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Identities for traces of singular moduli

by

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1. Introduction and statement of results. Let $q := e^{2\pi iz}$ and let

$$J(z) = q^{-1} + 196884q + 21493760q^2 + \cdots$$

be the Hauptmodul for $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Similarly, let

$$j_3^*(z) = q^{-1} + 783q + 8672q^2 + 65367q^3 + 371520q^4 + \cdots$$

be the Hauptmodul for $\Gamma_0^*(3)$, the extension of $\Gamma_0(3)$ by the Fricke involution W_3. The values of such functions at imaginary quadratic arguments are known as *singular moduli.*

There are general identities relating singular moduli for functions such as $J(z)$ and $j_3^*(z)$. To illustrate, observe that the evaluations

$$J\left(\frac{-1+\sqrt{-3}}{2}\right) = -744, \quad j_3^*\left(\frac{-3+\sqrt{-3}}{6}\right) = -42,$$
$$j_3^*\left(\frac{-7+\sqrt{-3}}{6}\right) = -234$$

imply that

$$J\left(\frac{-1+\sqrt{-3}}{2}\right) = j_3^*\left(\frac{-3+\sqrt{-3}}{6}\right) + 3j_3^*\left(\frac{-7+\sqrt{-3}}{6}\right). \tag{1.1}$$

Such identities follow from arithmetic relations between modular forms of different levels.

For a positive integer m, let $J_m(z)$ be the unique modular function on Γ with Fourier expansion

$$J_m(z) = q^{-m} + \sum_{n=1}^{\infty} c_m(n)q^n.$$

Zagier [Za] computed the generating functions for the "traces" of the $J_m(z)$ singular moduli, as well as several other classes of modular functions.

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To describe these traces, let $D \equiv 0, 3 \pmod 4$ be a positive integer, and let $\mathcal{Q}_D$ denote the positive definite integral binary quadratic forms $Q(a,b,c) = ax^2 + bxy + cy^2$ of discriminant $-D = b^2 - 4ac$. The group Γ acts on $\mathcal{Q}_D$ by

$$Q \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} := Q(\alpha x + \beta y, \gamma x + \delta y).$$

For each $Q \in \mathcal{Q}_D$, we let $\alpha_Q := (-b + \sqrt{b^2 - 4ac})/2a \in \mathbb{H}$ and $\omega_Q := |\Gamma_Q|$, where Γ_Q is the isotropy subgroup of Q in Γ. If m is a positive integer and $-D$ is a discriminant, then Zagier defined the trace of the singular moduli of discriminant $-D$ for $J_m(z)$ by

$$t_m(D) := \sum_{Q \in \mathcal{Q}_D/\Gamma} \frac{1}{\omega_Q} J_m(\alpha_Q). \tag{1.2}$$

He proved the striking fact that their generating functions are weight $3/2$ modular forms.

It turns out that (1.1) is a special case of a general phenomenon where sums of coefficients of certain weakly holomorphic weight $3/2$ modular forms are given in terms of such traces. A meromorphic modular form is *weakly holomorphic* if it is holomorphic on $\mathbb{H}$. Following Kohnen, for integers k let $\mathcal{M}^+_{k+1/2}(\Gamma_0(4))$ be the space of weakly holomorphic weight $k + 1/2$ modular forms on $\Gamma_0(4)$ with a Fourier expansion of the form

$$\sum_{\substack{n \gg -\infty \\ (-1)^k n \equiv 0,1 \pmod 4}} a(n)q^n. \tag{1.3}$$

Furthermore, if p is an odd prime and $\varepsilon = \pm 1$, then let $\mathcal{M}^{+,\varepsilon}_{k+1/2}(\Gamma_0(4p))$ be the space of those weight $k + 1/2$ weakly holomorphic modular forms $f(z) = \sum_{n \gg -\infty} a(n)q^n$ on $\Gamma_0(4p)$ whose Fourier coefficients satisfy

$$a(n) = 0 \quad \text{whenever } (-1)^k n \equiv 2, 3 \pmod 4 \text{ or } \left(\frac{(-1)^k n}{p}\right) = -\varepsilon. \tag{1.4}$$

The generic identities, for forms in $\mathcal{M}^{+,\varepsilon}_{3/2}(\Gamma_0(4p))$, are expressed in terms of a distinguished sequence of modular forms. To define them, for each positive $m \equiv 0, 1 \pmod 4$ let $g_m(z)$ be the unique form in $\mathcal{M}^+_{3/2}(\Gamma_0(4))$ with a Fourier expansion of the form

$$g_m(z) = q^{-m} + \sum_{n=0}^{\infty} B(m,n)q^n. \tag{1.5}$$

Remark. The uniqueness of the $g_m(z)$ essentially follows from the fact that there are no holomorphic forms in $\mathcal{M}^+_{3/2}(\Gamma_0(4))$ (see the discussion preceding Theorem 4 of [Za]).

THEOREM 1.1. *If p is an odd prime*, $\varepsilon = \pm 1$, *and*

$$g = \sum_{n \gg -\infty} a(n)q^n \in \mathcal{M}^{+,\varepsilon}_{3/2}(\Gamma_0(4p)),$$

then for every positive $n \equiv 0, 3 \pmod 4$ *we have*

$$a(n) + \varepsilon a(np^2) = \sum_{m \geq 1} (a(-m) + \varepsilon a(-mp^2))B(m, n).$$

REMARK. Since Zagier (see Section 7 of [Za]) proved that the integers $B(m, n)$ are traces or "twisted traces" of singular moduli, Theorem 1.1 provides identities expressing sums of coefficients of forms in $\mathcal{M}^{+,\varepsilon}_{3/2}(\Gamma_0(4))$ as sums of traces of singular moduli. For brevity and aesthetics, we do not give the general formulas here. Instead, we simply give Corollary 1.2 which relates the traces $t_m(D)$ to the level p traces defined below.

Recently, Bruinier and Funke [BF] have greatly generalized Zagier's results to include traces of singular moduli of modular functions on groups which do not necessarily have a Hauptmodul. A particularly elegant example of their general work corresponds to modular functions on $\Gamma_0(p)$ which are fixed by the Fricke involution $W_p = \left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$. To define these traces, let p be an odd prime, and let $\Gamma_0^*(p)$ be the projective image of the extension of $\Gamma_0(p)$ by the Fricke involution W_p in $\mathrm{PSL}_2(\mathbb{R})$. Let $\mathcal{Q}_{D,p}$ be the set of quadratic forms $Q \in \mathcal{Q}_D$ such that $a \equiv 0 \pmod p$. The group $\Gamma_0^*(p)$ acts on $\mathcal{Q}_{D,p}$, where the action for elements of $\Gamma_0(p)$ is defined as before, and $Q \circ W_p := Q'(x, y)$, where $Q'(x, y) = (pc, -b, a/p)$.

Suppose that $f(z)$ is in $\mathcal{M}_0(\Gamma_0^*(p))$, the set of weakly holomorphic modular functions for $\Gamma_0^*(p)$. For such f, the discriminant $-D$ trace is given by

$$t_f^*(D) := \sum_{Q \in \mathcal{Q}_{D,p}/\Gamma_0^*(p)} \frac{1}{|\Gamma_0^*(p)_Q|} f(\alpha_Q). \tag{1.6}$$

Here $\Gamma_0^*(p)_Q$ is the stabilizer of Q in $\Gamma_0^*(p)$. Applying Theorem 1.1 to the Bruinier–Funke generating functions gives the following corollary.

COROLLARY 1.2. *If p is an odd prime and* $f = \sum a(n)q^n \in \mathcal{M}_0(\Gamma_0^*(p))$, *with* $a(0) = 0$, *then for every positive integer* $D \equiv 0, 3 \pmod 4$ *we have*

$$t_f^*(D) + t_f^*(Dp^2) = \sum_{m \geq 1} \frac{b(m)}{m} \sum_{l \mid m} \mu(m/l)\, t_l(D).$$

Here $\mu(n)$ denotes the usual Möbius function, and

$$b(m) = \sum_{n \geq 1} (ma(-mn) + p^2 ma(-p^2 mn)).$$

REMARK. There is an alternate approach to proving Corollary 1.2. Using the weight zero analog of Lemma 2.2, one can obtain an identity of the form

$$\sum_{m=1}^{M} c(m)J_m(z) = f(z) + pf(z)|U(p).$$

Using this identity, one easily obtains sums of $t_m(n)$ as sums of singular moduli for $f(z)$. To prove Corollary 1.2, one needs to relate such sums to (1.6). This requires a somewhat lengthy calculation which aligns the corresponding CM points, counted with the correct multiplicity, with the points obtained from definition of the $U(p)$ operator. The extra information provided by Bruinier and Funke, and Zagier, allows us to argue instead with standard properties of half-integral weight modular forms.

EXAMPLE. If $f = j_3^*$, $D = 3$, and $p = 3$, then Corollary 1.2 implies that

$$t^*_{j_3^*}(3) + t^*_{j_3^*}(27) = t_1(3) = \frac{1}{3}\,J\left(\frac{-1+\sqrt{-3}}{2}\right) = -248.$$

It turns out that

$$\begin{aligned} t^*_{j_3^*}(3) &= \frac{1}{6}\,j_3^*\left(\frac{-3+\sqrt{-3}}{6}\right) = -\frac{42}{6} = -7,\\ t^*_{j_3^*}(27) &= j_3^*\left(\frac{-7+\sqrt{-3}}{6}\right) + \frac{1}{6}\,j_3^*\left(\frac{-9+\sqrt{-3}}{6}\right) = -234 - \frac{42}{6} = -241. \end{aligned}$$

Noting that

$$j_3^*\left(\frac{-3+\sqrt{-3}}{6}\right) = j_3^*\left(\frac{-9+\sqrt{-3}}{6}\right) = -42$$

gives (1.1).

In Section 2 we recall facts about half-integral weight modular forms. In Section 3 we discuss features of the "Kohnen spaces", recall the formulation of the Bruinier–Funke and the Zagier generating functions, and prove Theorem 1.1 and Corollary 1.2.

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2. Preliminaries on half-integral weight modular forms. We recall basic facts about half-integral weight modular forms (see [Sh] and [Ko]). Throughout, let k be an integer, and let $p = 1$ or an odd prime. We let $\mathfrak{G}_{k+1/2}$ be the group consisting of all pairs $(A, \phi(z))$, where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ and ϕ is a complex-valued holomorphic function on $\mathbb{H}$ satisfying

$$|\phi(z)| = (\det A)^{-k/2-1/4}|cz+d|^{k+1/2}.$$

The group law in $\mathfrak{G}_{k+1/2}$ is given by

$$(A,\phi(z))\cdot(B,\psi(z)) := (AB,\phi(Bz)\psi(z)). \tag{2.1}$$

The group algebra of $\mathfrak{G}_{k+1/2}$ over $\mathbb{C}$ acts on functions $f:\mathbb{H}\to\mathbb{C}$ by

$$f\,\Big|\sum_{\nu} c_\nu(A_\nu,\phi_\nu) = \sum_{\nu} c_\nu \phi_\nu(z)^{-1} f(A_\nu z).$$

We require the automorphy factor

$$\phi(z) := \chi(d)\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-k-1/2}(cz+d)^{k+1/2},$$

where χ is an even Dirichlet character modulo $4p$. As an abuse of notation, we shall let

$$A^* := (A,\phi). \tag{2.2}$$

Let $\Delta_0(4p,\chi)_{k+1/2}$ be the set of those pairs A^*, where $A\in\Gamma_0(4p)$. A meromorphic function $f:\mathbb{H}\to\mathbb{C}$ is a *weight* $k+1/2$ *meromorphic modular form with Nebentypus* χ if it is meromorphic at the cusps of $\Gamma_0(4p)$ and if $f|A^* = f$ for all $A^*\in\Delta_0(4p,\chi)_{k+1/2}$. We let $\mathcal{M}_{k+1/2}(\Gamma_0(4p),\chi)$ be the space of weakly holomorphic modular forms of weight $k+1/2$ with Nebentypus character χ on $\Gamma_0(4p)$. Furthermore, write $\mathcal{M}_{k+1/2}(\Gamma_0(4p))$ if χ is trivial.

REMARK. Although we only describe those forms with level $4p$, most of the following arguments hold for general levels. Furthermore, the facts we quote from works of Shimura and Kohnen were originally stated for holomorphic forms or cusp forms, although their proofs also apply for weakly holomorphic forms.

For $m\ge 1$, define the operators $U(m)$ and $V(m)$ on formal power series in q by

$$\Big(\sum_{n\in\mathbb{Z}} a(n)q^n\Big)\,\Big|\,U(m) := \sum_{n\in\mathbb{Z}} a(mn)q^n, \tag{2.3}$$

$$\Big(\sum_{n\in\mathbb{Z}} a(n)q^n\Big)\,\Big|\,V(m) := \sum_{n\in\mathbb{Z}} a(n)q^{nm}. \tag{2.4}$$

We also require the Atkin–Lehner operators $W(p)$ defined by

$$W(p) := \left(\begin{pmatrix} p & a\\ 4p & pb\end{pmatrix}, \left(\frac{-4}{p}\right)^{-k-1/2} p^{-k/2-1/4}(4pz+pb)^{k+1/2}\right), \tag{2.5}$$

where a and b are integers with $pb-4a=1$. The next proposition (see [Sh] and [Ko]) describes the modularity of these operators.

PROPOSITION 2.1. *Suppose that* $f\in\mathcal{M}_{k+1/2}(\Gamma_0(4p),\chi)$.

(1) $f|U(p), f|W(p)\in\mathcal{M}_{k+1/2}\big(\Gamma_0(4p),\big(\frac{p}{\cdot}\big)\chi\big)$.
(2) $f|V(p)\in\mathcal{M}_{k+1/2}\big(\Gamma_0(4p^2),\big(\frac{p}{\cdot}\big)\chi\big)$.

We shall employ the trace map $\mathrm{Tr}_4^{4p} : \mathcal{M}_{k+1/2}(\Gamma_0(4p)) \to \mathcal{M}_{k+1/2}(\Gamma_0(4))$ defined by

$$\mathrm{Tr}_4^{4p}(f) := \sum_{i=1}^{r} f|_{k+1/2}\gamma_i, \tag{2.6}$$

where $\{\gamma_1, \dots, \gamma_r\}$ is a set of coset representatives for $\Gamma_0(4p)\backslash\Gamma_0(4)$. The next lemma (see page 66 of [Ko]) gives a description of this map in terms of $W(p)$ and $U(p)$.

LEMMA 2.2. *If p is an odd prime and $f \in \mathcal{M}_{k+1/2}(\Gamma_0(4p))$, then*

$$\mathrm{Tr}_4^{4p}(f) = f + \left(\frac{-1}{p}\right)^{-(k+1/2)} p^{-k/2+3/4} f|_{k+1/2}W(p)|U(p).$$

3. Kohnen's spaces and the traces of singular moduli. Let $\mathcal{M}^{+}_{k+1/2}(\Gamma_0(4p))$ and $\mathcal{M}^{+,\varepsilon}_{k+1/2}(\Gamma_0(4p))$ be the spaces defined in (1.3) and (1.4).

3.1. *Trace generating functions.* Zagier showed how the forms $g_m(z)$ may be used to compute the generating functions for traces and twisted traces of singular moduli. For example, consider

$$\begin{aligned} g_1(z) &= q^{-1} + \sum_{D=0}^{\infty} B(1,D)q^D \\ &= q^{-1} - 2 + 248q^3 - 492q^4 + \cdots \in \mathcal{M}^{+}_{3/2}(\Gamma_0(4)). \end{aligned} \tag{3.1}$$

To make the connection to traces, for positive integers m, let

$$B_m(1,D) := \text{the coefficient of } q^D \text{ in } g_1(z)|T(m^2), \tag{3.2}$$

where $T(m^2)$ is the usual Hecke operator on $\mathcal{M}^{+}_{3/2}(\Gamma_0(4))$. Zagier's formulas for the traces $t_m(D)$ are given by the following theorem (see Theorem 5 of [Za]).

THEOREM 3.1. *If $m \geq 1$ and $0 \leq D \equiv 0, 3 \pmod 4$, then $t_m(D) = -B_m(1,D)$.*

More generally, Zagier (see Theorem 5 of [Za]) obtains the following formula:

$$B_m(1,D) = \sum_{l|m} lB(l^2,D). \tag{3.3}$$

Applying Möbius inversion to these formulas immediately gives the following lemma.

LEMMA 3.2. *If m is a positive integer and $1 \le D \equiv 0,3 \pmod 4$, then*

$$B(m^2, D) = -\frac{1}{m}\sum_{l\mid m} \mu(m/l) t_l(D).$$

The following is a special case of Bruinier and Funke's work (see Theorem 1.1 of [BF]).

THEOREM 3.3. *If p is an odd prime and $f = \sum a(n)q^n \in \mathcal{M}_0(\Gamma_0^*(p))$, with $a(0) = 0$, then*

$$\begin{aligned} G_p(f,z) := & -\sum_{m\ge 1}\sum_{n\ge 1} m a(-mn) q^{-m^2} \\ & + \sum_{n\ge 1}(\sigma_1(n) + p\sigma_1(n/p))a(-n) + \sum_{D>0} t_f^*(D) q^D \end{aligned}$$

is an element of $\mathcal{M}_{3/2}^{+,+}(\Gamma_0(4p))$, where $\sigma_1(n) := \sum_{t\mid n} t$ for positive integers n, and $\sigma_1(x) = 0$ for non-integral x.

3.2. *Operators on the Kohnen spaces.* Here we assume that p is an odd prime. To prove Theorem 1.1, our plan is to apply Lemma 2.2. This requires an explicit description of the action of $U(p)$ and $W(p)$. Although the next two results are proven for holomorphic forms by Kohnen in [Ko], we give their proofs for completeness.

PROPOSITION 3.4. *If $f \in \mathcal{M}_{k+1/2}(\Gamma_0(4p))$, then*

$$f|U(p)|W(p)^2 = \left(\frac{-1}{p}\right)^{k+1/2} f|U(p).$$

Proof. Using (2.1), we obtain the identity

$$\begin{aligned} W(p)^2 = & \left(\frac{-1}{p}\right)\left(\frac{b+1}{4a+pb^2}\right)\left(\frac{-1}{4a+pb^2}\right)^{-k-1/2}\left(\begin{pmatrix} p & 0 \\ 0 & p\end{pmatrix}, 1\right) \\ & \times \left(\begin{pmatrix} p+4a & a(1+b) \\ 4p(1+b) & 4a+pb^2\end{pmatrix}, \left(\frac{b+1}{4a+pb^2}\right)\left(\frac{-1}{4a+pb^2}\right)^{-k-1/2}\right. \\ & \left. \times (4p(1+b)z + (4a+pb^2))^{k+1/2}\right). \end{aligned}$$

Since $\left(\left(\begin{smallmatrix} p & 0 \\ 0 & p\end{smallmatrix}\right), 1\right)$ acts as the identity on $f|U(p)$, and since $\Delta_0\left(4p, \left(\frac{p}{\cdot}\right)\right)_{k+1/2}$ contains

$$\begin{aligned} & \left(\begin{pmatrix} p+4a & a(1+b) \\ 4p(1+b) & 4a+pb^2\end{pmatrix},\right. \\ & \left. \left(\frac{b+1}{4a+pb^2}\right)\left(\frac{-1}{4a+pb^2}\right)^{-k-1/2}(4p(1+b)z + (4a+pb^2))^{k+1/2}\right), \end{aligned}$$

Proposition 2.1 gives the identity

$$f|U(p)|W(p)^2 = \left(\frac{-1}{p}\right)\left(\frac{b+1}{4a+pb^2}\right)\left(\frac{-1}{4a+pb^2}\right)^{-k-1/2} f|U(p).$$

Since $\left(\frac{-1}{4a+pb^2}\right) = \left(\frac{-1}{p}\right)$, to complete the proof we have to show that

$$\left(\frac{b+1}{4a+pb^2}\right) = 1.$$

For this we choose $l \in \mathbb{N}$ maximal such that $2^l \mid (b+1)$. Moreover, we assume that $a \equiv 0 \pmod 2$ (i.e. $b \equiv p \pmod 8$). The Law of Quadratic Reciprocity implies that

$$(3.4) \qquad \left(\frac{b+1}{4a+pb^2}\right) = \left(\frac{2}{p}\right)^l (-1)^{\frac{1}{4}\left(\frac{b+1}{2^l}-1\right)(p-1)} \left(\frac{-1}{(b+1)/2^l}\right).$$

Here we used the fact that $4a - pb = -1$. To prove that the right-hand side of (3.4) equals 1, we distinguish the cases whether $l > 1$ or $l = 1$.

If $l > 1$, then $p \equiv b \equiv 3 \pmod 4$. Using the fact that in this case

$$(-1)^{\frac{1}{4}\left(\frac{b+1}{2^l}-1\right)(p-1)} = \left(\frac{-1}{(b+1)/2^l}\right),$$

we find from (3.4) that

$$\left(\frac{b+1}{4a+pb^2}\right) = \left(\frac{2}{p}\right)^l.$$

Thus the claim is clear if l is even. If l is odd, then $p \equiv b \equiv -1 \pmod 8$ and therefore $\left(\frac{2}{p}\right) = 1$. Thus we also obtain the claim.

If $l = 1$, then $p \equiv b \equiv 1, -3 \pmod 8$. If $p \equiv b \equiv 1 \pmod 8$, then we see from (3.4) that $\left(\frac{b+1}{4a+pb^2}\right) = 1$, since

$$\left(\frac{2}{p}\right) = 1, \quad (-1)^{\frac{1}{4}\left(\frac{b+1}{2}-1\right)(p-1)} = 1, \quad \left(\frac{-1}{(b+1)/2}\right) = 1.$$

If $p \equiv b \equiv 5 \pmod 8$, we also obtain $\left(\frac{b+1}{4a+pb^2}\right) = 1$ from (3.4), since

$$\left(\frac{2}{p}\right) = -1, \quad (-1)^{\frac{1}{4}\left(\frac{b+1}{2}-1\right)(p-1)} = 1, \quad \left(\frac{-1}{(b+1)/2}\right) = -1. \ \blacksquare$$

The following lemma relates the action of $U(p)$ and $W(p)$ on $\mathcal{M}^{+,\varepsilon}_{k+1/2}(\Gamma_0(4p))$.

Lemma 3.5. *If $f \in \mathcal{M}^{+,\varepsilon}_{k+1/2}(\Gamma_0(4p))$, where $\varepsilon \in \{\pm 1\}$, then*

$$f|p^{-k/2+1/4}U(p)|W(p)$$

is in $\mathcal{M}^{+,\varepsilon}_{k+1/2}(\Gamma_0(4p))$. *Moreover*,

$$f|W(p) = \varepsilon\left(\frac{-1}{p}\right)^{k+1/2} p^{-k/2+1/4} f|U(p).$$

Proof. Exactly as in [Ko, p. 41], we obtain for $f = \sum_{n\gg-\infty} a(n)q^n$ the following identity:

$$\begin{aligned}(3.5)\quad & f|p^{-k/2+1/4}U(p)|W(p) \\ &= \sum_{n\gg-\infty}\left(\frac{(-1)^k n}{p}\right)a(n)q^n + p^{-1/2} f\left|\left(\begin{pmatrix}1 & \nu_0\\ 0 & p\end{pmatrix}, p^{k/2+1/4}\right)\right|W(p),\end{aligned}$$

where ν_0 is an integer with $1+4\nu_0 \equiv 0 \pmod p$. Since $f \in \mathcal{M}^{+,\varepsilon}_{k+1/2}(\Gamma_0(4p))$, it is clear that the Fourier coefficients of the first term vanish unless

$$\left(\frac{(-1)^k n}{p}\right) = \varepsilon.$$

Moreover, the coefficients of the second term vanish unless $p \mid n$, which follows directly from the identity

$$\begin{aligned}(3.6)\quad & \left(\begin{pmatrix}1 & \nu_0\\ 0 & p\end{pmatrix}, p^{k/2+1/4}\right)W(p) \\ &= \left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \left(\frac{-1}{p}\right)^{k+1/2}\right)C^*\,W(p)\left(\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}, p^{-k/2-1/4}\right),\end{aligned}$$

where $C \in \Gamma_0(4p)$. Since Kohnen (see page 39 of [Ko]) proved that

$$f|p^{-k/2+1/4}U(p)|W(p) \in \mathcal{M}_{k+1/2}(\Gamma_0(4p)),$$

these calculations show that $f|p^{-k/2+1/4}U(p)|W(p) \in \mathcal{M}^{+,\varepsilon}_{k+1/2}(\Gamma_0(4p))$.

For the second part of the lemma, note that (3.5) and (3.6) imply

$$\begin{aligned} f|p^{-k/2+1/4}U(p)|W(p) &= \varepsilon f - \varepsilon f|U(p)|V(p) \\ &\qquad + \left(\frac{-1}{p}\right)^{-k-1/2} p^{k/2-1/4} f|W(p)|V(p).\end{aligned}$$

Since $f|p^{-k/2+1/4}U(p)|W(p) \in \mathcal{M}^{+,\varepsilon}_{k+1/2}(\Gamma_0(4p))$, we have

$$\begin{aligned} h^* := -\varepsilon f|U(p)|V(p) + \left(\frac{-1}{p}\right)^{-k-1/2} p^{k/2-1/4} f|W(p)|V(p) \\ \in \mathcal{M}^{+,\varepsilon}_{k+1/2}(\Gamma_0(4p)).\end{aligned}$$

If $h^* = \sum_{n\gg-\infty} c(n)q^n$, then the coefficients $c(n)$ vanish unless p divides n. A theorem of Serre and Stark (Theorem 1 of [SS]) then implies that $h^* = h_0|V(p)$, where $h_0 \in \mathcal{M}_{k+1/2}\big(\Gamma_0(4), \big(\frac{p}{\cdot}\big)\big)$. Obviously, there are no such non-zero forms, and so $h_0 = 0$, which in turn implies that

$$f|p^{-k/2+1/4}U(p)|W(p) = \varepsilon f.$$

The second claim in the lemma now follows by applying Proposition 3.4. ∎

Remark. The theorem of Serre and Stark employed above is stated for holomorphic half-integral weight forms. Note that we may apply their result to $h^*(z)\Delta(pz)^a$, where $\Delta(z)$ is the usual normalized weight 12 cusp form on $\mathrm{SL}_2(\mathbb{Z})$ and a is a sufficiently large positive integer. This results in the desired conclusion that $h^* = 0$.

3.3. *Proof of Theorem 1.1.* Now we prove Theorem 1.1 using the results from the previous two subsections. Lemmas 2.2 and 3.5 imply that

$$\mathrm{Tr}_4^{4p}(g) = g + \left(\frac{-1}{p}\right)^{-3/2} p^{1/4} \cdot g|W(p)|U(p) = g + \varepsilon g|U(p^2).$$

Therefore,

$$\mathrm{Tr}_4^{4p}(g) = \sum_{m\geq 1}(a(-m) + \varepsilon a(-mp^2))q^{-m} + \sum_{n\geq 0}(a(n) + \varepsilon a(np^2))q^n.$$

By its Fourier expansion and Lemma 2.2, we have $\mathrm{Tr}_4^{4p}(g) \in \mathcal{M}_{3/2}^+(\Gamma_0(4))$, and

$$\mathrm{Tr}_4^{4p}(g) = \sum_{m\geq 1}(a(-m) + \varepsilon a(-mp^2))g_m(z).$$

Here we require that the g_m are uniquely determined by their "principal parts", together with the fact that there are no holomorphic forms in $\mathcal{M}_{3/2}^+(\Gamma_0(4))$. ∎

Proof of Corollary 1.2. By applying Theorem 1.1 to the modular form $G_p(f,z)$ from Theorem 3.3, we obtain the identity

$$\mathrm{Tr}_4^{4p}(G_p(f,z)) = -\sum_{m\geq 1}\Big(\sum_{n\geq 1} ma(-mn) + p^2ma(-mp^2n)\Big)g_{m^2}(z).$$

The corollary now follows from Theorem 3.3 and Lemma 3.2. ∎

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