ACTA ARITHMETICA
103.1 (2002)

L-series and their 2-adic valuations at $s = 1$ attached to CM elliptic curves

by

Derong Qiu and Xianke Zhang (Beijing)

1. Introduction and statement of main results. Consider the family of elliptic curves:

$$E: \quad y^2 = x^3 - Dx \tag{1.0}$$

defined over a number field K. Assume, first, $K = \mathbb{Q}$ is the rational field and $D \in \mathbb{Z}$ are rational integers (nevertheless, we will consider $K = \mathbb{Q}(\sqrt{-1})$ in the following). This family of elliptic curves E have been studied thoroughly for a long time, having close relations with several problems of number theory. For example, they correlate intimately with the congruent number problem when D is a square in $\mathbb{Z}$ (see [Tun]). Curves E have complex multiplication by $\sqrt{-1}$, accordingly their complex L-series $L(E/\mathbb{Q}, s)$ (as curves over $\mathbb{Q}$) could be identified with the L-series $L(\psi, s)$ of Hecke characters ψ (i.e. Grössencharakter) of the field $K = \mathbb{Q}(\sqrt{-1})$ attached to E. (Furthermore, when E are considered as curves over the quadratic fields K above, the L-series satisfy the relation

$$L(E/K, s) = L(\psi, s)L(\overline{\psi}, s),$$

where $\overline{\psi}$ is the dual of ψ.) The "conjecture of Birch and Swinnerton-Dyer" (or "BSD conjecture" for brevity) for an elliptic curve E over $\mathbb{Q}$ asserts that the L-series $L(E/\mathbb{Q}, s)$ has a zero at $s = 1$ of order m equal to the rank r of the Mordell group $E(\mathbb{Q})$, and gives a formula for the limit of $(s-1)^{-r}L(E/\mathbb{Q}, s)$ at $s = 1$ involving arithmetic properties of E (see e.g. [Sil1, p. 362]). In particular, for elliptic curves E in (1.0) with complex multiplication, the BSD conjecture predicts that $L(E/K, 1)$ (after division by

2000 *Mathematics Subject Classification*: Primary 11G05; Secondary 11G15, 11G40, 12J20.

Key words and phrases: elliptic curve, complex multiplication, L-series and BSD conjecture, valuation.

an appropriate period) is divisible by a certain power of 2 (see e.g. [Raz] and [Tun]).

In 1997, C. L. Zhao considered the elliptic curves $E = E_D$ in (1.0) over the Gaussian field $\mathbb{Q}(\sqrt{-1})$ with the assumption that $D = (D_0)^2$ is a perfect square in $\mathbb{Z}[\sqrt{-1}]$, studying the 2-adic valuation of $L(\psi_D, 1)$, the value at $s = 1$ of the L-series of Hecke characters ψ_D attached to E_D. (Actually, the value of the L-series should be first divided by an appropriate period Ω; this normally will not be mentioned again in the following.) He gave a rigorous lower bound for the 2-adic valuation as well as a criterion of reaching this bound, and hence obtained nice results about congruent numbers and showed the BSD conjecture is true for some elliptic curves E_D over $\mathbb{Q}$.

Here we study $E = E_D$ in the general case: D is not necessarily a square. Consider the elliptic curves $E_D : y^2 = x^3 - Dx$ over the Gaussian field $\mathbb{Q}(\sqrt{-1})$ with $D = \pi_1 \ldots \pi_n$ and $D = \pi_1^2 \ldots \pi_r^2 \pi_{r+1} \ldots \pi_n$, where $\pi_1, \ldots, \pi_n$ are distinct Gaussian prime integers in $\mathbb{Z}[\sqrt{-1}]$. (In particular, when $r = n$, the second case turns out to be the case studied in [Zhao].) We will give a formula for the special values at $s = 1$ of the Hecke L-series attached to $E = E_D$ (expressed via the Weierstrass $\wp$-function), lower bounds for the 2-adic valuation of the values, and a criterion of reaching the bounds. These results develop the results for E_D with $D = (D_0)^2$ square in the Gaussian field of [Zhao]. Moreover, our results are consistent with the predictions of the BSD conjecture. We will further study the elliptic curves $E : y^2 = x^3 - D$ having complex multiplication by $\sqrt{-3}$ over the field $\mathbb{Q}(\sqrt{-3})$ in a separate paper, giving results similar to the above but for the 3-adic valuation.

Throughout the following, we put $I = \sqrt{-1}$, and let $K = \mathbb{Q}(\sqrt{-1})$ be the Gaussian number field, $O_K = \mathbb{Z}[\sqrt{-1}]$ the Gaussian integers (i.e. the ring of algebraic integers of K), $E_D : y^2 = x^3 - Dx$ an elliptic curve defined over K with complex multiplication by O_K. We let ψ_D be the Hecke character of K attached to E_D, and let $L(\overline{\psi}_D, s)$ denote the Hecke L-series of $\overline{\psi}_D$, the dual of ψ_D. (For the definition of such Hecke L-series attached to an elliptic curve, see [Sil2].)

(A) Consider $E_D : y^2 = x^3 - Dx$ with $D = \pi_1 \ldots \pi_n$, where $\pi_k \equiv 1 \pmod 4$ are distinct prime integers in O_K ($k=1, \ldots, n$). Set $S=\{\pi_1, \ldots, \pi_n\}$. For any subset T of $\{1, \ldots, n\}$, define

$$D_T = \prod_{k\in T} \pi_k, \qquad \widehat{D}_T = D/D_T,$$

and put $D_\emptyset = 1$ when $T = \emptyset$. Let ψ_{D_T} be the Hecke character (Grössencharakter) of the field K attached to the elliptic curve $E_{D_T} : y^2 = x^3 - D_T x$, and let $L_S(\overline{\psi}_{D_T}, s)$ be the Hecke L-series of $\overline{\psi}_{D_T}$ (the dual of ψ_{D_T}) with all Euler factors at primes in S omitted. We have the following formula for the special value $L_S(\overline{\psi}_{D_T}, 1)$ of the above L-series at $s = 1$ expressed as a finite sum of the values of the Weierstrass $\wp$-function $\wp(z)$.

THEOREM 1. *Let ψ_{D_T} be the Hecke character of the Gaussian field $\mathbb{Q}(\sqrt{-1})$ attached to the elliptic curve $E_{D_T} : y^2 = x^3 - D_T x$, where D_T is any factor of $D = \pi_1 \ldots \pi_n \in \mathbb{Q}(\sqrt{-1})$ as above. Then we have the following formula for the value of L-series*:

$$\begin{aligned}(1.1)\qquad &\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1)\\ &\qquad = \frac{I}{2}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \frac{1}{\wp(c\omega/D) - I} + \frac{1}{4}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4,\end{aligned}$$

where $\theta = 2+2I$, $(-)_4$ is the quartic residue symbol, $\mathcal{C}$ is any complete set of representatives of O_K modulo D which are relatively prime to D, $L_\omega = \omega O_K$ is the period lattice of the elliptic curve $E_1 : y^2 = x^3 - x$,

$$\omega = \int_1^\infty \frac{dx}{\sqrt{x^3 - x}} = 2.6220575\ldots,$$

$\wp(z)$ is the Weierstrass $\wp$-function associated to the lattice L_ω (i.e., $\wp(z)$ and its derivative $\wp'(z)$ satisfy the equation $\wp'(z)^2 = 4\wp(z)^3 - 4\wp(z)$).

REMARK 1.1. Formula (1.1) and its proof are developed from a famous formula and proof of Birch and Swinnerton-Dyer for elliptic curves over the rationals $\mathbb{Q}$ in [B-SD, Formula 3.14].

For any prime number p, we let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ at the p-adic valuation, $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ the algebraic closures of $\mathbb{Q}$ and $\mathbb{Q}_p$ respectively; and let v_p be the normalized p-adic exponential valuation of $\overline{\mathbb{Q}}_p$ (i.e. $v_p(p) = 1$). Fix an isomorphic embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Then $v_p(\alpha)$ is defined for any algebraic number α in $\overline{\mathbb{Q}}$. The value $v_p(\alpha)$ for $\alpha \in \overline{\mathbb{Q}}$ depends on the choice of the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, but this does not affect our discussion in this paper. We will discuss the case $p = 2$.

For any Gaussian integers α, β which are relatively prime, write $(\alpha/\beta)_4^2 = (\alpha/\beta)_2$ for short, and define $[\alpha/\beta]_2 = (1 - (\alpha/\beta)_2)/2$. Then $[\alpha\gamma/\beta]_2 = [\alpha/\beta]_2 + [\gamma/\beta]_2$ (regard $[-]_2$ as an $\mathbb{F}_2$-valued function, where $\mathbb{F}_2$ is the finite field with two elements). For $D = \pi_1 \ldots \pi_n$ as above, put

$$(1.2)\qquad S^*(D) = \frac{I}{2}\sum_{c\in\mathcal{C}} \frac{1}{\wp(c\omega/D) - I}\sum_T \left(\frac{c}{D_T}\right)_4.$$

We will show that $v_2(S^*(D)) \geq (n-1)/2$ (see Lemma 2.4). Accordingly we define an $\mathbb{F}_2$-valued function ε_n as follows ($n = n(D)$ is the number of distinct prime factors of D):

$$\varepsilon_n(D) = \begin{cases} 1 & \text{if } v_2(S^*(D)) = (n-1)/2, \\ 0 & \text{if } v_2(S^*(D)) > (n-1)/2. \end{cases}$$

Then for Gaussian prime integers π, π_k congruent to 1 modulo 4 ($1 \leq k \in \mathbb{Z}$) (and their products), we could define the $\mathbb{F}_2$-valued functions s_1 and δ_n ($n = 1, 2, \ldots$) inductively as follows:

$$\begin{aligned} s_1(\pi) &= \begin{cases} 1 & \text{if } v_2(\pi - 1) = 2, \\ 0 & \text{if } v_2(\pi - 1) > 2, \end{cases} \\ \delta_1(\pi) &= s_1(\pi) + \varepsilon_1(\pi), \\ \delta_n(D) &= \delta_n(\pi_1, \ldots, \pi_n) \\ &= \varepsilon_n(D) + \sum_{\emptyset \neq T \subsetneq \{1,\ldots,n\}} \Bigg(\prod_{k \notin T} \left[\frac{D_T}{\pi_k}\right]_2 \Bigg) \delta_t(D_T) \qquad (n \geq 2), \end{aligned}$$

where $t = \sharp T$ is the cardinal of T.

THEOREM 2. *Let ψ_D be the Hecke character of $\mathbb{Q}(\sqrt{-1})$ attached to the elliptic curve $E_D : y^2 = x^3 - Dx$, where $D = \pi_1 \ldots \pi_n$ with $\pi_k \equiv 1$* (mod 4) *distinct Gaussian prime integers* ($k = 1, \ldots, n$). *Then for the* 2*-adic valuation of the values of the L-series $L(\overline{\psi}_D, s)$ at $s = 1$ we have*:

(i) $v_2(L(\overline{\psi}_D, 1)/\omega) \geq (n-1)/2$;

(ii) *Equality holds in* (i) *if and only if* $\delta_n(D) = 1$.

THEOREM 3. *Let $D = \pm p_1 \ldots p_m \equiv 1 \pmod 4$ with $p_k \not\equiv 5 \pmod 8$ distinct positive rational prime numbers* ($k = 1, \ldots, m$). *If $\delta_n(D) = 1$, then the first part of the BSD conjecture is true for the elliptic curve $E_D : y^2 = x^3 - Dx$, that is,*

$$\operatorname{rank}(E_D(\mathbb{Q})) = \operatorname{Ord}_{s=1}(L(E_D/\mathbb{Q}, s)) = 0, \tag{1.3}$$

where $n = n(D)$ is the number of distinct Gaussian prime factors of D.

(B) Consider the elliptic curves $E_D : y^2 = x^3 - Dx$ with

$$D = \pi_1^2 \ldots \pi_r^2 \pi_{r+1} \ldots \pi_n,$$

where $\pi_k \equiv 1 \pmod 4$ are distinct prime integers in $\mathbb{Z}[\sqrt{-1}]$ ($k = 1, \ldots, n$). Let $\Delta = \pi_1 \ldots \pi_n$ and $S = \{\pi_1, \ldots, \pi_n\}$. For any subset T of $\{1, \ldots, n\}$, define

$$D_T = \prod_{r \geq k \in T} \pi_k^2 \prod_{r < j \in T} \pi_j.$$

Let $L_S(\overline{\psi}_{D_T}, s)$ denote the Hecke L-series of $\overline{\psi}_{D_T}$ (omitting all Euler factors corresponding to primes in S), where ψ_{D_T} is the Hecke character of $K = \mathbb{Q}(\sqrt{-1})$ attached to $E_{D_T} : y^2 = x^3 - D_T x$.

THEOREM 4. *For any factor D_T of $D = \pi_1^2 \dots \pi_r^2 \pi_{r+1} \dots \pi_n \in \mathbb{Q}(\sqrt{-1})$ as above, we have*

$$(1.4)\quad \frac{\Delta}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1) = \frac{I}{2}\sum_{c\in\mathcal{C}} \left(\frac{c}{D_T}\right)_4 \frac{1}{\wp(c\omega/\Delta) - I} + \frac{1}{4}\sum_{c\in\mathcal{C}} \left(\frac{c}{D_T}\right)_4,$$

where $\mathcal{C}$ is any complete set of representatives of O_K modulo Δ relatively prime to Δ, $\theta = 2 + 2I$, $L_\omega = \omega O_K$, ω and $\wp(z)$ are as in Theorem 1.

THEOREM 5. *Let $D = \pi_1^2 \dots \pi_r^2 \pi_{r+1} \dots \pi_n$, where n, r are positive integers $(1 \le r \le n)$ (if $r = n$ then $D = \pi_1^2 \dots \pi_n^2$) and $\pi_k \equiv 1 \pmod 4$ are distinct prime Gaussian integers $(k = 1, \dots, n)$. Then for the 2-adic valuation of the values of the L-series we have*

$$(1.5)\quad v_2(L(\overline{\psi}_D, 1)/\omega) \ge \frac{n}{2} - 1,$$

where ψ_D is the Hecke character of $\mathbb{Q}(\sqrt{-1})$ attached to the elliptic curve $E_D : y^2 = x^3 - Dx$.

2. Proofs of the theorems. We need the following results.

PROPOSITION A. *Let E be an elliptic curve defined over an imaginary quadratic field K with complex multiplication by O_K (the ring of integers of K), $L = \Omega O_K$ be its period lattice, $\Omega \in \mathbb{C}^\times$ a complex number, and ϕ be the Hecke character of K attached to E. Assume $\mathfrak{g}$ is an integral ideal of K, $E_\mathfrak{g}$ is the group of $\mathfrak{g}$-divisible points on E. Let $\mathbf{B}$ be a set of integral ideals of K relatively prime to $\mathfrak{g}$ such that*

$$\{\sigma_\flat \mid \flat \in \mathbf{B}\} = \mathrm{Gal}(K(E_\mathfrak{g})/K), \qquad \sigma_\flat \neq \sigma_{\flat'} \quad \text{if } \flat \neq \flat',$$

where $\sigma_\flat = \left(\frac{K(E_\mathfrak{g})/K}{\flat}\right)$ is the Artin symbol. Assume $\varrho \in \Omega K^\times$ is a complex number such that $\varrho\Omega^{-1}O_K = \mathfrak{g}^{-1}\mathfrak{h}$ for some integral ideal $\mathfrak{h}$ of K which is relatively prime to $\mathfrak{g}$. Then

$$\frac{\phi^k(\mathfrak{h})}{N(\mathfrak{h})^{k-s}} \cdot \frac{\overline{\varrho}^k}{|\varrho|^{2s}} \cdot L_\mathfrak{g}(\overline{\phi}^k, s) = \sum_{\flat\in\mathbf{B}} H_k(\phi(\flat)\varrho, 0, s, L)$$

$(\mathrm{Re}(s) > 1 + k/2)$, *where k is a positive integer and N denotes the norm map from K to $\mathbb{Q}$,*

$$L_\mathfrak{g}(\overline{\phi}^k, s) = \prod_{\wp\nmid\mathfrak{g}} (1 - \overline{\phi}^k(\wp)N(\wp)^{-s})^{-1} \qquad (\mathrm{Re}(s) > 1 + k/2),$$

$$H_k(z, 0, s, L) = \sum{}' \frac{(\overline{z} + \overline{\alpha})^k}{|z + \alpha|^{2s}} \qquad (\mathrm{Re}(s) > 1 + k/2),$$

here $\sum'$ is taken over $\alpha \in L$ other than $-z$ if $z \in L$ (see [Go-Sch]).

LEMMA B. *Let the elliptic curve E, field K, Hecke character ϕ, and $\mathfrak{g}$ be as in Proposition* A. *If the conductor f_ϕ of ϕ divides $\mathfrak{g}$, then the ray class field of K modulo $\mathfrak{g}$ is $K(E_{\mathfrak{g}})$, the extension of K obtained by adding the coordinates of all $\mathfrak{g}$-division points of E to K* (see [Go-Sch]).

Now we consider Theorem 1 and let K, E_D, D_T and $L_S(\overline{\psi}_{D_T}, s)$ be as there. Then by definition (see [B-SD], [Ire-Ro]) we have

LEMMA 2.1.

$$L_S(\overline{\psi}_{D_T}, s) = \begin{cases} L(\overline{\psi}_{D_T}, s) & \text{if } \prod_{\pi_k \in S} \pi_k = D_T, \\ L(\overline{\psi}_{D_T}, s) \prod_{\pi_k \mid \widehat{D}_T} \left(1 - \left(\frac{D_T}{\pi_k}\right)_4 \cdot \overline{\pi}_k / (\pi_k \overline{\pi}_k)^s\right) & \text{otherwise.} \end{cases}$$

Proof of Theorem 1. Assume $L = \Omega O_K$ is the period lattice of $E_{D_T}\colon y^2 = x^3 - D_T x$, where $\Omega = \alpha\omega$, $\alpha \in \mathbb{C}^\times$. (Obviously $\Omega = \omega/\sqrt[4]{D_T}$.) From [Bir-Ste] we know that the conductor of ψ_{D_T} is (θD_T). Now, in Proposition A, let $k = 1$, $\varrho = \Omega/(\theta D)$, $\mathfrak{g} = (\theta D)$, $\mathfrak{h} = O_K$. We have

$$\frac{\overline{\varrho}}{|\varrho|^{2s}} L_{\mathfrak{g}}(\overline{\psi}_{D_T}, s) = \sum_{\flat \in \mathbf{B}} H_1(\psi_{D_T}(\flat)\varrho, 0, s, L) \quad (\mathrm{Re}(s) > 3/2). \tag{2.1}$$

Since the conductor of ψ_{D_T} is θD_T, and $(\theta D_T) | (\theta D) = \mathfrak{g}$, by Lemma B the ray class field of K modulo (θD) is $K((E_{D_T})_{(\theta D)})$, the extension of K obtained by adding the coordinates of θD-division points of E_{D_T} to K. In particular we have the following isomorphism via the Artin map:

$$(O_K/(\theta D))^\times / \mu_4 \cong \mathrm{Gal}(K((E_{D_T})_{(\theta D)})/K),$$

where μ_4 is the group of quartic roots of unity, and $\mu_4 \cong (O_K/\theta)^\times$. So we may take the set

$$\mathbf{B} = \{(c\theta + D) \mid c \in \mathcal{C}\}, \tag{2.2}$$

where $\mathcal{C}$ is as in Theorem 1, a set of representatives of $(O_K/(D))^\times$; thus

$$\frac{\overline{\varrho}}{|\varrho|^{2s}} L_{\mathfrak{g}}(\overline{\psi}_{D_T}, s) = \sum_{c \in \mathcal{C}} H_1(\psi_{D_T}(c\theta + D)\varrho, 0, s, L) \quad (\mathrm{Re}(s) > 3/2). \tag{2.3}$$

Note that the analytic continuation of $H_1(z, 0, 1, L)$ can be given by the Eisenstein E^*-function (see [Zhao] or [We]):

$$H_1(z, 0, 1, L) = E^*_{0,1}(z, L) = E^*_1(z, L).$$

So by (2.3) we have

$$\frac{\theta D}{\Omega} L_{(\theta D)}(\overline{\psi}_{D_T}, 1) = \sum_{c \in \mathcal{C}} E_1^*\left(\psi_{D_T}(c\theta + D)\frac{\Omega}{\theta D}, \Omega O_K\right). \tag{2.4}$$

Since $D \equiv 1 \pmod 4$, we get $c\theta + D \equiv 1 \pmod \theta$ for any $c \in \mathcal{C}$. Thus by the definition of ψ_{D_T} and quartic reciprocity law,

$$\begin{aligned}\psi_{D_T}(c\theta + D) &= \overline{\left(\frac{D_T}{c\theta + D}\right)_4}(c\theta + D) = \overline{\left(\frac{c\theta + D}{D_T}\right)_4}(c\theta + D) \\ &= \overline{\left(\frac{c\theta}{D_T}\right)_4}(c\theta + D).\end{aligned}$$

Then by (2.4) and the fact that $L_{(\theta D)}(\overline{\psi}_{D_T}, 1) = L_S(\overline{\psi}_{D_T}, 1)$, we have

$$\frac{\theta D}{\alpha\omega} L_S(\overline{\psi}_{D_T}, 1) = \sum_{c \in \mathcal{C}} E_1^*\left(\left(\frac{c\omega}{D} + \frac{\omega}{\theta}\right)\alpha\overline{\left(\frac{c\theta}{D_T}\right)_4}, \alpha\omega O_K\right). \tag{2.5}$$

Put $\lambda = \alpha\overline{\left(\frac{c\theta}{D_T}\right)_4}$. Since $E_1^*(\lambda z, \lambda L) = \lambda^{-1}E_1^*(z, L)$, we have

$$\begin{aligned}E_1^*&\left(\left(\frac{c\omega}{D} + \frac{\omega}{\theta}\right)\alpha\overline{\left(\frac{c\theta}{D_T}\right)_4}, \alpha\overline{\left(\frac{c\theta}{D_T}\right)_4}\omega O_K\right) \\ &= \frac{1}{\alpha}\left(\frac{c\theta}{D_T}\right)_4 E_1^*\left(\frac{c\omega}{D} + \frac{\omega}{\theta}, \omega O_K\right).\end{aligned}$$

So by (2.5),

$$\frac{\theta D}{\omega} L_S(\overline{\psi}_{D_T}, 1) = \left(\frac{\theta}{D_T}\right)_4 \sum_{c \in \mathcal{C}} \left(\frac{c}{D_T}\right)_4 E_1^*\left(\frac{c\omega}{D} + \frac{\omega}{\theta}, \omega O_K\right). \tag{2.6}$$

For the period lattice $L_\omega = \omega O_K$ mentioned above, denote the corresponding Weierstrass $\wp$-function by $\wp(z, L_\omega)$ and the corresponding Weierstrass zeta-function by $\zeta(z, L_\omega)$. Then $\wp'(z)^2 = 4\wp(z)^3 - 4\wp(z)$. So by results in [Go-Sch] we have

$$\begin{aligned}&E_1^*\left(\frac{c\omega}{D} + \frac{\omega}{\theta}, \omega O_K\right) \\ &= \zeta\left(\frac{c\omega}{D}, L_\omega\right) + \zeta\left(\frac{\omega}{\theta}, L_\omega\right) + \frac{1}{2} \cdot \frac{\wp'(c\omega/D) - (2 - 2I)}{\wp(c\omega/D) - I} - \frac{\pi}{\omega}\overline{\left(\frac{c}{D} + \frac{1}{\theta}\right)}.\end{aligned} \tag{2.7}$$

We choose $\mathcal{C}$ in such a way that c and $-c$ both are in $\mathcal{C}$. Obviously $\left(\frac{-c}{D_T}\right)_4 = \left(\frac{c}{D_T}\right)_4$. Since $\zeta(z, L_\omega)$ and $\wp'(z, L_\omega)$ are odd functions, and $\wp(z, L_\omega)$ is even, by (2.6) we have

$$
\begin{aligned}
&\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T},1) \\
&= \frac{1}{\theta}\Big\{\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \zeta\left(\frac{c\omega}{D},L_\omega\right) - \frac{\pi}{\omega}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \frac{\overline{c}}{\overline{D}} \\
&\quad + \frac{1}{2}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \frac{\wp'(c\omega/D)}{\wp(c\omega/D)-I} - (1-I)\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \frac{1}{\wp(c\omega/D)-I}\Big\} \\
&\quad + \frac{1}{\theta}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \left(\zeta\left(\frac{\omega}{\theta},L_\omega\right) - \frac{\pi}{\omega\overline{\theta}}\right) \\
&= -\frac{1-I}{\theta}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \frac{1}{\wp(c\omega/D)-I} + \frac{1}{\theta}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \left(\zeta\left(\frac{\omega}{\theta},L_\omega\right) - \frac{\pi}{\omega\overline{\theta}}\right).
\end{aligned}
$$

That is,

$$
\begin{aligned}
(2.8)\quad &\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T},1) \\
&= \frac{I}{2}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \frac{1}{\wp(c\omega/D)-I} + \frac{1}{\theta}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \left(\zeta\left(\frac{\omega}{\theta},L_\omega\right) - \frac{\pi}{\omega\overline{\theta}}\right).
\end{aligned}
$$

By [Zhao] we know that

$$\zeta\left(\frac{\omega}{\theta},L_\omega\right) - \frac{\pi}{\omega\overline{\theta}} = \frac{\theta}{4},$$

so

$$\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T},1) = \frac{I}{2}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \frac{1}{\wp(c\omega/D)-I} + \frac{1}{4}\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4.$$

This proves Theorem 1.

Lemma 2.2. *We have*

$$\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 = \begin{cases} \sharp\mathcal{C} & \text{if } T=\emptyset, \\ 0 & \text{if } T\neq\emptyset. \end{cases}$$

Proof. Easy by the definition of quartic residue symbol.

Lemma 2.3. *Let $D=\pi_1\ldots\pi_n$ where $\pi_k\equiv 1 \pmod 4$ are distinct Gaussian prime ($k=1,\ldots,n$). Let c be any Gaussian integer relatively prime to D. Then*

(1) $\sum_T \left(\frac{c}{D_T}\right)_4 = \mu(1+I)^t$ *or* 0, *where* $\mu\in\{\pm1,\pm I\}$, *t is an integer with $n\le t\le 2n$.*

(2) $\sum_T \left(\frac{c}{D_T}\right)_4 = 0$ *if and only if* $\left(\frac{c}{\pi_k}\right)_4 = -1$ (*for some* $k\in\{1,\ldots,n\}$).

(3) *Suppose that* $\left(\frac{c}{\pi_k}\right)_4 \neq -1$ *for any* $k \in \{1, \dots, n\}$. *Then*

$$\sum_T \left(\frac{c}{D_T}\right)_4 = \mu(1+I)^{n+s}, \qquad \textit{where } \mu \textit{ is as in } (1) \textit{ above},$$

$$s = \sharp\left\{\pi_k : \pi_k \mid D \textit{ and } \left(\frac{c}{\pi_k}\right)_4 = 1,\ k = 1, \dots, n\right\}.$$

In particular,

$$\sum_T \left(\frac{c}{D_T}\right)_4 = 2^n \quad \text{if and only if} \quad \left(\frac{c}{\pi_1}\right)_4 = \dots = \left(\frac{c}{\pi_n}\right)_4 = 1;$$

$$\sum_T \left(\frac{c}{D_T}\right)_4 = \mu(1+I)^n \quad \text{if and only if} \quad \left(\frac{c}{\pi_k}\right)_4 \in \{I, -I\},\ k = 1, \dots, n,$$

where the sum $\sum_T$ is taken over all subsets T of $\{1, \dots, n\}$.

Proof of Lemma 2.3. In fact we have

$$\sum_T \left(\frac{c}{D_T}\right)_4 = \left(1 + \left(\frac{c}{\pi_1}\right)_4\right) \cdots \left(1 + \left(\frac{c}{\pi_n}\right)_4\right),$$

from which the results could be deduced.

LEMMA 2.4. $v_2(S^*(D)) \geq (n-1)/2$.

Proof. By results of [Zhao] or [B-SD], we know that

$$v_2\left(\wp\left(\frac{c\omega}{D}\right) - I\right) = \frac{3}{4}$$

for any Gaussian integer c relatively prime to D. By Lemma 2.3 we have

$$v_2\left(\sum_T \left(\frac{c}{D_T}\right)_4\right) = v_2(\mu(1+I)^t) = \frac{t}{2} \geq \frac{n}{2}.$$

(Here we regard $v_2(0)$ as ∞.) Thus by properties of valuation and our choice of $\mathcal{C}$ with the property $c, -c \in \mathcal{C}$, we have

$$v_2(S^*(D)) \geq -\frac{3}{4} + \frac{n}{2}.$$

Since $\pi_k \equiv 1 \pmod 4$ $(k = 1, \dots, n)$, it follows that

$$N(D_T) \equiv N(D) \equiv 1 \pmod 8, \qquad \left(\frac{I}{D_T}\right)_4 = I^{(N(D_T)-1)/4} = \pm 1.$$

Also

$$\sharp(O_K/(D))^\times = \sharp\mathcal{C} = \prod_{k=1}^n (N(\pi_k) - 1) \equiv 0 \pmod 8,$$

so we can choose $\mathcal{C}$ such that $\pm c, \pm Ic \in \mathcal{C}$ (when $c \in \mathcal{C}$). Put

$$V = \{c \in \mathcal{C} : c \equiv 1 \pmod{\theta}\}, \qquad V' = V \cup IV.$$

Then $\mathcal{C} = V' \cup (-V')$. Since $IO_K = O_K$, we have $IL_\omega = I(\omega O_K) = \omega O_K = L_\omega$. Thus by the definition of Weierstrass $\wp$-function,

$$\begin{aligned}\wp(Iz, IL_\omega) &= \frac{1}{(Iz)^2} + \sum_{\alpha \in IL_\omega} \left(\frac{1}{(Iz-\alpha)^2} - \frac{1}{\alpha^2} \right) \\ &= \frac{1}{(Iz)^2} + \sum_{\alpha' \in L_\omega} \left(\frac{1}{(Iz - I\alpha')^2} - \frac{1}{(I\alpha')^2} \right) = -\wp(z, L_\omega).\end{aligned}$$

In particular,

$$\wp\left(\frac{Ic\omega}{D}, L_\omega\right) = -\wp\left(\frac{c\omega}{D}, L_\omega\right),$$

$$\begin{aligned}S^*(D) &= \frac{I}{2} \sum_{c \in \mathcal{C}} \frac{1}{\wp(c\omega/D) - I} \sum_T \left(\frac{c}{D_T}\right)_4 \\ &= I \sum_{c \in V'} \frac{1}{\wp(c\omega/D) - I} \sum_T \left(\frac{c}{D_T}\right)_4 \\ &= I \sum_{c \in V} \left[\frac{1}{\wp(c\omega/D) - I} \sum_T \left(\frac{c}{D_T}\right)_4 + \frac{1}{\wp(Ic\omega/D) - I} \sum_T \left(\frac{Ic}{D_T}\right)_4 \right] \\ &= I \sum_{c \in V} \left[\sum_T \left(\frac{1}{\wp(c\omega/D) - I} - \left(\frac{I}{D_T}\right)_4 \frac{1}{\wp(c\omega/D) + I} \right) \left(\frac{c}{D_T}\right)_4 \right] \\ &= I \sum_{c \in V} \frac{2B}{(\wp(c\omega/D))^2 + 1} \sum_T \left(\frac{c}{D_T}\right)_4,\end{aligned}$$

where $B = I$ or $\wp(c\omega/D)$.

Note that $v_2(\wp(c\omega/D) - I) = 3/4$, so $v_2(\wp(c\omega/D) + I) = 3/4$. Hence

$$v_2\left(\left(\wp\left(\frac{c\omega}{D}\right)\right)^2 + 1\right) = v_2\left(\wp\left(\frac{c\omega}{D}\right) - I\right) + v_2\left(\wp\left(\frac{c\omega}{D}\right) + I\right) = \frac{3}{4} + \frac{3}{4} = \frac{3}{2},$$

$$v_2\left(\frac{2B}{(\wp(c\omega/D))^2 + 1}\right) = 1 - \frac{3}{2} = -\frac{1}{2}$$

(and obviously we have $v_2(B) = 0$). Therefore

$$v_2(S^*(D)) \geq -\frac{1}{2} + v_2\left(\sum_T \left(\frac{c}{D_T}\right)_4\right) \geq -\frac{1}{2} + \frac{n}{2} = \frac{n-1}{2}.$$

This proves Lemma 2.4.

Proof of Theorem 2. First let us prove

$$v_2(L(\overline{\psi}_D,1)/\omega) \geq (n-1)/2.$$

Taking sums of both sides of formula (1.1) over subsets T of $\{1,\ldots,n\}$, we have

$$\sum_T \frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T},1)$$
$$= \frac{I}{2}\sum_T\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4 \frac{1}{\wp(c\omega/D)-I} + \frac{1}{4}\sum_T\sum_{c\in\mathcal{C}}\left(\frac{c}{D_T}\right)_4.$$

So by Lemma 2.2 and (1.2), we obtain

$$\sum_T \frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T},1) = S^*(D) + \frac{\sharp\mathcal{C}}{4}, \tag{2.9}$$

$$v_2\left(\frac{\sharp\mathcal{C}}{4}\right) = v_2\left(\frac{\prod_{k=1}^n(\pi_k\overline{\pi}_k-1)}{4}\right) \geq 3n-2 \geq n,$$

and by Lemma 2.4 we have

$$v_2\left(\sum_T \frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T},1)\right) \geq \frac{n-1}{2}.$$

By Lemma 2.1 we know that $L_S(\overline{\psi}_{D_T},1) = L(\overline{\psi}_D,1)$ when $T=\{1,\ldots,n\}$; and when $T=\emptyset$ we have

$$L_S(\overline{\psi}_{D_T},1) = L_S(\overline{\psi}_1,1) = L(\overline{\psi}_1,1)\prod_{k=1}^n\left(1-\frac{\overline{\pi}_k}{\pi_k\overline{\pi}_k}\right) = L(\overline{\psi}_1,1)\prod_{k=1}^n\left(1-\frac{1}{\pi_k}\right).$$

By [B-SD] or [Zhao] we know that $L(\overline{\psi}_1,1)=\omega/4$, so

$$L_S(\overline{\psi}_1,1) = \frac{\omega}{4}\prod_{k=1}^n\left(1-\frac{1}{\pi_k}\right),$$

$$v_2(L_S(\overline{\psi}_1,1)/\omega) = v_2\left(\frac{1}{4}\prod_{k=1}^n\left(1-\frac{1}{\pi_k}\right)\right) \geq 2n-2 \quad (\text{since } v_2(\pi_k-1)\geq 2).$$

Now we use induction on n to prove our assertion $v_2(L(\overline{\psi}_D,1)/\omega)\geq(n-1)/2$. If $n=1$, then $D=\pi_1$, $L_S(\overline{\psi}_1,1)=(\omega/4)\cdot(\pi_1-1)/\pi_1$. Since $\pi_1\equiv 1 \pmod 4$, we get $v_2(L_S(\overline{\psi}_1,1)/\omega)\geq 0$. By the above analysis we have

$$v_2\left(\frac{\pi_1}{\omega}\left(\frac{\theta}{1}\right)_4 L_S(\overline{\psi}_1,1) + \frac{\pi_1}{\omega}\overline{\left(\frac{\theta}{\pi_1}\right)_4} L_S(\overline{\psi}_{\pi_1},1)\right) \geq \frac{1-1}{2} = 0.$$

Therefore

$$v_2(L(\overline{\psi}_{\pi_1},1)/\omega) = v_2(L_S(\overline{\psi}_{\pi_1},1)/\omega) \geq 0.$$

Now assume our assertion is true for $1, \ldots, n-1$, and consider $D = \pi_1 \ldots \pi_n$. For any subset T of $\{1, \ldots, n\}$, set $t = t(T) = \sharp T$. By Lemma 2.1,

$$\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1) = \frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L(\overline{\psi}_{D_T}, 1) \prod_{\pi_k | \widehat{D}_T} \left(1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{1}{\pi_k}\right).$$

Since $(D_T/\pi_k)_4 = \pm 1, \pm I$, we get

$$1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{1}{\pi_k} = \frac{\pi_k - \mu}{\pi_k}, \qquad \mu \in \{\pm 1, \pm I\}.$$

Note that $\pi_k \equiv 1 \pmod 4$, so $v_2(\pi_k - \mu) \geq 1/2$; moreover equality holds if and only if $(D_T/\pi_k)_4^2 = -1$. Thus when T is non-trivial (i.e. $1 \leq t < n$), by our inductive assumption,

$$\begin{aligned} v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1)\right) &= v_2(L(\overline{\psi}_{D_T}, 1)/\omega) + \sum_{\pi_k | \widehat{D}_T} v_2\left(1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{1}{\pi_k}\right) \\ &\geq \frac{t-1}{2} + \frac{1}{2} \cdot \sharp\{\pi_k : \pi_k \mid \widehat{D}_T\} = \frac{t-1}{2} + \frac{n-t}{2} = \frac{n-1}{2}. \end{aligned}$$

Also when $T = \emptyset$ we have

$$L_S(\overline{\psi}_{D_T}, 1) = L_S(\overline{\psi}_1, 1) = L(\overline{\psi}_1, 1) \prod_{k=1}^{n} \left(1 - \frac{1}{\pi_k}\right) = \frac{\omega}{4} \prod_{k=1}^{n} \left(1 - \frac{1}{\pi_k}\right),$$

therefore

$$\begin{aligned} &v_2(L_S(\overline{\psi}_1, 1)/\omega) \geq 2n - 2 \geq (n-1)/2, \\ &v_2(L(\overline{\psi}_D, 1)/\omega) = v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D}\right)_4} L(\overline{\psi}_D, 1)\right) \\ &\qquad = v_2\Bigg(\sum_T \frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1) \\ &\qquad\qquad - \sum_{\emptyset \neq T \subsetneq \{1,\ldots,n\}} \frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1) - \frac{D}{\omega} L_S(\overline{\psi}_1, 1)\Bigg) \\ &\qquad \geq (n-1)/2. \end{aligned}$$

Thus we have proved our assertion for any positive integer n.

Now we consider the condition for equality to hold, using also induction on n. If $n = 1$, then $D = \pi_1$, and by (2.9) we obtain

$$\frac{\pi_1}{\omega}\overline{\left(\frac{\theta}{1}\right)_4} L_{\pi_1}(\overline{\psi}_1, 1) + \frac{\pi_1}{\omega}\overline{\left(\frac{\theta}{\pi_1}\right)_4} L_{\pi_1}(\overline{\psi}_{\pi_1}, 1) = S^*(\pi_1) + \frac{\pi_1\overline{\pi}_1 - 1}{4},$$

that is,

$$\frac{1}{4}(\pi_1 - 1) + \frac{\pi_1}{\omega}\overline{\left(\frac{\theta}{\pi_1}\right)_4} L(\overline{\psi}_{\pi_1}, 1) = S^*(\pi_1) + \frac{\pi_1\overline{\pi}_1 - 1}{4}.$$

Since

$$v_2\left(\frac{\pi_1\overline{\pi}_1 - 1}{4}\right) = v_2(\pi_1\overline{\pi}_1 - 1) - 2 \geq 1, \qquad v_2(S^*(\pi_1)) \geq \frac{1-1}{2} = 0$$

(Lemma 2.4), the equality

$$\begin{aligned} v_2(L(\overline{\psi}_{\pi_1}, 1)/\omega) &= v_2\left(\frac{\pi_1}{\omega}\overline{\left(\frac{\theta}{\pi_1}\right)_4} L_{\pi_1}(\overline{\psi}_{\pi_1}, 1)\right) \\ &= v_2\left(S^*(\pi_1) + \frac{\pi_1\overline{\pi}_1 - 1}{4} - \frac{1}{4}(\pi_1 - 1)\right) = 0 \end{aligned}$$

holds if and only if one of the following conditions is true:

(1) $v_2(\pi_1 - 1) = 2$ when $v_2(S^*(\pi_1)) > 0$;
(2) $v_2(\pi_1 - 1) > 2$ when $v_2(S^*(\pi_1)) = 0$.

Thus

$$v_2(L(\overline{\psi}_{\pi_1}, 1)/\omega) = 0 \quad \text{if and only if} \quad \delta_1(\pi_1) = s_1(\pi_1) + \varepsilon_1(\pi_1) = 1.$$

Assume our result is true for $1, \ldots, n-1$, and let $D = \pi_1 \ldots \pi_n$. When $T = \emptyset$, we have

$$\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1) = \frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_D(\overline{\psi}_1, 1) = \frac{D}{\omega} L(\overline{\psi}_1, 1) \prod_{k=1}^{n}\left(1 - \frac{1}{\pi_k}\right),$$

$$\begin{aligned} v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1)\right) &= v_2(L(\overline{\psi}_1, 1)/\omega) + \sum_{k=1}^{n} v_2(\pi_k - 1) \\ &= v_2(1/4) + \sum_{k=1}^{n} v_2(\pi_k - 1) \geq 2n - 2 \geq (n-1)/2. \end{aligned}$$

When $\emptyset \neq T \subsetneq \{1, \ldots, n\}$, we have

$$\begin{aligned} v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1)\right) &= v_2(L_S(\overline{\psi}_{D_T}, 1)/\omega) \\ &= v_2\left(\frac{L(\overline{\psi}_{D_T}, 1)}{\omega} \prod_{\pi_k \mid \widehat{D}_T}\left(1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{1}{\pi_k}\right)\right) \\ &= v_2(L(\overline{\psi}_{D_T}, 1)/\omega) + \sum_{\pi_k \mid \widehat{D}_T} v_2\left(1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{1}{\pi_k}\right). \end{aligned}$$

Since $(D_T/\pi_k)_4 = \pm 1, \pm I$, we have

$$1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{1}{\pi_k} = \frac{\pi_k - \mu}{\pi_k} = \frac{\pi_k - 1 + (1-\mu)}{\pi_k}, \qquad \mu \in \{\pm 1, \pm I\}.$$

Therefore

$$v_2\left(1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{1}{\pi_k}\right) \geq \frac{1}{2},$$

and equality holds if and only if $(D_T/\pi_k)_4 = \pm I$, i.e. $(D_T/\pi_k)_4^2 = -1$, that is, $[D_T/\pi_k]_2 = 1$. Thus

$$v_2\left(1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{1}{\pi_k}\right) = \frac{1}{2} \quad \text{if and only if} \quad \left[\frac{D_T}{\pi_k}\right]_2 = 1.$$

By the proof of the first part of the theorem we know that

$$v_2(L(\overline{\psi}_{D_T}, 1)/\omega) \geq (t(T) - 1)/2, \qquad t(T) = \sharp T,$$

and by our inductive assumption, equality holds if and only if $\delta_t(D_T) = 1$, $t = t(T)$. Thus

$$v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1)\right) \geq \frac{t(T)-1}{2} + \frac{n - t(T)}{2} = \frac{n-1}{2},$$

and equality holds if and only if $[D_T/\pi_k]_2 = 1$ (for any $\pi_k \mid \widehat{D}_T$) and $\delta_t(D_T) = 1$. That is to say,

$$v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1)\right) = \frac{n-1}{2}$$

if and only if

$$\left(\prod_{\pi_k \mid \widehat{D}_T} \left[\frac{D_T}{\pi_k}\right]_2\right) \delta_t(D_T) = 1.$$

For the elliptic curve E_{D_T}: $y^2 = x^3 - D_T x$ and Hecke characters ψ_{D_T}, by [Ru1,2] we know that $L(\overline{\psi}_{D_T}, 1)/\varOmega \in K = \mathbb{Q}(I)$, and also we have $\varOmega = \omega/\sqrt[4]{D_T}$, so

$$\begin{aligned} L(\overline{\psi}_{D_T}, 1)/\omega &= (\sqrt[4]{D_T})^{-1} \cdot L(\overline{\psi}_{D_T}, 1)/\frac{\omega}{\sqrt[4]{D_T}} \\ &= (\sqrt[4]{D_T})^{-1} \cdot L(\overline{\psi}_{D_T}, 1)/\varOmega \in K(\sqrt[4]{D_T}), \end{aligned}$$

i.e. $L(\overline{\psi}_{D_T}, 1)/\omega \in K(\sqrt[4]{D_T})$. Thus by Lemma 2.1 we get

$$\begin{aligned} &\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4} L_S(\overline{\psi}_{D_T}, 1) \\ &\quad = D\overline{\left(\frac{\theta}{D_T}\right)_4} \prod_{\pi_k \mid \widehat{D}_T} \left(1 - \left(\frac{D_T}{\pi_k}\right)_4 \frac{1}{\pi_k}\right) \cdot L(\overline{\psi}_{D_T}, 1)/\omega \in K(\sqrt[4]{D_T}), \end{aligned}$$

and if

$$v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1)\right)=\frac{n-1}{2},$$

then

$$\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1)=(1+I)^{n-1}\alpha_T\sqrt[4]{D_T^3},$$

where $\alpha_{T}\in K$, and $v_2(\alpha_{T})=0$ (since $v_2(\sqrt[4]{D_T^3})=\frac{3}{4}v_2(D_T)=0$). For any subsets T and T' of $\{1,\dots,n\}$, if $v_2(\alpha_{T})=v_2(\alpha_{T'})=0$, then it can be easily verified that

$$v_2(\alpha_{T}\sqrt[4]{D_T^3}+\alpha_{T'}\sqrt[4]{D_{T'}^3})>0.$$

Thus, consider the terms in the sum

$$\sum_{\emptyset\neq T\subsetneq\{1,\dots,n\}}\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1).\tag{2.10}$$

For any two terms with 2-adic valuations equal to $(n-1)/2$, the 2-adic valuation of their sum is greater than $(n-1)/2$. Also when $n>1$ we have

$$v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_\emptyset}\right)_4}L_S(\overline{\psi}_{D_\emptyset},1)\right)\geq 2n-2\geq n>\frac{n-1}{2}.$$

Hence $v_2(L(\overline{\psi}_D,1)/\omega)=(n-1)/2$ if and only if one of the following statements is true:

(1) when $v_2(S^*(D))>(n-1)/2$, in the above sum (2.10), the number of terms with 2-adic valuation $(n-1)/2$ is odd;

(2) when $v_2(S^*(D))=(n-1)/2$, in the above sum (2.10), the number of terms with 2-adic valuation $(n-1)/2$ is even.

Statement (1) above means: if $\varepsilon_n(D)=0$, then

$$\begin{aligned}\sharp\left\{\emptyset\neq T\subsetneq\{1,\dots,n\}:v_2\left(\frac{D}{\omega}\overline{\left(\frac{\theta}{D_T}\right)_4}L_S(\overline{\psi}_{D_T},1)\right)=\frac{n-1}{2}\right\}\\ =\sharp\left\{\emptyset\neq T\subsetneq\{1,\dots,n\}:\left(\prod_{\pi_k\mid\widehat{D}_T}\left[\frac{D_T}{\pi_k}\right]_2\right)\delta_t(D_T)=1\right\}\\ \equiv\sum_{\emptyset\neq T\subsetneq\{1,\dots,n\}}\left(\prod_{\pi_k\mid\widehat{D}_T}\left[\frac{D_T}{\pi_k}\right]_2\right)\delta_t(D_T)\equiv 1\pmod 2,\end{aligned}$$

$$\delta_n(D)=\varepsilon_n(D)+\sum_{\emptyset\neq T\subsetneq\{1,\dots,n\}}\left(\prod_{\pi_k\mid\widehat{D}_T}\left[\frac{D_T}{\pi_k}\right]_2\right)\delta_t(D_T)\equiv 1\pmod 2.$$

And (2) means: if $\varepsilon_n(D) = 1$ then

$$\sum_{\emptyset \neq T \subsetneq \{1,\dots,n\}} \Bigg(\prod_{\pi_k | \widehat{D}_T} \left[\frac{D_T}{\pi_k}\right]_2 \Bigg) \delta_t(D_T) \equiv 0 \pmod 2,$$

so

$$\begin{aligned}\delta_n(D) &= \varepsilon_n(D) + \sum_{\emptyset \neq T \subsetneq \{1,\dots,n\}} \Bigg(\prod_{\pi_k | \widehat{D}_T} \left[\frac{D_T}{\pi_k}\right]_2 \Bigg) \delta_t(D_T) \\ &\equiv 1 + 0 \equiv 1 \pmod 2.\end{aligned}$$

Hence $v_2(L(\overline{\psi}_D, 1)/\omega) = (n-1)/2$ if and only if $\delta_n(D) = 1$. This proves the theorem.

Proof of Theorem 3. This theorem follows from Theorem 2 and the main result of Coates–Wiles in [Co-Wi].

For the elliptic curve $E_D : y^2 = x^3 - Dx$ with $D = \pi_1^2 \dots \pi_r^2 \pi_{r+1} \dots \pi_n$, where $\pi_k \equiv 1 \pmod 4$ are distinct Gaussian prime integers ($k = 1, \dots, n$), we could prove Theorems 4 and 5 similarly to Theorems 1 and 2.

References

[Bir-Ste] B. J. Birch and N. M. Stephens, *The parity of the rank of the Mordell–Weil group*, Topology 5 (1966), 295–299.

[B-SD] B. J. Birch and H. P. F. Swinnerton-Dyer, *Notes on elliptic curves II*, J. Reine Angew. Math. 218 (1965), 79–108.

[Co-Wi] J. Coates and A. Wiles, *On the conjecture of Birch and Swinnerton-Dyer*, Invent. Math. 39 (1977), 223–251.

[Go-Sch] C. Goldstein et N. Schappacher, *Séries d'Eisenstein et fonction L de courbes ellipliques à multiplication complexe*, J. Reine Angew. Math. 327 (1981), 184–218.

[Ire-Ro] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Grad. Texts in Math. 84, Springer, New York, 1990.

[Raz] M. Razar, *The nonvanishing of $L(1)$ for certain elliptic curves with no first descent*, Amer. J. Math. 96 (1974), 104–126.

[Ru1] K. Rubin, *Tate–Shafarevich groups and L-functions of elliptic curves with complex multiplication*, Invent. Math. 89 (1987), 527–560.

[Ru2] —, *The "main conjectures" of Iwasawa theory for imaginary quadratic fields*, ibid. 103 (1991), 25–68.

[Sil1] J. H. Silverman, *The Arithmetic of Elliptic Curves*, Grad. Texts in Math. 106, Springer, New York, 1986.

[Sil2] —, *Advanced Topics in the Arithmetic of Elliptic Curves*, Grad. Texts in Math. 151, Springer, 1994.

[Tun] J. B. Tunnell, *A classical Diophantine problem and modular forms of weight* $\frac{3}{2}$, Invent. Math. 72 (1983), 323–334.

[We] A. Weil, *Elliptic Functions According to Eisenstein and Kronecker*, Springer, 1976.

[Zhao] C. L. Zhao, *A criterion for elliptic curves with lowest 2-power in* $L(1)$, Math. Proc. Cambridge Philos. Soc. 121 (1997), 385–400.

Center for Advanced Study
Tsinghua University
Beijing 100084, P.R. China
E-mail: derong@castu.tsinghua.edu.cn

Department of Mathematical Sciences
Tsinghua University
Beijing 100084, P.R. China
E-mail: xianke@tsinghua.edu.cn

Received on 12.4.2001
and in revised form on 14.7.2001 (4013)