On some new estimates for $h^-(\mathbb{Q}(\zeta_p))$

by

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1. Introduction. Let $p$ be an odd prime number and $m = (p - 1)/2$. Let $h_p$ resp. $h_p^+$ denote the class numbers of the cyclotomic field $\mathbb{Q}(\zeta_p)$, resp. the maximal real subfield $\mathbb{Q}(\zeta_p)^+$ of this field. The Dirichlet class number formula for the class number $h_p = h(\mathbb{Q}(\zeta_p))$ is

$$h_p = \frac{p^{p/2}}{2^{m-1}\pi^m R} \prod_{\chi \neq 1} L(1, \chi),$$

where the product is taken over all nonprincipal characters of $\mathbb{Q}(\zeta_p)$. It is well known that $h_p^+ | h_p$ (see Theorem 4.10 in [4]). We have $h_p = h_p^+ h_p^-$, where

$$h_p^- = \frac{1}{2^{m-1}} p^{(p+3)/4} \frac{1}{\pi^m} \prod_{\chi \text{ odd}} L(1, \chi) = \frac{1}{(2p)^{m-1}} \prod_{\chi \text{ odd}} \sum_{k=1}^{p-1} k\chi(k)$$

(see Theorems 4.17 and 4.9 in [4]).

We consider two types of sequences $(a_i)_{1 \leq i \leq m}$ over $\mathbb{Z}$: $a_i = m + i$, $i = 1, \ldots, m$, or $a_i = r^i$, $i = 1, \ldots, m$, where $p \equiv 1 \pmod{4}$ and $r$ is a primitive root modulo $p$, or $p \equiv 3 \pmod{4}$ and $r$ generates the group of quadratic residues modulo $p$. For the sequences $\{a_i\}_{1 \leq i \leq m}$, if $1 \leq j \leq m$ there exists $1 \leq i \leq m$ such that $a_i \equiv j \pmod{p}$ or $a_i \equiv -j \pmod{p}$.

In [2] and [3] it is proved that

$$h_p^- \leq 2p \left( \frac{p}{24} \right)^{m/2}.$$

We prove the estimates

$$h_p^- < 3.492 \cdot p \left( \frac{p}{32} \right)^{m/2},$$

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provided \( p \equiv 1 \pmod{4} \) and \( r = 2 \) is a primitive root modulo \( p \) or \( p \equiv 3 \pmod{4} \) and \( r = 2 \) generates the group of quadratic residues modulo \( p \). Analogously, if we replace \( r = 2 \) by \( r = 3 \) resp. \( r = 5 \) we obtain the estimates

\[
h_p^- < 1.502 \cdot p \left( \frac{p}{27} \right)^{m/2} \quad \text{and} \quad h_p^- < 2p \left( \frac{p}{25} \right)^{m/2}.
\]

In the proofs, we make use of two types of matrices \( A = (A_{ij})_{1 \leq i,j \leq m} \) or \( B = (B_{ij})_{1 \leq i,j \leq m} \) over \( \mathbb{Z} \) associated to the sequences \( (a_i)_{1 \leq i \leq m} \):

\[
A_{ij} = \left[ a_i (m + j) / p \right],
\]

for \( a_i = m + i \) (here as usual \([x]\) denotes the integral part of \( x \), and for \( a_i = r^i \), \( B_{ij} = 1 \) and

\[
B_{ij} = \left[ a_i (m + j) / p \right] - r [a_{i-1} (m + j) / p] \quad \text{if} \ i \geq 2.
\]

2. Some relations between the matrices \( A \) and \( h_p^- \). Let \( \chi \) be a generator of the group of characters of the field \( \mathbb{Q}(\zeta_p) \). Then odd characters of this field are odd powers of \( \chi \). Moreover, it is well-known that for \( \chi \) odd,

\[
L(1, \chi) = \frac{\pi i \tau(\chi)}{p^2} \sum_{j=1}^{p-1} j \overline{\chi}(j),
\]

where \( \tau(\chi) \) as usual denotes the Gauss sum (see Theorem 4.9 in [4]). After some manipulation the formula can be rewritten as

\[
L(1, \chi) = \frac{\pi i \tau(\chi)}{p(\overline{\chi}(2) - 2)} \sum_{j=1}^{m} \overline{\chi}(j).
\]

Therefore formula (1) can be rewritten as

\[
h_p^- = \left| \frac{p}{2m-1} \prod_{j=1}^{m} \frac{1}{\overline{\chi}^{2j-1}(2) - 2} \sum_{k=1}^{m} \chi^{2j-1}(k) \right|.
\]

Let \( [x]^* = [x] - 1/2 \) if \( x \in \mathbb{Z} \) and \( [x]^* = [x] \) otherwise. It is well-known that

\[
[x]^* = x - \frac{1}{2} + \sum_{j=1}^{\infty} \frac{\sin(2jx\pi)}{\pi j}.
\]

Lemma 1. Let \( \chi \) be an odd Dirichlet character modulo \( p \), and \( a \) be a natural number. Then

\[
\sum_{j=1}^{m} \left[ \frac{a_j}{p} \right] \chi(j) = \frac{1}{2} \left( \frac{a - \overline{\chi}(a)}{\overline{\chi}(2) - 2} + a - 1 \right) \sum_{j=1}^{m} \chi(j),
\]

\[
\sum_{j=m+1}^{p-1} \left[ \frac{a_j}{p} \right] \chi(j) = \frac{1}{2} \left( \frac{a - \overline{\chi}(a)}{\overline{\chi}(2) - 2} - a + 1 \right) \sum_{j=1}^{m} \chi(j).
\]
Proof. From the formulas before the lemma, (2), (3) and well-known properties of the Gauss sum we obtain

\[
\sum_{k=1}^{p-1} \left[ \frac{ak}{p} \right] \chi(k) = \sum_{k=1}^{p-1} \left[ \frac{ak^*}{p} \right] \chi(k) = \sum_{k=1}^{p-1} \chi(k) \left( \frac{ak}{p} - \frac{1}{2} + \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{1}{j} \sin \frac{2\pi akj}{p} \right)
\]

\[
= \sum_{k=1}^{p-1} \chi(k) \left( \frac{ak}{p} - \frac{1}{2} + \frac{1}{2\pi i} \sum_{j=1}^{\infty} \frac{1}{j} (\zeta_p^{akj} - \zeta_p^{-akj}) \right)
\]

\[
= \frac{a}{p} \sum_{k=1}^{p-1} k \chi(k) + \frac{\tau(\chi)\overline{\chi}(a)}{\pi i} \sum_{j=1}^{\infty} \frac{\overline{\chi}(j)}{j}
\]

\[
= \frac{p(a - \overline{\chi}(a))}{\pi i \tau(\overline{\chi})} L(1, \overline{\chi}) = \frac{a - \overline{\chi}(a)}{\overline{\chi}(2)} - \frac{m}{2} \sum_{k=1}^{m} \chi(k).
\]

Lemma 1 now follows from

\[
\left[ \frac{ai}{p} \right] + \left[ \frac{a(p - i)}{p} \right] = a - 1.
\]

Let \( s \) be a rational \( p \)-integer number and let \( \chi \) be a Dirichlet character modulo \( p \). Define \( \chi(s) = \chi(n) \) where \( n \in \mathbb{Z} \) and \( s \equiv n \pmod{p} \). For \( \chi \) odd we have

\[
\sum_{j=1}^{m} \chi^{2j-1}(s) = \begin{cases} 0 & \text{if } s \not\equiv \pm 1 \pmod{p}, \\ \pm m & \text{if } s \equiv \pm 1 \pmod{p}. \end{cases}
\]

**THEOREM 1.** Let \( p \) be an odd prime and \( m = (p - 1)/2 \). For the matrix \( A \) defined in the Introduction we have

\[
|\det(A)| = h_p^{-}.
\]

Proof. Let \( \chi \) be a generator of the group of characters of the field \( \mathbb{Q}(\zeta_p) \). Set \( K = (K_{ij})_{1 \leq i,j \leq m} \), where \( K_{ij} = \chi^{2j-1}(a_i) \). Let as usual \( K^T \) denote the transpose of \( K \). Write \( M = KK^T = (M_{ij})_{1 \leq i,j \leq m} \). Then by (5) we obtain

\[
M_{ij} = \sum_{k=1}^{m} \chi^{2k-1}(a_i a_j) = \begin{cases} 0 & \text{if } a_i a_j \not\equiv \pm 1 \pmod{p}, \\ \pm m & \text{if } a_i a_j \equiv \pm 1 \pmod{p}, \end{cases}
\]

and consequently

\[
\det(M) = \pm m^m.
\]

On the other hand, applying Lemma 1 and (4) gives

\[
AK = \frac{1}{2} h_p^{-} p^{m-1} C,
\]
where $C = \{C_{ij}\}$ with

$$C_{ij} = 3m + 3i - 2 - \chi^{2j-1}\left(\frac{1}{m+i}\right) + (-m - i + 1)\chi^{2j-1}\left(\frac{1}{2}\right).$$

Moreover, by (5) we have

$$CK^T = \begin{pmatrix}
-\frac{m}{m+1} & \ast & \ldots & \ast & \ast \\
0 & -\frac{m}{m+1} & \ldots & \ast & \ast \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & -m(3-2m) & -m(6m-5) \\
0 & 0 & \ldots & -m(1-2m) & -m(-1+6m)
\end{pmatrix}.$$

Hence

$$\text{det}(CK^T) = \pm 2pm^m.$$  

**Theorem 2.** Let $p$ be an odd prime and let $m = (p-1)/2$. Let $1 \leq n < m$ and $\varepsilon_0 = \pm 1$ be the unique integers satisfying $r^n \equiv 2\varepsilon_0 \pmod{p}$. Write $\varepsilon = \varepsilon_0(r/p)$. For the matrix $B$ defined in the Introduction we have

$$|\text{det}(B)| = \frac{2r^{m-1} - \varepsilon r^{n-1}}{p} h_{p^{-}}.$$

**Proof.** Let $K$ be the matrix defined in the proof of Theorem 1. Applying Lemma 1 and (4) gives

$$BK = \frac{1}{2} h_{p^{-}} p^{m-1} D,$$

where $D = (D_{ij})_{1 \leq i, j \leq m}$ with $D_{1j} = 4 - 2\chi^{2j-1}(2)$ and

$$D_{ij} = a_i - \chi^{2j-1}(a_i) - (a_i - 1)(\chi^{2j-1}(2) - 2)$$

$$- r(a_{i-1} - \chi^{2j-1}(a_{i-1}) - (a_{i-1} - 1)(\chi^{2j-1}(2) - 2))$$

$$= - (2 - 2r) + (1 - r)\chi^{2j-1}\left(\frac{1}{2}\right) - \chi^{2j-1}\left(\frac{1}{a_i}\right) + r\chi^{2j-1}\left(\frac{1}{a_{i-1}}\right),$$

if $i \geq 2$. Write $R = DK^T = (R_{ij})_{1 \leq i, j \leq m}$. Then we have

$$R_{1k} = 4 \sum_{j=1}^{m} \chi^{2j-1}(a_k) - 2 \sum_{j=1}^{m} \chi^{2j-1}\left(\frac{a_k}{2}\right) \quad \text{for } k = 1, \ldots, m,$$

and

$$R_{ik} = - (2 - 2r) \sum_{j=1}^{m} \chi^{2j-1}(a_k) + (1 - r) \sum_{j=1}^{m} \chi^{2j-1}\left(\frac{a_k}{2}\right)$$

$$- \sum_{j=1}^{m} \chi^{2j-1}\left(\frac{a_k}{a_i}\right) + r \sum_{j=1}^{m} \chi^{2j-1}\left(\frac{a_k}{a_{i-1}}\right),$$
where $i \geq 2$. Define $F = (F_{ik})_{1 \leq i,k \leq m}$, where $F_{1k} = R_{1k}$ and for $i \geq 2$,

$$F_{ik} = R_{ik} + \frac{1-r}{2} R_{1k} = -\sum_{j=1}^{m} \chi^{2j-1} \left( \frac{a_k}{a_i} \right) + r \sum_{j=1}^{m} \chi^{2j-1} \left( \frac{a_k}{a_{i-1}} \right).$$

Applying (5) gives

$$F = \begin{pmatrix} \ast & \ast & \ldots & \ast & 4m(r/p) \\ rm & -m & \ldots & 0 & 0 \\ 0 & rm & \ldots & 0 & 0 \\ \ldots \ldots \ldots \ldots \ldots \ldots \\ 0 & 0 & \ldots & rm & -m \end{pmatrix},$$

where $F_{1n} = -2\varepsilon_2 m$, $F_{1m} = 4m(r/p)$, and all remaining entries vanish. It follows that

$$\det(F) = \pm 2pm^m \frac{2r^{m-1} - \varepsilon r^{n-1}}{p}$$

where $2r^{m-1} - \varepsilon r^{n-1} \equiv 0 \pmod{p}$, which completes the proof.

3. Applications. Let $X = (X_{ij})_{1 \leq i,j \leq m}$ be a real matrix and let $\| \cdot \|$ denote the Euclidean matrix norm defined as

$$\|X\| = \left( \sum_{i,j} X_{ij}^2 \right)^{1/2}.$$

By Hadamard’s inequality and the inequality between geometric and arithmetic means we have

$$|\det(X)| \leq \left( \frac{\|X\|}{n} \right)^{n/2}. \quad (7)$$

**Theorem 3** (Schur 1909, see [1, Theorem 7.3.1, p. 202]). Let $X$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \|X\|^2.$$

**Corollary to Theorem 2.** Let $p$ be a prime number and $r$ be a natural number such that either $p \equiv 1 \pmod{4}$ and $r$ is a primitive root modulo $p$, or $p \equiv 3 \pmod{4}$ and $r$ generates the group of quadratic residues modulo $p$. We have

1. If $r = 2$ and $p > 23$,

$$h_p^- < 3.492 \cdot p \left( \frac{p}{32} \right)^{m/2}.$$
2. If \( r = 3 \) and \( p > 100 \),
\[
    h_{p}^{-} \leq 1.502 \cdot p \left( \frac{p}{27} \right)^{m/2}.
\]

3. If \( r = 5 \),
\[
    h_{p}^{-} \leq 2p \left( \frac{p}{25} \right)^{m/2}.
\]

**Proof.** Denote by \( x_{i} \) (\( 1 \leq i \leq m \)) the \( i \)th row of the matrix \( B \). Let as usual \((x, y)\) denote the scalar product. Then Theorem 3 implies the inequality
\[
(8) \quad |\det(B)| \leq \left( \frac{Q}{m} \right)^{m/2}, \quad \text{where} \quad Q = \sum_{i=1}^{m} (x_{i}, x_{i}).
\]

1. If \( r = 2 \) the matrix \( B \) is a (0-1) matrix. Applying the Gram–Schmidt orthogonalization process we pass from the vectors \((x_{i})_{1 \leq i \leq m}\) to an orthogonal system of vectors \((y_{i})_{1 \leq i \leq m}\):
\[
y_{1} = x_{1} \quad \text{and} \quad y_{i} = x_{i} - \sum_{j=1}^{i-1} \frac{(x_{i}, y_{j})}{(y_{j}, y_{j})} y_{j} \quad \text{if} \quad i \geq 2.
\]

We have
\[
(y_{1}, y_{1}) = (x_{1}, x_{1}) \quad \text{and} \quad (y_{i}, y_{i}) = (x_{i}, x_{i}) - \sum_{j=1}^{i-1} \frac{(x_{i}, y_{j})^{2}}{(y_{j}, y_{j})} \quad \text{if} \quad i \geq 2.
\]

Moreover, Theorem 2 for \( r = 2 \) together with (8) implies the inequality
\[
(9) \quad \frac{2m - \left( \frac{2}{p} \right) h_{p}^{-}}{p} = |\det(B)| \leq \left( \frac{Q}{m} \right)^{m/2}, \quad \text{where} \quad Q = \sum_{i=1}^{m} (y_{i}, y_{i}).
\]

If \( t_{i} \) denotes the number of 1’s in the \( i \)th row, then
\[
    Q = \sum_{i=1}^{m} (y_{i}, y_{i}) < \sum_{i=1}^{m} (x_{i}, x_{i}) - \frac{1}{m} \sum_{i=2}^{m} (x_{i}, x_{1})^{2} = \sum_{i=1}^{m} t_{i} - \frac{1}{m} \sum_{i=2}^{m} t_{i}^{2}
\]
\[
\leq m + (m - 1) \frac{m}{2} - \frac{1}{m} (m - 1) \left( \frac{m}{2} \right)^{2} = m + m \frac{m - 1}{4},
\]

therefore
\[
    \frac{Q}{m} < 1 + \frac{m - 1}{4} = \frac{m + 3}{4} = p + \frac{5}{8}.
\]

Hence and by (9) for \( m \geq 14 \) we obtain
\[
    \frac{2^{m} \cdot 2^{-0.0001}}{p} h_{p}^{-} < \frac{2^{m} - \left( \frac{2}{p} \right)}{p} h_{p}^{-} \leq \left( \frac{p + 5}{8} \right)^{m/2} < e^{5/4} \left( \frac{p}{8} \right)^{m/2},
\]
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because

$$\lim_{n \to \infty} \left(1 + \frac{5}{4n}\right)^n = e^{5/4}.$$ 

This gives the corollary for $r = 2$ at once.

2. For $i \geq 2$ subtract the first row of $B$ from its $i$th row for $i = 2, \ldots, m$ and denote the resulting matrix by $E$. The number of entries in the $i$th row of $E$ for $i = 2, \ldots, m$ that are equal to $\pm 1$ is $[p/3]$. Therefore

$$\|E\| = m + (m - 1)[p/3] \leq m + (m - 1)\frac{2m}{3}$$

and so

$$\frac{\|E\|}{m} \leq 1 + \frac{2(m - 1)}{3} = \frac{p}{3}.$$ 

Consequently, by (7) and Theorem 2 for $r = 3$ we obtain

$$\frac{2 \cdot 3^{m-1} - 3^{m-7}}{p} h_p^- < \frac{2 \cdot 3^{m-1} - \varepsilon 3^{n-1}}{p} h_p^- = |\det(B)| = |\det(E)|$$

$$\leq \left(\frac{\|E\|}{m}\right)^{m/2} \leq \left(\frac{p}{3}\right)^{m/2},$$

because for $p > 100$ we have

$$2 \cdot 3^{m-1} - \varepsilon 3^{n-1} > 2 \cdot 3^{m-1} - 3^{m-7}.$$ 

The above inequality is obvious if $\varepsilon = -1$ or $\varepsilon = 1$ and $m - n > 6$. If $n = m - k$, $k \leq 6$ and $p > 100$, we have

$$0 \equiv 2 \cdot 3^{m-1} - \varepsilon 3^{n-1} \equiv 3^{m-k-1}(2 \cdot 3^k - 1) \not\equiv 0 \pmod{p},$$

because

$$\prod_{k=1}^{6}(2 \cdot 3^k - 1) = 5^2 \cdot 7 \cdot 17 \cdot 23 \cdot 31 \cdot 47 \cdot 53 \cdot 97 \not\equiv 0 \pmod{p},$$

if $p > 100$; a contradiction.

Now from (10) we have

$$3^{c-7} h_p^- < p \left(\frac{p}{27}\right)^{m/2},$$

where $c = \log_3(2 \cdot 3^6 - 1)$. Hence the corollary follows in the case when $r = 3$.

3. For $r = 5$ analysis analogous to that in the proof of Corollary in the case $r = 3$ gives the Metsänkyla–Lepistö type inequality

$$h_p^- \leq 2p \left(\frac{p}{25}\right)^{m/2}.$$
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