B_{N_m}$ the disjoint closed balls of radius ϱ which cover $C_K(\alpha)$. We know that each automorphism $\sigma \in G_K$ produces a permutation of these balls. Let $H = \{\sigma \in G_K : \sigma(B_1) = B_1\}$. Then H is a closed subgroup of finite index in G_K. In fact, $[G_K : H] = N_m$. Indeed, if one chooses for any $j \in \{1, \ldots, N_m\}$ an automorphism $\sigma_j \in G_K$ such that $\sigma_j(B_1) = B_j$, then for each $\tau \in G_K$ there are $j \in \{1, \ldots, N_m\}$ and $\sigma \in H$ uniquely determined such that $\tau = \sigma_j \sigma$, which proves the claim. By Galois theory, H corresponds to a subfield $K \subseteq L \subset \overline{K}$ such that $H = G_L = \mathrm{Gal}(\overline{K}/L)$. Moreover,

$$[L : K] = [G_K : G_L] = N_m. \tag{4.7}$$

Note that since $C_L(\alpha) = \{\sigma(\alpha) : \sigma \in G_L = H\} \subset B_1 = B[\alpha, \varrho]$, one has

$$\operatorname{diam} C_L(\alpha) \leq \varrho. \tag{4.8}$$

By Sen [7] and Ax [4] it follows that

$$d(\alpha, L) \leq c_p \varrho, \tag{4.9}$$

where as usual c_p is the Sen–Ax constant. Therefore for any $\varepsilon > 0$ there is an element $\beta \in L$ such that

$$|\beta - \alpha| \le (c_p + \varepsilon)\varrho. \tag{4.10}$$

We now choose the unique pair $\binom{t_n}{D_n}$ from $\mathcal{D}_K(\alpha)$ for which

$$t_{n-1} > |\beta - \alpha| \ge t_n, \tag{4.11}$$

where for $n = 1$ we set $t_0 = \infty$. From (4.10) and (4.11) we get

$$t_n \le (c_p + \varepsilon)\varrho. \tag{4.12}$$

Recall that for $|\beta - \alpha| < t < t_{n-1}$ the function $t \mapsto D(K, B[\alpha, t])$ is constantly D_n. For any such t one has $\beta \in B[\alpha, t]$, and from Theorem 1 we know that $\deg_K \beta$ is a multiple of $D(K, B[\alpha, t]) = D_n$. Since $\beta \in L$ it follows that $[L : K]$ is also a multiple of D_n, and from (4.7) one obtains (4.6). So far we proved that for any $\varrho \in (\varepsilon_m, \varepsilon_{m-1})$ and any $\varepsilon > 0$ there is an $n = n(\varrho, \varepsilon)$ for which (4.12) and (4.6) hold. We now let $\varepsilon \to 0$ and $\varrho \to \varepsilon_m$ through a sequence of values for which $n(\varrho, \varepsilon)$ is constant, and obtain a pair $\binom{t_n}{D_n}$ in $\mathcal{D}_K(\alpha)$ which satisfies (4.5) and (4.6). This completes the proof of Theorem 7.

Remark. For some classes of elements $\alpha \in \overline{K}$ one can choose β in the above proof such that instead of (4.10) one has the sharper inequality

$$|\beta - \alpha| \le \varrho. \tag{4.13}$$

For instance, if the characteristic of the residue field of K does not divide $\deg_K \alpha$ then one can choose

$$\beta = \frac{\mathrm{Tr}_{K(\alpha)/L}(\alpha)}{[K(\alpha) : L]},$$

and then (4.13) will hold true. Similarly, if the residue field of K is finite and $K(\alpha)/K$ is unramified then as in [5, Proposition 3.4], one can truncate in a suitable way the expansion

$$\alpha = a_0 + a_1\pi + \ldots + a_k\pi^k + \ldots,$$

where π is a uniformizing element of K and the a_j's are roots of unity in $K(\alpha)$, in order to find an element $\beta \in L$ which satisfies (4.13). In such cases when (4.13) holds, the above proof gives $\mathcal{N}_K(\alpha) \overset{1}{\le} \mathcal{D}_K(\alpha)$, and since we also have $\mathcal{D}_K(\alpha) \overset{1}{\le} \mathcal{N}_K(\alpha)$, the chains $\mathcal{D}_K(\alpha)$ and $\mathcal{N}_K(\alpha)$ will coincide by Lemma 3.

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