L-series and their 2-adic valuations at \( s = 1 \) attached to CM elliptic curves

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1. Introduction and statement of main results. Consider the family of elliptic curves:

\[
E : \quad y^2 = x^3 - Dx
\]

defined over a number field \( K \). Assume, first, \( K = \mathbb{Q} \) is the rational field and \( D \in \mathbb{Z} \) are rational integers (nevertheless, we will consider \( K = \mathbb{Q}(\sqrt{-1}) \) in the following). This family of elliptic curves \( E \) have been studied thoroughly for a long time, having close relations with several problems of number theory. For example, they correlate intimately with the congruent number problem when \( D \) is a square in \( \mathbb{Z} \) (see [Tun]). Curves \( E \) have complex multiplication by \( \sqrt{-1} \), accordingly their complex \( L \)-series \( L(E/\mathbb{Q}, s) \) (as curves over \( \mathbb{Q} \)) could be identified with the \( L \)-series \( L(\psi, s) \) of Hecke characters \( \psi \) (i.e. Grössencharakter) of the field \( K = \mathbb{Q}(\sqrt{-1}) \) attached to \( E \). Furthermore, when \( E \) are considered as curves over the quadratic fields \( K \) above, the \( L \)-series satisfy the relation

\[
L(E/K, s) = L(\psi, s)L(\overline{\psi}, s),
\]

where \( \overline{\psi} \) is the dual of \( \psi \). The “conjecture of Birch and Swinnerton-Dyer” (or “BSD conjecture” for brevity) for an elliptic curve \( E \) over \( \mathbb{Q} \) asserts that the \( L \)-series \( L(E/\mathbb{Q}, s) \) has a zero at \( s = 1 \) of order \( m \) equal to the rank \( r \) of the Mordell group \( E(\mathbb{Q}) \), and gives a formula for the limit of \((s - 1)^{-r}L(E/\mathbb{Q}, s)\) at \( s = 1 \) involving arithmetic properties of \( E \) (see e.g. [Sil1, p. 362]). In particular, for elliptic curves \( E \) in (1.0) with complex multiplication, the BSD conjecture predicts that \( L(E/K, 1) \) (after division by

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an appropriate period) is divisible by a certain power of 2 (see e.g. [Raz] and [Tun]).

In 1997, C. L. Zhao considered the elliptic curves $E = E_D$ in (1.0) over the Gaussian field $\mathbb{Q}(\sqrt{-1})$ with the assumption that $D = (D_0)^2$ is a perfect square in $\mathbb{Z}[\sqrt{-1}]$, studying the 2-adic valuation of $L(\psi_D, 1)$, the value at $s = 1$ of the $L$-series of Hecke characters $\psi_D$ attached to $E_D$. (Actually, the value of the $L$-series should be first divided by an appropriate period $\Omega$; this normally will not be mentioned again in the following.) He gave a rigorous lower bound for the 2-adic valuation as well as a criterion of reaching this bound, and hence obtained nice results about congruent numbers and showed the BSD conjecture is true for some elliptic curves $E_D$ over $\mathbb{Q}$.

Here we study $E = E_D$ in the general case: $D$ is not necessarily a square. Consider the elliptic curves $E_D : y^2 = x^3 - Dx$ over the Gaussian field $\mathbb{Q}(\sqrt{-1})$ with $D = \pi_1 \ldots \pi_n$ and $D = \pi_1^2 \ldots \pi_r^2 \pi_{r+1} \ldots \pi_n$, where $\pi_1, \ldots, \pi_n$ are distinct Gaussian prime integers in $\mathbb{Z}[\sqrt{-1}]$. (In particular, when $r = n$, the second case turns out to be the case studied in [Zhao].) We will give a formula for the special values at $s = 1$ of the Hecke $L$-series attached to $E = E_D$ (expressed via the Weierstrass $\wp$-function), lower bounds for the 2-adic valuation of the values, and a criterion of reaching the bounds. These results develop the results for $E_D$ with $D = (D_0)^2$ square in the Gaussian field of [Zhao]. Moreover, our results are consistent with the predictions of the BSD conjecture. We will further study the elliptic curves $E : y^2 = x^3 - D$ having complex multiplication by $\mathbb{Z}[\sqrt{-3}]$ in a separate paper, giving results similar to the above but for the 3-adic valuation.

Throughout the following, we put $I = \sqrt{-1}$, and let $K = \mathbb{Q}(\sqrt{-1})$ be the Gaussian number field, $O_K = \mathbb{Z}[\sqrt{-1}]$ the Gaussian integers (i.e. the ring of algebraic integers of $K$), $E_D : y^2 = x^3 - Dx$ an elliptic curve defined over $K$ with complex multiplication by $O_K$. We let $\psi_D$ be the Hecke character of $K$ attached to $E_D$, and let $L(\overline{\psi}_D, s)$ denote the Hecke $L$-series of $\overline{\psi}_D$, the dual of $\psi_D$. (For the definition of such Hecke $L$-series attached to an elliptic curve, see [Sil2].)

(A) Consider $E_D : y^2 = x^3 - Dx$ with $D = \pi_1 \ldots \pi_n$, where $\pi_k \equiv 1$ (mod 4) are distinct prime integers in $O_K$ ($k = 1, \ldots, n$). Set $S = \{\pi_1, \ldots, \pi_n\}$. For any subset $T$ of $\{1, \ldots, n\}$, define

$$D_T = \prod_{k \in T} \pi_k, \quad \widehat{D}_T = D/D_T,$$

and put $D_\emptyset = 1$ when $T = \emptyset$. Let $\psi_{D_T}$ be the Hecke character (Grössencharakter) of the field $K$ attached to the elliptic curve $E_{D_T} : y^2 = x^3 - D_T x$, and let $L_S(\overline{\psi}_{D_T}, s)$ be the Hecke $L$-series of $\overline{\psi}_{D_T}$ (the dual of $\psi_{D_T}$) with all Euler factors at primes in $S$ omitted. We have the following formula for the
special value $L_S(\psi_{D_T}, 1)$ of the above $L$-series at $s = 1$ expressed as a finite sum of the values of the Weierstrass $\wp$-function $\wp(z)$.

**Theorem 1.** Let $\psi_{D_T}$ be the Hecke character of the Gaussian field $\mathbb{Q}(\sqrt{-1})$ attached to the elliptic curve $E_{D_T}: y^2 = x^3 - D_T x$, where $D_T$ is any factor of $D = \pi_1 \ldots \pi_n \in \mathbb{Q}(\sqrt{-1})$ as above. Then we have the following formula for the value of $L$-series:

$$L_S(\psi_{D_T}, 1) = \frac{1}{2} \sum_{c \in \mathcal{C}} \left( \frac{c}{D_T} \right)_4 \wp(c\omega / D) - I + \frac{1}{4} \sum_{c \in \mathcal{C}} \left( \frac{c}{D_T} \right)_4,$$

where $\theta = 2 + 2I, (-)_4$ is the quartic residue symbol, $\mathcal{C}$ is any complete set of representatives of $O_K$ modulo $D$ which are relatively prime to $D$, $L_\omega = \omega O_K$ is the period lattice of the elliptic curve $E_1: y^2 = x^3 - x$,

$$\omega = \int_1^\infty \frac{dx}{\sqrt{x^3 - x}} = 2.6220575\ldots,$$

$\wp(z)$ is the Weierstrass $\wp$-function associated to the lattice $L_\omega$ (i.e., $\wp(z)$ and its derivative $\wp'(z)$ satisfy the equation $\wp'(z)^2 = 4\wp(z)^3 - 4\wp(z)$).

**Remark 1.1.** Formula (1.1) and its proof are developed from a famous formula and proof of Birch and Swinnerton-Dyer for elliptic curves over the rationals $\mathbb{Q}$ in [B-SD, Formula 3.14].

For any prime number $p$, we let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ at the $p$-adic valuation, $\overline{\mathbb{Q}}$ and $\overline{\mathbb{Q}}_p$ the algebraic closures of $\mathbb{Q}$ and $\mathbb{Q}_p$ respectively; and let $v_p$ be the normalized $p$-adic exponential valuation of $\overline{\mathbb{Q}}_p$ (i.e. $v_p(p) = 1$). Fix an isomorphic embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Then $v_p(\alpha)$ is defined for any algebraic number $\alpha$ in $\overline{\mathbb{Q}}$. The value $v_p(\alpha)$ for $\alpha \in \overline{\mathbb{Q}}$ depends on the choice of the embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, but this does not affect our discussion in this paper. We will discuss the case $p = 2$.

For any Gaussian integers $\alpha, \beta$ which are relatively prime, write $(\alpha/\beta)_2$ for short, and define $[\alpha/\beta]_2 = (1 - (\alpha/\beta)_2)/2$. Then $[\alpha\gamma/\beta]_2 = [\alpha/\beta]_2 + [\gamma/\beta]_2$ (regard $[-]_2$ as an $\mathbb{F}_2$-valued function, where $\mathbb{F}_2$ is the finite field with two elements). For $D = \pi_1 \ldots \pi_n$ as above, put

$$S^*(D) = \frac{1}{2} \sum_{c \in \mathcal{C}} \frac{1}{\wp(c\omega / D) - I} \sum_{T} \left( \frac{c}{D_T} \right)_4.$$
distinct prime factors of $D$):

$$
\varepsilon_n(D) = \begin{cases} 
1 & \text{if } v_2(S^*(D)) = (n - 1)/2, \\
0 & \text{if } v_2(S^*(D)) > (n - 1)/2.
\end{cases}
$$

Then for Gaussian prime integers $\pi_k$ congruent to 1 modulo 4 ($1 \leq k \in \mathbb{Z}$) (and their products), we could define the $\mathbb{F}_2$-valued functions $s_1$ and $\delta_n$ ($n = 1, 2, \ldots$) inductively as follows:

$$
s_1(\pi) = \begin{cases} 
1 & \text{if } v_2(\pi - 1) = 2, \\
0 & \text{if } v_2(\pi - 1) > 2,
\end{cases}
$$

$$
\delta_1(\pi) = s_1(\pi) + \varepsilon_1(\pi),
$$

\[
\delta_n(D) = \delta_n(\pi_1, \ldots, \pi_n)
= \varepsilon_n(D) + \sum_{\emptyset \neq T \subset \{1, \ldots, n\}} \left( \prod_{k \in T} \left[ \frac{D_T}{\pi_k} \right] \right) \delta_t(D_T) \quad (n \geq 2),
\]

where $t = \# T$ is the cardinal of $T$.

**Theorem 2.** Let $\psi_D$ be the Hecke character of $\mathbb{Q}(\sqrt{-1})$ attached to the elliptic curve $E_D : y^2 = x^3 - Dx$, where $D = \pi_1 \ldots \pi_n$ with $\pi_k \equiv 1 \pmod{4}$ (mod 4) distinct Gaussian prime integers ($k = 1, \ldots, n$). Then for the 2-adic valuation of the values of the L-series $L(\psi_D, s)$ at $s = 1$ we have:

(i) $v_2(L(\psi_D, 1)/\omega) \geq (n - 1)/2$;

(ii) Equality holds in (i) if and only if $\delta_n(D) = 1$.

**Theorem 3.** Let $D = \pm p_1 \ldots p_m \equiv 1 \pmod{4}$ with $p_k \not\equiv 5 \pmod{8}$ distinct positive rational prime numbers ($k = 1, \ldots, m$). If $\delta_n(D) = 1$, then the first part of the BSD conjecture is true for the elliptic curve $E_D : y^2 = x^3 - Dx$, that is,

\[
\text{rank}(E_D(\mathbb{Q})) = \text{Ord}_{s=1}(L(E_D/\mathbb{Q}, s)) = 0, \tag{1.3}
\]

where $n = n(D)$ is the number of distinct Gaussian prime factors of $D$.

(B) Consider the elliptic curves $E_D : y^2 = x^3 - Dx$ with

$$
D = \pi_1^2 \ldots \pi_r^2 \pi_{r+1} \ldots \pi_n,
$$

where $\pi_k \equiv 1 \pmod{4}$ are distinct prime integers in $\mathbb{Z}[\sqrt{-1}]$ ($k = 1, \ldots, n$). Let $\Delta = \pi_1 \ldots \pi_n$ and $S = \{\pi_1, \ldots, \pi_n\}$. For any subset $T$ of $\{1, \ldots, n\}$, define

$$
D_T = \prod_{r \geq k \in T} \pi_k^2 \prod_{r < j \in T} \pi_j.
$$

Let $L_S(\overline{\psi}_{D_T}, s)$ denote the Hecke L-series of $\overline{\psi}_{D_T}$ (omitting all Euler factors corresponding to primes in $S$), where $\psi_{D_T}$ is the Hecke character of $K = \mathbb{Q}(\sqrt{-1})$ attached to $E_{D_T} : y^2 = x^3 - D_T x$. 
Theorem 4. For any factor $D_T$ of $D = \pi_1^2 \ldots \pi_r^2 \pi_{r+1}^2 \ldots \pi_n \in \mathbb{Q}(\sqrt{-1})$ as above, we have

\[ \frac{\Delta}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_S(\overline{\psi_{D_T}}, 1) = \frac{I}{2} \sum_{c \in \mathcal{C}} \left( \frac{c}{D_T} \right)_4 \frac{1}{\phi(c \omega / \Delta)} - I + \frac{1}{4} \sum_{c \in \mathcal{C}} \left( \frac{c}{D_T} \right)_4, \]

where $\mathcal{C}$ is any complete set of representatives of $\mathcal{O}_K$ modulo $\Delta$ relatively prime to $\Delta$, $\theta = 2 + 2i$, $L_\omega = \omega \mathcal{O}_K$, $\omega$ and $\phi(z)$ are as in Theorem 1.

Theorem 5. Let $D = \pi_1^2 \ldots \pi_r^2 \pi_{r+1}^2 \ldots \pi_n$, where $n, r$ are positive integers ($1 \leq r \leq n$) (if $r = n$ then $D = \pi_1^2 \ldots \pi_n^2$) and $\pi_k \equiv 1 \mod{4}$ are distinct prime Gaussian integers ($k = 1, \ldots, n$). Then for the 2-adic valuation of the values of the $L$-series we have

\[ v_2(L(\overline{\psi_D}, 1)/\omega) \geq \frac{n}{2} - 1, \]

where $\psi_D$ is the Hecke character of $\mathbb{Q}(\sqrt{-1})$ attached to the elliptic curve $E_D : y^2 = x^3 - Dx$.

2. Proofs of the theorems. We need the following results.

Proposition A. Let $E$ be an elliptic curve defined over an imaginary quadratic field $K$ with complex multiplication by $\mathcal{O}_K$ (the ring of integers of $K$), $L = \Omega \mathcal{O}_K$ be its period lattice, $\Omega \in \mathbb{C}^\times$ a complex number, and $\phi$ be the Hecke character of $K$ attached to $E$. Assume $\mathfrak{g}$ is an integral ideal of $K$, $E_\mathfrak{g}$ is the group of $\mathfrak{g}$-divisible points on $E$. Let $\mathcal{B}$ be a set of integral ideals of $K$ relatively prime to $\mathfrak{g}$ such that

\[ \{ \sigma_b \mid b \in \mathcal{B} \} = \text{Gal}(K(E_\mathfrak{g})/K), \quad \sigma_b \neq \sigma_{b'}, \text{ if } b \neq b', \]

where $\sigma_b = \left( \frac{K(E_\mathfrak{g})/K}{\mathfrak{g}} \right)$ is the Artin symbol. Assume $\phi \in \Omega K^\times$ is a complex number such that $\phi \Omega^{-1} \Omega_K = \mathfrak{g}^{-1} \mathfrak{h}$ for some integral ideal $\mathfrak{h}$ of $K$ which is relatively prime to $\mathfrak{g}$. Then

\[ \frac{\phi^k(\mathfrak{h})}{N(\mathfrak{h})^{k-s}} \cdot \frac{\phi^k}{|\phi|^{2s}} \cdot L_\mathfrak{g}(\overline{\phi}^{-k}, s) = \sum_{b \in \mathcal{B}} H_k(\phi(b), 0, s, L) \]

$(\text{Re}(s) > 1 + k/2)$, where $k$ is a positive integer and $N$ denotes the norm map from $K$ to $\mathbb{Q}$,

\[ L_\mathfrak{g}(\overline{\phi}^{-k}, s) = \prod_{\phi \in \mathcal{B}} (1 - \overline{\phi}^{-k}(\phi)N(\phi)^{-s})^{-1} \quad (\text{Re}(s) > 1 + k/2), \]

\[ H_k(z, 0, s, L) = \sum' \frac{(z + \alpha)^k}{|z + \alpha|^{2s}} \quad (\text{Re}(s) > 1 + k/2), \]

here $\sum'$ is taken over $\alpha \in L$ other than $-z$ if $z \in L$ (see [Go-Sch]).
Lemma B. Let the elliptic curve $E$, field $K$, Hecke character $\phi$, and $g$ be as in Proposition A. If the conductor $f_\phi$ of $\phi$ divides $g$, then the ray class field of $K$ modulo $g$ is $K(E_g)$, the extension of $K$ obtained by adding the coordinates of all $g$-division points of $E$ to $K$ (see [Go-Sch]).

Now we consider Theorem 1 and let $K$, $E_D$, $D_T$ and $L_S(D_T, s)$ be as there. Then by definition (see [B-SD], [Ire-Ro]) we have

Lemma 2.1.

$$L_S(\overline{\psi}_{D_T}, s) = \left\{ \begin{array}{ll}
L(\overline{\psi}_{D_T}, s) & \text{if } \prod_{\pi_k \in S} \pi_k = D_T, \\
L(\overline{\psi}_{D_T}, s) \prod_{\pi_k | D_T} \left( 1 - \left( \frac{D_T}{\pi_k} \right)_4 \cdot \overline{\pi_k}/(\pi_k \overline{\pi_k})^s \right) & \text{otherwise.}
\end{array} \right.$$

Proof of Theorem 1. Assume $L = \Omega O_K$ is the period lattice of $E_{D_T}: y^2 = x^3 - D_T x$, where $\Omega = \alpha \omega, \alpha \in \mathbb{C}^\times$. (Obviously $\Omega = \omega/\sqrt[4]{D_T}$.) From [Bir-Ste] we know that the conductor of $\psi_{D_T}$ is $(\theta D_T)$. Now, in Proposition A, let $k = 1$, $g = \Omega/(\theta D)$, $g = (\theta D)$, $\mathfrak{h} = O_K$. We have

$$2 \left( \frac{D_T}{\omega} \right)_4 L_g(\overline{\psi}_{D_T}, s) = \sum_{\psi \in \mathcal{B}} H_1(\psi_{D_T}(\psi)|0, s, L) \quad (\text{Re}(s) > 3/2).$$

Since the conductor of $\psi_{D_T}$ is $\theta D_T$, and $(\theta D_T)|(\theta D) = g$, by Lemma B the ray class field of $K$ modulo $(\theta D)$ is $K((E_{D_T})_{(\theta D)})$, the extension of $K$ obtained by adding the coordinates of $\theta D$-division points of $E_{D_T}$ to $K$. In particular we have the following isomorphism via the Artin map:

$$(O_K/(\theta D))^\times/\mu_4 \cong \text{Gal}(K((E_{D_T})_{(\theta D)})/K),$$

where $\mu_4$ is the group of quartic roots of unity, and $\mu_4 \cong (O_K/\theta)^\times$. So we may take the set

$$\mathcal{B} = \{(c\theta + D) \mid c \in \mathcal{C}\},$$

where $\mathcal{C}$ is as in Theorem 1, a set of representatives of $(O_K/(D))^\times$; thus

$$2 \left( \frac{D_T}{\omega} \right)_4 L_g(\overline{\psi}_{D_T}, s) = \sum_{c \in \mathcal{C}} H_1(\psi_{D_T}(c\theta + D)|\theta, 0, s, L) \quad (\text{Re}(s) > 3/2).$$

Note that the analytic continuation of $H_1(z, 0, 1, L)$ can be given by the Eisenstein $E^*$-function (see [Zhao] or [We]):

$$H_1(z, 0, 1, L) = E_{0,1}^*(z, L) = E_1^*(z, L).$$

So by (2.3) we have
(2.4) \[ \frac{\theta D}{\Omega} L_{(\theta D)}(\psi_{D_T}, 1) = \sum_{c \in \mathbb{C}} E_1^*(\psi_{D_T}(c\theta + D) \frac{\Omega}{\theta D}, \Omega O_K). \]

Since \( D \equiv 1 \pmod{4} \), we get \( c\theta + D \equiv 1 \pmod{\theta} \) for any \( c \in \mathbb{C} \). Thus by the definition of \( \psi_{D_T} \) and quartic reciprocity law,

\[
\psi_{D_T}(c\theta + D) = \left( \frac{D_T}{c\theta + D} \right)_4 (c\theta + D) = \left( \frac{c\theta + D}{D_T} \right)_4 (c\theta + D) = \left( \frac{c\theta}{D_T} \right)_4 (c\theta + D).
\]

Then by (2.4) and the fact that \( L_{(\theta D)}(\psi_{D_T}, 1) = L_S(\psi_{D_T}, 1) \), we have

(2.5) \[ \frac{\theta D}{\alpha \omega} L_S(\psi_{D_T}, 1) = \sum_{c \in \mathbb{C}} E_1^*(\left( \frac{c\omega + \omega}{\theta} \right) \alpha \left( \frac{c\theta}{D_T} \right)_4, \alpha \omega O_K). \]

Put \( \lambda = \alpha \left( \frac{c\theta}{D_T} \right)_4 \). Since \( E_1^*(\lambda z, \lambda L) = \lambda^{-1} E_1^*(z, L) \), we have

\[
E_1^*(\left( \frac{c\omega + \omega}{\theta} \right) \alpha \left( \frac{c\theta}{D_T} \right)_4, \alpha \left( \frac{c\theta}{D_T} \right)_4 \omega O_K) = \frac{1}{\alpha} \left( \frac{c\theta}{D_T} \right)_4 E_1^*(\frac{c\omega + \omega}{\theta}, \omega O_K).
\]

So by (2.5),

(2.6) \[ \frac{\theta D}{\omega} L_S(\psi_{D_T}, 1) = \left( \frac{\theta}{D_T} \right)_4 \sum_{c \in \mathbb{C}} \left( \frac{c}{D_T} \right)_4 E_1^*\left( \frac{c\omega + \omega}{\theta}, \omega O_K \right). \]

For the period lattice \( L_\omega = \omega O_K \) mentioned above, denote the corresponding Weierstrass \( \wp \)-function by \( \wp(z, L_\omega) \) and the corresponding Weierstrass zeta-function by \( \zeta(z, L_\omega) \). Then \( \wp'(z)^2 = 4\wp(z)^3 - 4\wp(z) \). So by results in [Go-Sch] we have

(2.7) \[ E_1^*\left( \frac{c\omega}{D} + \frac{\omega}{\theta}, \omega O_K \right) \]

\[ = \zeta\left( \frac{c\omega}{D}, L_\omega \right) + \zeta\left( \frac{\omega}{\theta}, L_\omega \right) + \frac{1}{2} \cdot \frac{\wp'(c\omega/D) - (2 - 2I \varphi(c\omega,D) - I}{\varphi(c\omega,D) - I} - \frac{\pi i}{\omega} \left( \frac{c}{D} \right)_4 \left( \frac{1}{\theta} \right).\]

We choose \( \mathbb{C} \) in such a way that \( c \) and \( -c \) both are in \( \mathbb{C} \). Obviously \( \left( \frac{-c}{D_T} \right)_4 = \left( \frac{c}{D_T} \right)_4 \). Since \( \zeta(z, L_\omega) \) and \( \wp'(z, L_\omega) \) are odd functions, and \( \wp(z, L_\omega) \) is even, by (2.6) we have
\[
\frac{D}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_S(\overline{\psi}_{D_T}, 1)
\]
\[
= \frac{1}{\theta} \left\{ \sum_{c \in \mathbb{C}} \left( \frac{c}{D_T} \right)_4 \zeta(\frac{c\omega}{D}, L_\omega) - \frac{\pi}{\omega} \sum_{c \in \mathbb{C}} \left( \frac{c}{D_T} \right)_4 \frac{\bar{c}}{D} \right\}
\]
\[
+ \frac{1}{2} \sum_{c \in \mathbb{C}} \left( \frac{c}{D_T} \right)_4 \frac{\varphi'(c\omega/D)}{\varphi(c\omega/D) - I} - (1 - I) \sum_{c \in \mathbb{C}} \left( \frac{c}{D_T} \right)_4 \frac{1}{\varphi(c\omega/D) - I} \left\{ \right\}
\]
\[
+ \frac{1}{\theta} \sum_{c \in \mathbb{C}} \left( \frac{c}{D_T} \right)_4 \zeta \left( \frac{\omega}{\theta}, L_\omega \right) - \frac{\pi}{\omega \theta} \right\}
\]
\[
= - \frac{1 - I}{\theta} \sum_{c \in \mathbb{C}} \left( \frac{c}{D_T} \right)_4 \frac{1}{\varphi(c\omega/D) - I} + \frac{1}{\theta} \sum_{c \in \mathbb{C}} \left( \frac{c}{D_T} \right)_4 \zeta \left( \frac{\omega}{\theta}, L_\omega \right) - \frac{\pi}{\omega \theta} \right) .
\]

That is,

\[
(2.8) \quad \frac{D}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_S(\overline{\psi}_{D_T}, 1)
\]
\[
= \frac{I}{2} \sum_{c \in \mathbb{C}} \left( \frac{c}{D_T} \right)_4 \frac{1}{\varphi(c\omega/D) - I} + \frac{1}{\theta} \sum_{c \in \mathbb{C}} \left( \frac{c}{D_T} \right)_4 \zeta \left( \frac{\omega}{\theta}, L_\omega \right) - \frac{\pi}{\omega \theta} \right) .
\]

By [Zhao] we know that

\[
\zeta \left( \frac{\omega}{\theta}, L_\omega \right) - \frac{\pi}{\omega \theta} = \frac{\theta}{4},
\]
so

\[
\frac{D}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_S(\overline{\psi}_{D_T}, 1) = \frac{I}{2} \sum_{c \in \mathbb{C}} \left( \frac{c}{D_T} \right)_4 \frac{1}{\varphi(c\omega/D) - I} + \frac{1}{4} \sum_{c \in \mathbb{C}} \left( \frac{c}{D_T} \right)_4 .
\]

This proves Theorem 1.

**Lemma 2.2.** We have

\[
\sum_{c \in \mathbb{C}} \left( \frac{c}{D_T} \right)_4 = \begin{cases} \#\mathbb{C} & \text{if } T = \emptyset, \\ 0 & \text{if } T \neq \emptyset. \end{cases}
\]

**Proof.** Easy by the definition of quartic residue symbol.

**Lemma 2.3.** Let \( D = \pi_1 \ldots \pi_n \) where \( \pi_k \equiv 1 \pmod{4} \) are distinct Gaussian prime \( (k = 1, \ldots, n) \). Let \( c \) be any Gaussian integer relatively prime to \( D \). Then

1. \( \sum_T \left( \frac{c}{D_T} \right)_4 = \mu(1 + I)^t \) or 0, where \( \mu \in \{ \pm 1, \pm I \} \), \( t \) is an integer with \( n \leq t \leq 2n \).
2. \( \sum_T \left( \frac{c}{D_T} \right)_4 = 0 \) if and only if \( \left( \frac{c}{\pi_k} \right)_4 = -1 \) (for some \( k \in \{1, \ldots, n\} \)).
(3) Suppose that \((\frac{c}{\pi_k})_4 \neq -1\) for any \(k \in \{1, \ldots, n\}\). Then
\[
\sum_T \left( \frac{c}{D_T} \right)_4 = \mu(1 + I)^{n+s}, \quad \text{where } \mu \text{ is as in (1) above},
\]
\[s = \# \left\{ \pi_k : \pi_k \mid D \text{ and } \left( \frac{c}{\pi_k} \right)_4 = 1, \ k = 1, \ldots, n \right\}.
\]
In particular,
\[
\sum_T \left( \frac{c}{D_T} \right)_4 = 2^n \quad \text{if and only if} \quad \left( \frac{c}{\pi_1} \right)_4 = \ldots = \left( \frac{c}{\pi_n} \right)_4 = 1;
\]
\[
\sum_T \left( \frac{c}{D_T} \right)_4 = \mu(1 + I)^n \quad \text{if and only if} \quad \left( \frac{c}{\pi_k} \right)_4 \in \{I, -I\}, \ k = 1, \ldots, n,
\]
where the sum \(\sum_T\) is taken over all subsets \(T\) of \(\{1, \ldots, n\}\).

**Proof of Lemma 2.3.** In fact we have
\[
\sum_T \left( \frac{c}{D_T} \right)_4 = \left(1 + \left( \frac{c}{\pi_1} \right)_4 \right) \ldots \left(1 + \left( \frac{c}{\pi_n} \right)_4 \right),
\]
from which the results could be deduced.

**Lemma 2.4.** \(v_2(S^*(D)) \geq (n - 1)/2\).

**Proof.** By results of [Zhao] or [B-SD], we know that
\[
v_2 \left( \varphi \left( \frac{c \omega}{D} \right) - I \right) = \frac{3}{4}
\]
for any Gaussian integer \(c\) relatively prime to \(D\). By Lemma 2.3 we have
\[
v_2 \left( \sum_T \left( \frac{c}{D_T} \right)_4 \right) = v_2(\mu(1 + I)^t) = \frac{t}{2} \geq \frac{n}{2}.
\]
(Here we regard \(v_2(0)\) as \(\infty\).) Thus by properties of valuation and our choice of \(\mathcal{C}\) with the property \(c, -c \in \mathcal{C}\), we have
\[
v_2(S^*(D)) \geq -\frac{3}{4} + \frac{n}{2}.
\]
Since \(\pi_k \equiv 1 \pmod{4}\) \((k = 1, \ldots, n)\), it follows that
\[
N(D_T) \equiv N(D) \equiv 1 \pmod{8}, \quad \left( \frac{I}{D_T} \right)_4 = I^{(N(D_T) - 1)/4} = \pm 1.
\]
Also
\[
\#(O_K/(D))^{\times} = \#\mathcal{C} = \prod_{k=1}^{n} (N(\pi_k) - 1) \equiv 0 \pmod{8},
\]
so we can choose $C$ such that $\pm c, \pm Ic \in C$ (when $c \in C$). Put

$$V = \{c \in C : c \equiv 1 \pmod{\theta}\}, \quad V' = V \cup IV.$$ 

Then $C = V' \cup (-V')$. Since $IO_K = O_K$, we have $IL_\omega = I(\omega O_K) = \omega O_K = L_\omega$. Thus by the definition of Weierstrass $\wp$-function,

$$\wp(Iz, IL_\omega) = \frac{1}{(Iz)^2} + \sum_{\alpha \in IL_\omega} \left( \frac{1}{(Iz-\alpha)^2} - \frac{1}{\alpha^2} \right) = \frac{1}{(Iz)^2} + \sum_{\alpha' \in L_\omega} \left( \frac{1}{(Iz-I\alpha')^2} - \frac{1}{(I\alpha')^2} \right) = -\wp(z, L_\omega).$$

In particular,

$$\wp\left(\frac{Ic\omega}{D}, L_\omega\right) = -\wp\left(\frac{c\omega}{D}, L_\omega\right),$$

$$S^*(D) = \frac{I}{2} \sum_{c \in C} \frac{1}{\wp(c\omega/D) - I} \sum_T \left( \frac{c}{DT} \right)_4$$

$$= I \sum_{c \in V} \frac{1}{\wp(c\omega/D) - I} \sum_T \left( \frac{c}{DT} \right)_4$$

$$= I \sum_{c \in V} \left[ \sum_T \left( \frac{1}{\wp(c\omega/D) - I} \right) \left( \frac{Ic}{DT} \right)_4 \right]$$

$$= I \sum_{c \in V} \left[ \sum_T \left( \frac{1}{\wp(c\omega/D) - I} \frac{I}{DT} \right)_4 \frac{1}{\wp(c\omega/D) + I} \left( \frac{c}{DT} \right)_4 \right]$$

$$= I \sum_{c \in V} \left( \frac{2B}{(\wp(c\omega/D))^2 + 1} + \sum_T \left( \frac{c}{DT} \right)_4 \right),$$

where $B = I$ or $\wp(c\omega/D)$.

Note that $v_2(\wp(c\omega/D) - I) = 3/4$, so $v_2(\wp(c\omega/D) + I) = 3/4$. Hence

$$v_2\left( \left( \wp\left(\frac{c\omega}{D}\right) \right)_4^2 + 1 \right) = v_2\left( \wp\left(\frac{c\omega}{D}\right) - I \right) + v_2\left( \wp\left(\frac{c\omega}{D}\right) + I \right) = \frac{3}{4} + \frac{3}{4} = \frac{3}{2},$$

$$v_2\left( \frac{2B}{(\wp(c\omega/D))^2 + 1} \right) = 1 - \frac{3}{2} = -\frac{1}{2}$$

(and obviously we have $v_2(B) = 0$). Therefore

$$v_2(S^*(D)) \geq -\frac{1}{2} + v_2\left( \sum_T \left( \frac{c}{DT} \right)_4 \right) \geq -\frac{1}{2} + \frac{n}{2} = \frac{n-1}{2}.$$ 

This proves Lemma 2.4.
Proof of Theorem 2. First let us prove
\[ v_2(L(\psi_D, 1)/\omega) \geq (n - 1)/2. \]

Taking sums of both sides of formula (1.1) over subsets \( T \) of \( \{1, \ldots, n\} \), we have
\[
\sum_T \frac{D}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_S(\psi_{D_T}, 1) = \frac{1}{2} \sum_T \sum_{c \in \mathcal{C}} \left( \frac{c}{D_T} \right)_4 \frac{1}{\varphi(c\omega/D)} - I + \frac{1}{4} \sum_T \sum_{c \in \mathcal{C}} \left( \frac{c}{D_T} \right)_4.
\]

So by Lemma 2.2 and (1.2), we obtain
\[
(2.9) \quad \sum_T \frac{D}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_S(\psi_{D_T}, 1) = S^*(D) + \frac{\#\mathcal{C}}{4},
\]

\[ v_2 \left( \frac{\#\mathcal{C}}{4} \right) = v_2 \left( \prod_{k=1}^{n} \left( \frac{\pi_k}{\pi_k \pi_k} - 1 \right) \right) \geq 3n - 2 \geq n,
\]

and by Lemma 2.4 we have
\[ v_2 \left( \sum_T \frac{D}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_S(\psi_{D_T}, 1) \right) \geq \frac{n - 1}{2}.
\]

By Lemma 2.1 we know that \( L_S(\psi_{D_T}, 1) = L(\psi_D, 1) \) when \( T = \{1, \ldots, n\} \); and when \( T = \emptyset \) we have
\[
L_S(\psi_{D_T}, 1) = L_S(\psi_1, 1) = L(\psi_1, 1) \prod_{k=1}^{n} \left( 1 - \frac{\pi_k}{\pi_k \pi_k} \right) = L(\psi_1, 1) \prod_{k=1}^{n} \left( 1 - \frac{1}{\pi_k} \right).
\]

By [B-SD] or [Zhao] we know that \( L(\psi_1, 1) = \omega/4 \), so
\[
L_S(\psi_1, 1) = \frac{\omega}{4} \prod_{k=1}^{n} \left( 1 - \frac{1}{\pi_k} \right),
\]

\[ v_2(L_S(\psi_1, 1)/\omega) = v_2 \left( \frac{1}{4} \prod_{k=1}^{n} \left( 1 - \frac{1}{\pi_k} \right) \right) \geq 2n - 2 \quad \text{(since } v_2(\pi_k - 1) \geq 2).\]

Now we use induction on \( n \) to prove our assertion \( v_2(L(\psi_D, 1)/\omega) \geq (n - 1)/2 \). If \( n = 1 \), then \( D = \pi_1 \), \( L_S(\psi_1, 1) = (\omega/4) \cdot (\pi_1 - 1)/\pi_1 \). Since \( \pi_1 \equiv 1 \pmod{4} \), we get \( v_2(L_S(\psi_1, 1)/\omega) \geq 0 \). By the above analysis we have
\[
v_2 \left( \frac{\pi_1}{\omega} \left( \frac{\theta}{1} \right)_4 L_S(\psi_1, 1) + \frac{\pi_1}{\omega} \left( \frac{\theta}{\pi_1} \right)_4 L_S(\psi_{\pi_1}, 1) \right) \geq \frac{1 - 1}{2} = 0.
\]

Therefore
\[ v_2(L(\psi_{\pi_1}, 1)/\omega) = v_2(L_S(\psi_{\pi_1}, 1)/\omega) \geq 0.\]
Now assume our assertion is true for 1, ..., \( n-1 \), and consider \( D = \pi_1 \ldots \pi_n \).

For any subset \( T \) of \( \{1, \ldots, n\} \), set \( t = t(T) = \#T \). By Lemma 2.1,

\[
\frac{D}{\omega} \left( \frac{\theta}{D_T} \right) L_S(\overline{\psi}_{D_T}, 1) = \frac{D}{\omega} \left( \frac{\theta}{D_T} \right) L(\overline{\psi}_{D_T}, 1) \prod_{\pi_k \mid D_T} \left( 1 - \left( \frac{D_T}{\pi_k} \right) \frac{1}{4 \pi_k} \right).
\]

Since \( (D_T/\pi_k)_4 = \pm 1, \pm i \), we get

\[
1 - \left( \frac{D_T}{\pi_k} \right) \frac{1}{4 \pi_k} = \frac{\pi_k - \mu}{\pi_k}, \quad \mu \in \{\pm 1, \pm i\}.
\]

Note that \( \pi_k \equiv 1 \pmod{4} \), so \( v_2(\pi_k - \mu) \geq 1/2 \); moreover equality holds if and only if \( (D_T/\pi_k)_4^2 = -1 \). Thus when \( T \) is non-trivial (i.e. \( 1 \leq t < n \)), by our inductive assumption,

\[
v_2 \left( \frac{D}{\omega} \left( \frac{\theta}{D_T} \right) L_S(\overline{\psi}_{D_T}, 1) \right) = v_2(L(\overline{\psi}_{D_T}, 1)/\omega) + \sum_{\pi_k \mid D_T} v_2 \left( 1 - \left( \frac{D_T}{\pi_k} \right) \frac{1}{4 \pi_k} \right)
\]

\[
\geq \frac{t-1}{2} + \frac{1}{2} \cdot \#\{\pi_k : \pi_k \mid D_T\} = \frac{t-1}{2} + \frac{n-t}{2} = \frac{n-1}{2}.
\]

Also when \( T = \emptyset \) we have

\[
L_S(\overline{\psi}_{D_T}, 1) = L_S(\overline{\psi}_1, 1) = L(\overline{\psi}_1, 1) \prod_{k=1}^{n} \left( 1 - \frac{1}{\pi_k} \right) = \omega \prod_{k=1}^{n} \left( 1 - \frac{1}{\pi_k} \right),
\]

therefore

\[
v_2(L_S(\overline{\psi}_1, 1)/\omega) \geq 2n - 2 \geq (n-1)/2,
\]

\[
v_2(L(\overline{\psi}_{D_T}, 1)/\omega) = v_2 \left( \frac{D}{\omega} \left( \frac{\theta}{D_T} \right) L(\overline{\psi}_{D_T}, 1) \right)
\]

\[
= v_2 \left( \sum_T \frac{D}{\omega} \left( \frac{\theta}{D_T} \right) L_S(\overline{\psi}_{D_T}, 1) \right)
\]

\[
- \sum_{T \neq \emptyset \subseteq \{1, \ldots, n\}} \frac{D}{\omega} \left( \frac{\theta}{D_T} \right) L_S(\overline{\psi}_{D_T}, 1) - \frac{D}{\omega} L_S(\overline{\psi}_1, 1) \right)
\]

\[
\geq (n-1)/2.
\]

Thus we have proved our assertion for any positive integer \( n \).

Now we consider the condition for equality to hold, using also induction on \( n \). If \( n = 1 \), then \( D = \pi_1 \), and by (2.9) we obtain

\[
\frac{\pi_1}{\omega} \left( \frac{\theta}{1} \right) L_{\pi_1}(\overline{\psi}_1, 1) + \frac{\pi_1}{\omega} \left( \frac{\theta}{\pi_1} \right) L_{\pi_1}(\overline{\psi}_{\pi_1}, 1) = S^*(\pi_1) + \frac{\pi_1 \pi_1 - 1}{4},
\]
that is,
\[
\frac{1}{4}(\pi_1 - 1) + \frac{\pi_1}{\omega} \left( \frac{\theta}{\pi_1} \right)_4 L(\overline{\psi}_{\pi_1}, 1) = S^*(\pi_1) + \frac{\pi_1 \overline{\psi}_1 - 1}{4}.
\]

Since
\[
v_2 \left( \frac{\pi_1 \overline{\psi}_1 - 1}{4} \right) = v_2(\pi_1 \overline{\psi}_1 - 1) - 2 \geq 1, \quad v_2(S^*(\pi_1)) \geq \frac{1 - \frac{1}{2}}{2} = 0
\]
(Lemma 2.4), the equality
\[
v_2(L(\overline{\psi}_{\pi_1}, 1)/\omega) = v_2 \left( \frac{\pi_1}{\omega} \left( \frac{\theta}{\pi_1} \right)_4 L(\overline{\psi}_{\pi_1}, 1) \right)
= v_2 \left( S^*(\pi_1) + \frac{\pi_1 \overline{\psi}_1 - 1}{4} - \frac{1}{4}(\pi_1 - 1) \right) = 0
\]
holds if and only if one of the following conditions is true:

1. \( v_2(\pi_1 - 1) = 2 \) when \( v_2(S^*(\pi_1)) > 0 \);
2. \( v_2(\pi_1 - 1) > 2 \) when \( v_2(S^*(\pi_1)) = 0 \).

Thus
\[
v_2(L(\overline{\psi}_{\pi_1}, 1)/\omega) = 0 \quad \text{if and only if} \quad \delta_1(\pi_1) = s_1(\pi_1) + \varepsilon_1(\pi_1) = 1.
\]

Assume our result is true for \( 1, \ldots, n - 1 \), and let \( D = \pi_1 \ldots \pi_n \). When \( T = \emptyset \), we have
\[
\frac{D}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_S(\overline{\psi}_{D_T}, 1) = \frac{D}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_D(\overline{\psi}_1, 1) = \frac{D}{\omega} L(\overline{\psi}_1, 1) \prod_{k=1}^n \left( 1 - \frac{1}{\pi_k} \right),
\]
\[
v_2 \left( \frac{D}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_S(\overline{\psi}_{D_T}, 1) \right) = v_2(L(\overline{\psi}_1, 1)/\omega) + \sum_{k=1}^n v_2(\pi_k - 1)
= v_2(1/4) + \sum_{k=1}^n v_2(\pi_k - 1) \geq 2n - 2 \geq (n - 1)/2.
\]

When \( \emptyset \neq T \subsetneq \{1, \ldots, n\} \), we have
\[
v_2 \left( \frac{D}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_S(\overline{\psi}_{D_T}, 1) \right) = v_2(L_S(\overline{\psi}_{D_T}, 1)/\omega)
= v_2 \left( \frac{L(\overline{\psi}_{D_T}, 1)}{\omega} \prod_{\pi_k \mid D_T} \left( 1 - \left( \frac{D_T}{\pi_k} \right)_4 \frac{1}{\pi_k} \right) \right)
= v_2(L(\overline{\psi}_{D_T}, 1)/\omega) + \sum_{\pi_k \mid D_T} v_2 \left( 1 - \left( \frac{D_T}{\pi_k} \right)_4 \frac{1}{\pi_k} \right).
\]
Since \((D_T/\pi_k)^4 = \pm 1, \pm I\), we have
\[
1 - \left( \frac{D_T}{\pi_k} \right) \frac{1}{4} \frac{\pi_k - \mu}{\pi_k} = \frac{\pi_k - 1 + (1 - \mu)}{\pi_k}, \quad \mu \in \{\pm 1, \pm I\}.
\]
Therefore
\[
v_2 \left( 1 - \left( \frac{D_T}{\pi_k} \right) \frac{1}{4} \frac{\pi_k}{\pi_k} \right) \geq \frac{1}{2},
\]
and equality holds if and only if \((D_T/\pi_k)^4 = \pm I\), i.e. \((D_T/\pi_k)^2 = -1\), that is, \([D_T/\pi_k]_2 = 1\). Thus
\[
\left( D_T \right)^\frac{1}{4} \left( \frac{\pi_k}{\pi_k} \right) = 1\quad \text{if and only if} \quad \left[ \frac{D_T}{\pi_k} \right]_2 = 1.
\]
By the proof of the first part of the theorem we know that
\[
v_2(\psi_{D_T}, 1) / \omega \geq (t(T) - 1) / 2, \quad t(T) = \# T,
\]
and by our inductive assumption, equality holds if and only if \(\delta_t(D_T) = 1\), \(t = t(T)\). Thus
\[
v_2 \left( \frac{D}{\omega} \left( \frac{\theta}{D_T} \right) \frac{1}{4} \frac{L_S(\psi_{D_T}, 1)}{L_S(\psi_{D_T}, 1)} \right) \geq \frac{t(T) - 1}{2} + \frac{n - t(T)}{2} = \frac{n - 1}{2},
\]
and equality holds if and only if \([D_T/\pi_k]_2 = 1\) (for any \(\pi_k \mid \widehat{D_T}\)) and \(\delta_t(D_T) = 1\). That is to say,
\[
v_2 \left( \frac{D}{\omega} \left( \frac{\theta}{D_T} \right) \frac{1}{4} \frac{L_S(\psi_{D_T}, 1)}{L_S(\psi_{D_T}, 1)} \right) = \frac{n - 1}{2}
\]
if and only if
\[
\left( \prod_{\pi_k \mid D_T} \left[ \frac{D_T}{\pi_k} \right]_2 \right) \delta_t(D_T) = 1.
\]
For the elliptic curve \(E_{D_T}: y^2 = x^3 - D_T x\) and Hecke characters \(\psi_{D_T}\), by [Ru1,2] we know that \(L(\psi_{D_T}, 1) / \Omega \subseteq K = \mathbb{Q}(I)\), and also we have \(\Omega = \omega / \sqrt[4]{D_T}\), so
\[
L(\psi_{D_T}, 1) / \omega = \left( \frac{\sqrt[4]{D_T}}{D_T} \right)^{-1} \cdot L(\psi_{D_T}, 1) / \sqrt[4]{D_T}
\]
and equality holds if and only if \(L(\psi_{D_T}, 1) / \omega \subseteq K(\sqrt[4]{D_T})\), i.e. \(L(\psi_{D_T}, 1) / \omega \in K(\sqrt[4]{D_T})\). Thus by Lemma 2.1 we get
\[
\frac{D}{\omega} \left( \frac{\theta}{D_T} \right) \frac{1}{4} \frac{L_S(\psi_{D_T}, 1)}{L_S(\psi_{D_T}, 1)} = \frac{D}{\omega} \left( \frac{\theta}{D_T} \right) \frac{1}{4} \prod_{\pi_k \mid D_T} \left( 1 - \left( \frac{D_T}{\pi_k} \right) \frac{1}{4} \frac{\pi_k}{\pi_k} \right) \cdot L(\psi_{D_T}, 1) / \omega \in K(\sqrt[4]{D_T}),
\]
and if
\[ v_2 \left( \frac{D}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_S(\psi_{D_T}, 1) \right) = \frac{n - 1}{2}, \]
then
\[ \frac{D}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_S(\psi_{D_T}, 1) = (1 + I)^{n-1} \alpha_T \frac{4}{D_T^3}, \]
where \( \alpha_T \in K \), and \( v_2(\alpha_T) = 0 \) (since \( v_2(\sqrt{D_T^3}) = \frac{3}{4} v_2(D_T) = 0 \)). For any subsets \( T \) and \( T' \) of \( \{1, \ldots, n\} \), if \( v_2(\alpha_T) = v_2(\alpha_{T'}) = 0 \), then it can be easily verified that
\[ v_2(\alpha_T \sqrt{D_T^3} + \alpha_{T'} \sqrt{D_{T'}^3}) > 0. \]

Thus, consider the terms in the sum
\[ (2.10) \sum_{\emptyset \neq T \subsetneq \{1, \ldots, n\}} \frac{D}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_S(\psi_{D_T}, 1). \]

For any two terms with 2-adic valuations equal to \( (n - 1)/2 \), the 2-adic valuation of their sum is greater than \( (n - 1)/2 \). Also when \( n > 1 \) we have
\[ v_2 \left( \frac{D}{\omega} \left( \frac{\theta}{D_0} \right)_4 L_S(\psi_{D_0}, 1) \right) \geq 2n - 2 \geq n > \frac{n - 1}{2}. \]

Hence \( v_2(L(\psi_D, 1)/\omega) = (n - 1)/2 \) if and only if one of the following statements is true:

(1) when \( v_2(S^*(D)) > (n - 1)/2 \), in the above sum (2.10), the number of terms with 2-adic valuation \( (n - 1)/2 \) is odd;

(2) when \( v_2(S^*(D)) = (n - 1)/2 \), in the above sum (2.10), the number of terms with 2-adic valuation \( (n - 1)/2 \) is even.

Statement (1) above means: if \( \varepsilon_n(D) = 0 \), then
\[ 2 \left\{ \emptyset \neq T \subsetneq \{1, \ldots, n\} : v_2 \left( \frac{D}{\omega} \left( \frac{\theta}{D_T} \right)_4 L_S(\psi_{D_T}, 1) \right) = \frac{n - 1}{2} \right\} \]
\[ = 2 \left\{ \emptyset \neq T \subsetneq \{1, \ldots, n\} : \left( \prod_{\pi_k | D_T} \left[ \frac{D_T}{\pi_k} \right]_2 \right) \delta_t(D_T) = 1 \right\} \]
\[ \equiv \sum_{\emptyset \neq T \subsetneq \{1, \ldots, n\}} \left( \prod_{\pi_k | D_T} \left[ \frac{D_T}{\pi_k} \right]_2 \right) \delta_t(D_T) \equiv 1 \pmod{2}, \]
\[ \delta_n(D) = \varepsilon_n(D) + \sum_{\emptyset \neq T \subsetneq \{1, \ldots, n\}} \left( \prod_{\pi_k | D_T} \left[ \frac{D_T}{\pi_k} \right]_2 \right) \delta_t(D_T) \equiv 1 \pmod{2}. \]
And (2) means: if $\varepsilon_n(D) = 1$ then
\[
\sum_{\emptyset \neq T \subseteq \{1, \ldots, n\}} \left( \prod_{\pi_k | D_T} \left[ \frac{D_T}{\pi_k} \right] \right) \delta_1(D_T) \equiv 0 \pmod{2},
\]
so
\[
\delta_n(D) = \varepsilon_n(D) + \sum_{\emptyset \neq T \subseteq \{1, \ldots, n\}} \left( \prod_{\pi_k | D_T} \left[ \frac{D_T}{\pi_k} \right] \right) \delta_1(D_T)
\]
\[
\equiv 1 + 0 \equiv 1 \pmod{2}.
\]
Hence $v_2(L(\overline{\psi}_D, 1)/\omega) = (n - 1)/2$ if and only if $\delta_n(D) = 1$. This proves the theorem.

**Proof of Theorem 3.** This theorem follows from Theorem 2 and the main result of Coates–Wiles in [Co-Wi].

For the elliptic curve $E_D : y^2 = x^3 - Dx$ with $D = \pi_1^2 \ldots \pi_r^2 \pi_{r+1} \ldots \pi_n$, where $\pi_k \equiv 1 \pmod{4}$ are distinct Gaussian prime integers ($k = 1, \ldots, n$), we could prove Theorems 4 and 5 similarly to Theorems 1 and 2.

**References**


L-series attached to elliptic curves


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