ACTA ARITHMETICA
119.4 (2005)

The equation $x + y = \alpha$ in multiplicatively dependent unknowns

by

P. Habegger (Basel)

1. Introduction and results. In this paper we give essentially optimal upper bounds for the height of multiplicatively dependent algebraic solutions of the inhomogeneous linear equation $x + y = \alpha$ in two unknowns x and y where α is any non-zero algebraic number. Furthermore we will study the case where $\alpha \geq 2$ is a rational power of a non-zero integer and derive a better bound for the height of the solution. We will also see that this bound is best possible in the case where $\alpha \geq 2$ is a rational power of 2 and even that the maximal height value is isolated if α is also assumed to be an integer. Furthermore for non-zero rational α we give a bound independent of α for the number of solutions of $x + y = \alpha$ if the unknowns are algebraic units in the union of all number fields which have unit group of rank 1.

Two non-zero elements x, y of a field are called *multiplicatively dependent* if there are $r, s \in \mathbb{Z}$ not both zero such that

$$x^r y^s = 1. \tag{1}$$

When τ is algebraic then $H(\tau)$ will denote the absolute Weil height of τ (see Section 2).

Let α be a non-zero algebraic number. We will show in Theorem 1 that the heights of multiplicatively dependent algebraic numbers x, y satisfying

$$x + y = \alpha \tag{2}$$

are effectively bounded in terms of the height of α. More precisely:

Theorem 1. *Let α be a non-zero algebraic number, and let x, y be non-zero multiplicatively dependent algebraic numbers with $x + y = \alpha$. Then*

$$\max\{H(x), H(y)\} \leq 2H(\alpha)^2, \tag{3}$$

$$\min\{H(x), H(y)\} \leq 2H(\alpha). \tag{4}$$

2000 *Mathematics Subject Classification*: Primary 11G50; Secondary 14G40.

The general strategy in the proof of Theorem 1 is the following: Given non-zero algebraic numbers x, y, α as in Theorem 1, there are integers r, s not both zero such that $x^r y^s = 1$. Define the rational function $f = T^r(T-\alpha)^s$ and let d be the degree of f. Note that $f(x) = \pm 1$. For any algebraic τ not a pole of f we will derive a lower bound for $H(f(\tau))H(\tau)^{-d}$ in terms of $H(\alpha)$ and d. By substituting $\tau = x$ we will get an upper bound for $H(x)$ independent of r, s, and d. Here it will be essential that the lower bound has optimal dependence on d; we will achieve this through careful estimates of certain positive local minima, together with the use of the product formula to avoid zero values.

Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers. Theorem 1 makes explicit a special case of

THEOREM ([BMZ99, p. 1120]). *Let $\mathcal{C}$ be a closed absolutely irreducible curve in $\mathbb{G}_m^n$, $n \geq 2$, defined over $\overline{\mathbb{Q}}$ and not contained in a translate of a proper subtorus of $\mathbb{G}_m^n$. Then the algebraic points of $\mathcal{C}$ which lie in the union of all proper algebraic subgroups of $\mathbb{G}_m^n$ form a set of bounded Weil height.*

Indeed, (2) defines a line $\mathcal{C}$ in $\mathbb{G}_m^2$ which is not contained in a translate of a proper subtorus because $\alpha \neq 0$. And any proper algebraic subgroup of $\mathbb{G}_m^2$ is contained in some set (1). Finally, the Weil height used in [BMZ99] was the expression $H(x)H(y) \geq \max\{H(x), H(y)\}$.

In the special case $\alpha = 1$ we get

$$\max\{H(x), H(y)\} \leq 2.$$

This result has already been proved by Cohen and Zannier with different methods in [CZ00, Theorem 1]. It is easily seen to be sharp after setting $x = y = 1/2$. More generally, pick any $\phi \in \mathbb{Q}$ with $\phi \geq 0$ and set $x = y = 2^{-\phi-1}$, $\alpha = 2^{-\phi}$. Then $x + y = \alpha$. Further, by using the standard properties of the height function stated in the next section one obtains $\min\{H(x), H(y)\} = 2^{\phi+1} = 2H(\alpha)$. So upper bounds for $H(x), H(y)$ have to be at least linear in $H(\alpha)$ and (4) is sharp. We ask: can the bound for the height in (3) be improved? If so, is the upper bound linear in $H(\alpha)$? The next theorem gives a positive answer to the first question. It is proved using simple ideas from diophantine approximation.

THEOREM 2. *Let α be a non-zero algebraic number, and let x, y be non-zero multiplicatively dependent algebraic numbers with $x + y = \alpha$. Then*

$$\max\{H(x), H(y)\} \leq 14H(\alpha)\log(3H(\alpha)). \tag{5}$$

Note that the estimate given in Theorem 2 is asymptotically much better for $H(\alpha) \to \infty$ than the one given in Theorem 1, even though it is worse for small $H(\alpha) < 31$.

Still the logarithm in (5) seems a bit disturbing, as it does not answer the second question posed above. In fact a linear inequality like

$$\max\{H(x), H(y)\} \leq CH(\alpha)$$

with an absolute constant C is impossible—even if x, y, and α are restricted to $\mathbb{Q}$—as the following theorem shows.

THEOREM 3. *For each pair of reals θ, Θ with $0 < \theta < 1$ and $\Theta > 1$, there are non-zero $\alpha \in \mathbb{Q}$ with $H(\alpha) > \Theta$, and non-zero multiplicatively dependent $x, y \in \mathbb{Q}$ with $x + y = \alpha$ such that*

$$\max\{H(x), H(y)\} > \theta \frac{H(\alpha)\log(3H(\alpha))}{(\log\log(3H(\alpha)))^2}.$$

So the bound given in Theorem 2 is optimal in the sense that it cannot be replaced by $c(\varepsilon)H(\alpha)(\log(3H(\alpha)))^{1-\varepsilon}$ for some $\varepsilon > 0$ (and similarly the bound given in Theorem 3 cannot be replaced by $c(\varepsilon)H(\alpha)(\log(3H(\alpha)))^{1+\varepsilon}$).

Until now we have considered any non-zero algebraic α, but if we restrict α to special values, then we are able to improve (5) to a linear bound.

THEOREM 4. *Let $\alpha = n^{\phi}$ where n is a positive integer and ϕ a positive rational, and suppose that $\alpha \geq 2$. Then for all non-zero multiplicatively dependent algebraic numbers x, y with $x + y = \alpha$ we have*

$$\max\{H(x), H(y)\} \leq 2H(\alpha) \tag{6}$$

with equality if and only if α is a rational power of 2 and $x = 2\alpha$ or $y = 2\alpha$.

We already know that inequality (6) holds for $\alpha = 1$ by our Theorem 1 or by the result of Cohen and Zannier. So (6) is valid for every non-zero integer α.

Once we have a sharp upper bound, say $B(\alpha)$ for $\max\{H(x), H(y)\}$ as in Theorem 4, it is an interesting problem to determine if this upper bound is isolated in the sense that there exists $\varepsilon(\alpha) > 0$ such that either

$$H(x) = B(\alpha) \quad \text{or} \quad H(x) \leq B(\alpha) - \varepsilon(\alpha). \tag{7}$$

This kind of problem was first studied by Cohen and Zannier ([CZ00, Proposition 1]) who proved isolation for $\alpha = 1$ with $B(\alpha) = 2$. They use Bilu's Equidistribution Theorem and up to now no one has published an explicit $\varepsilon(1)$. We will give an effective result if $\alpha = 2^{\phi}$ with $\phi \in \mathbb{N}$.

THEOREM 5. *Let $\alpha = 2^{\phi}$ where ϕ is a positive integer. Then for all non-zero multiplicatively dependent algebraic numbers x, y with $x + y = \alpha$ we have either*

$$\max\{H(x), H(y)\} = 2H(\alpha) \quad \textit{or} \quad \max\{H(x), H(y)\} \leq 1.98H(\alpha). \tag{8}$$

Theorem 5 shows not only that our $\varepsilon(\alpha)$ in (7) can be chosen independent of α, but that it can even be chosen such that $\varepsilon(\alpha) \to \infty$ if $H(\alpha) \to \infty$.

We move on to an application. Given $M \subset \overline{\mathbb{Q}}$ we define $S_M(\alpha)$ to be the set of pairs $(x, y) \in M^2$ which satisfy (2) and where x, y are also required to be units in the ring of algebraic integers. Equations of this type are called *unit equations* and have been studied extensively for example if $M = K$ is a number field. In this case it is a well known result that $S_K(\alpha)$ is finite and even $\#S_K(\alpha) \leq c(K)$, i.e. the cardinality is bounded independently of α. For example Evertse ([Eve84, Theorem 1]) shows that $\#S_K(\alpha) \leq 3 \cdot 7^{2[K:\mathbb{Q}]+r}$ where r is the number of real embeddings of K. Further bounds for $\#S_K(\alpha)$ have been obtained by Beukers and Schlickewei ([BS96, Theorem 1.1]).

Now let K be a number field with unit group of rank 1 and $\alpha \in \mathbb{Q}^*$. If x and y solve the unit equation, then they are multiplicatively dependent and so by Theorem 1 their height is bounded effectively in terms of $H(\alpha)$. Define $\mathcal{F}$ to be the union of all number fields with unit group of rank 1. Then the same argument just given leads to an effective height bound for x, y with $(x, y) \in S_{\mathcal{F}}(\alpha)$. In both cases Dirichlet's Unit Theorem shows that the degrees of x and y do not exceed 4. So Northcott's Theorem implies that $S_K(\alpha)$ and $S_{\mathcal{F}}(\alpha)$ are finite sets. In fact we will prove a uniform result in the spirit of Evertse.

THEOREM 6. *Let α be a non-zero rational number and K a number field with unit group of rank* 1 *and $\mathcal{F}$ as above. Then $\#S_K(\alpha) \leq 292$ and $\#S_{\mathcal{F}}(\alpha) \leq 755 \cdot 10^6$.*

The first inequality is merely a numerical improvement of a special case of Evertse's bound which leads to $\#S_K(\alpha) \leq 3 \cdot 7^8 = 17294403$. The second result is best possible in the sense that if $\mathcal{F}$ is replaced by $\mathcal{F}'$, which is the union of all number fields with unit group of rank 2, then $S_{\mathcal{F}'}(\alpha)$ is infinite. Indeed, let $a \in \mathbb{Z}$ and let x be a zero of the polynomial $T(1-T)(a-T)-1 \in \mathbb{Z}[T]$ which is easily seen to be irreducible. Then x and $1-x$ are units. Clearly there are infinitely many such x as a runs over all rational integers. This is not a contradiction to the theorem above because for a large enough x lies in a cubic number field with unit group of rank 2.

In Section 2 we introduce notation used throughout the paper and state the required results on lower bounds for the height of values of special rational functions. In Section 3 we prove the statements made in Section 2. These results are then used in Section 4 to prove Theorems 1 and 2. In Section 5 we prove Theorem 3 by construction. Finally, in Sections 6, 7, and 8 we prove Theorems 4, 5, and 6 respectively.

I thank Professor D. W. Masser for many helpful comments and discussions. I would also like to thank the referee for carefully reading the manuscript and giving valuable suggestions.

2. Notation and auxiliary results on rational functions. First we introduce the notation used throughout the paper. Let K be a number field with ring of algebraic integers $\mathcal{O}_K$, and define M_K to be the set of absolute values of K such that their restriction to $\mathbb{Q}$ is the usual p-adic absolute value or the standard complex absolute value. If $v \in M_K$ extends $w \in M_{\mathbb{Q}}$ we will write $v \mid w$. An extension $v \in M_K$ of the standard complex absolute value will be called an *infinite prime*, or $v \mid \infty$ for short. All other $v \in M_K$ will be called *finite primes*. It is well known that there are one-to-one correspondences between infinite primes and embeddings $K \to \mathbb{C}$ up to complex conjugation on the one hand, and between finite primes and non-zero prime ideals of $\mathcal{O}_K$ on the other hand. For $v \in M_K$ define $d_v = [K_v : \mathbb{Q}_v]$ for the completions K_v, $\mathbb{Q}_v$ of K, $\mathbb{Q}$ with respect of v. Then we have the product formula ([Lan83, pp. 19–20])

$$\prod_{v \in M_K} |\tau|_v^{d_v} = 1 \quad \text{for any } \tau \in K^* = K \backslash \{0\}.$$

For integers n it is sometimes useful to define

$$\delta_v(n) = \max\{1, |n|_v\}.$$

For $\tau \in K$, the *height* of τ is defined as

$$H(\tau) = \prod_{v \in M_K} \max\{1, |\tau|_v^{d_v/[K:\mathbb{Q}]}\}.$$

This definition is independent of the field K containing τ ([HS00, p. 176]) and the height function is thus defined on the algebraic closure $\overline{\mathbb{Q}}$.

For the reader's convenience we recall some basic properties of the height function:

(i) $H(\tau^n) = H(\tau)^{|n|}$ for all $\tau \in \overline{\mathbb{Q}}^*$ and $n \in \mathbb{Z}$ ([Lan83, p. 51]).
(ii) $H(\tau) = 1$ if and only if $\tau \in \overline{\mathbb{Q}}$ is a root of unity or zero.
(iii) $H(\zeta\tau) = H(\tau)$ if $\tau \in \overline{\mathbb{Q}}$ and ζ is a root of unity.
(iv) $H(\tau\mu) \le H(\tau)H(\mu)$ if $\tau, \mu \in \overline{\mathbb{Q}}$ ([Lan83, p. 51]).
(v) $H(\tau + \mu) \le 2H(\tau)H(\mu)$ if $\tau, \mu \in \overline{\mathbb{Q}}$.
(vi) $H(\tau) = \max\{|a|, |b|\}$ if $\tau = a/b \in \mathbb{Q}$, $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and $(a, b) = 1$ ([Lan83, p. 52]).
(vii) For $C, D \in \mathbb{R}$ there exist only finitely many $\tau \in \overline{\mathbb{Q}}$ with $H(\tau) \le C$ and $[\mathbb{Q}(\tau) : \mathbb{Q}] \le D$.

Property (ii) is often referred to as Kronecker's Theorem ([HS00, B.2.3.1, p. 178]). Property (iii) follows directly from the fact that $|\zeta| = 1$ if ζ is a root of unity and $|\cdot|$ is any absolute value. And (v) is a consequence of the estimate $\max\{1, |\tau + \mu|_v\} \le \delta_v(2) \max\{1, |\tau|_v\} \max\{1, |\mu|_v\}$. Finally, property (vii) is a special case of Northcott's Theorem ([HS00, B.2.3, p. 177]).

Let $f = f(T)$ be a rational function with algebraic coefficients and let $\tau \in \overline{\mathbb{Q}}$. We are interested in how $H(f(\tau))$ depends on $H(\tau)$ and f (provided τ

is not a pole of f). Recall that the *degree* of f is defined as $\max\{\deg P, \deg Q\}$ for any two coprime polynomials P, Q with $f = P/Q$. Classically it is known that

$$C \le \frac{H(f(\tau))}{H(\tau)^{\deg f}} \le C' \tag{9}$$

with positive constants C, C' that are independent of τ (for example it follows easily from the [Lan83, Theorem 1.8, p. 81]).

As pointed out in the introduction we are particularly interested in a sharp lower bound of (9) for certain rational functions. The first function we will investigate has the form $f = T^r(T-\alpha)^s$ with α a non-zero algebraic number. Its degree is $|r|+|s|$ if $rs \ge 0$, and $\max\{|r|,|s|\}$ if $rs < 0$.

PROPOSITION 1. *Let r, s be integers and α a non-zero algebraic number and define the rational function $f = T^r(T-\alpha)^s$ of degree d. Put*

$$e(f) = \begin{cases} 2|s|-|r| & (rs<0,\ |r|<|s|), \\ |r| & (rs<0, |r|\ge|s|), \\ |r|+|s| & (rs \ge 0). \end{cases}$$

Then for all algebraic $\tau \ne 0, \alpha$ we have

$$\frac{H(f(\tau))}{H(\tau)^d} \ge \frac{1}{2^d}\frac{1}{H(\alpha)^{e(f)}} \ge \frac{1}{2^d}\frac{1}{H(\alpha)^{2d}}. \tag{10}$$

The exponent $e(f)$ in (10) looks strange, but in fact it is best possible for each value of r and $s \ne 0$.

We will also study lower bounds for certain types of polynomials.

PROPOSITION 2. *Let m, n be integers with $n > m \ge 0$, let β be an algebraic number, and define the polynomial $P = T^n + \beta T^m$. Put $\theta = m/n$. Then for all algebraic τ we have*

$$\frac{H(P(\tau))}{H(\tau)^n} \ge \frac{1-\theta}{2}\frac{1}{H(\beta)^{1/(1-\theta)}} \ge \frac{1}{2n}\frac{1}{H(\beta)^n}. \tag{11}$$

Furthermore, if $n = m > 0$ and $\beta \ne -1$, then

$$\frac{H(P(\tau))}{H(\tau)^n} \ge \frac{1}{2n}\frac{1}{H(\beta)^n}.$$

If β is a root of unity, then Proposition 2 reduces to $H(P(\tau))H(\tau)^{-n} \ge (2n)^{-1}$. In Proposition 3 we use diophantine approximation to get a lower bound independent of n. This improvement comes at a price: the denominator on the left-hand side of (11) has to be replaced by $X/\log X$ for large $X = H(\tau)^n$. So strictly speaking we are not in the situation of (9) anymore. Nevertheless Proposition 3 is essential for the proof of Theorem 2.

PROPOSITION 3. *Let m, n be integers with $n > m \ge 0$, let ζ be a root of unity, and define the polynomial $P = T^n + \zeta T^m$. Then for all algebraic τ*

we have

$$\frac{H(P(\tau))}{H(\tau)^n}\max\{1,\log(H(\tau)^n)\} \ge \frac{1}{2e}$$

with $e = 2.71828\ldots$.

Compare this lower bound with the easy upper bound

$$H(P(\tau)) = H(\tau^m(\tau^{n-m}+\zeta)) \le 2H(\tau^m)H(\tau^{n-m}) = 2H(\tau)^n.$$

By combining the previous inequality with the inequality from Proposition 3, we will deduce the following amusing consequence for the logarithmic height $h(\tau) = \log H(\tau)$.

Corollary. *For $\sigma \in \overline{\mathbb{Q}}$ and $\phi \in \mathbb{Q}$ with $0 \le \phi \le 1$ take any determination of σ^ϕ. Then*

$$|h(\sigma+\sigma^\phi) - h(\sigma)| \le 1 + \log 2 + \log\max\{1, h(\sigma)\}.$$

3. Proofs of Propositions 1–3 and Corollary. Our general strategy in estimating the heights $H(f(\tau))$ where f and τ are defined as in Proposition 1 is to consider each local factor separately. In the first step we will consider the case $rs \ge 0$. It will actually suffice to suppose $r \ge 0$ and $s \ge 0$, so that f is a polynomial. Because some results proved below remain valid in a more general context we will work with an arbitrary field K containing α and τ equipped with an absolute value $|\cdot|$. Recall that in general, an absolute value $|\cdot|$ on a field K satisfies the triangle inequality $|x+y| \le |x|+|y|$ for all $x, y \in K$, and that it is called *ultrametric* if it satisfies $|x+y| \le \max\{|x|,|y|\}$ for all $x, y \in K$.

For an absolute value $|\cdot|$ we define

$$m_f(\tau) = \frac{\max\{1, |\tau|^r|\tau-\alpha|^s\}}{\max\{1,|\tau|^d\}} \tag{12}$$

with $d = \deg f = r + s$. The subscript f will be omitted if it is clear from the context what function is meant.

Lemma 1. *Suppose $r, s \ge 0$. Let $|\cdot|$ be an ultrametric absolute value on a field K. Then for any $\tau \in K$,*

$$m_f(\tau) \ge \frac{1}{\max\{1,|\alpha|\}^d}.$$

Proof. The case $|\tau| \le 1$ is trivial because then $\max\{1, |\tau|^r|\tau-\alpha|^s\} \ge 1 \ge \max\{1,|\alpha|\}^{-d}$. So let $|\tau| > 1$. Now the assertion can be proved by a simple study of cases. First consider $|\alpha| < |\tau|$. This implies $|\tau-\alpha| = |\tau|$, so $m(\tau) = 1$. Next if $|\alpha| = |\tau|$ we get $m(\tau) = \max\{|\alpha|^{-d}, |\alpha|^{-s}|\tau-\alpha|^s\} \ge |\alpha|^{-d} = \max\{1,|\alpha|^d\}^{-1}$. Finally, if $|\alpha| > |\tau|$ we conclude that $m(\tau) = \max\{|\tau|^{-d}, |\alpha/\tau|^s\} \ge 1$ because $|\alpha/\tau| > 1$. ∎

We now treat the case that our absolute value is not ultrametric but satisfies the weaker triangle inequality. In that case, we obtain an estimate slightly worse than that of Lemma 1.

LEMMA 2. *Suppose $r, s \geq 0$. Let $|\cdot|$ be an absolute value on a field K. Then for any $\tau \in K$,*

$$m_f(\tau) \geq \frac{1}{(1+|\alpha|)^d} \geq \frac{1}{2^d \max\{1, |\alpha|\}^d}.$$

Proof. First assume $|\tau| \leq 1 + |\alpha|$; then

$$m(\tau) \geq \frac{1}{\max\{1, |\tau|\}^d} \geq \frac{1}{(1+|\alpha|)^d} \geq \frac{1}{2^d \max\{1, |\alpha|\}^d}$$

as desired. If on the other hand $|\tau| \geq 1 + |\alpha| \geq 1$ then $|\tau - \alpha| \geq 1$ so one has

$$\begin{aligned} m(\tau) &= \frac{|\tau-\alpha|^s}{|\tau|^s} = \left|1 - \frac{\alpha}{\tau}\right|^s \geq \left(1 - \left|\frac{\alpha}{\tau}\right|\right)^s \geq \left(1 - \frac{|\alpha|}{1+|\alpha|}\right)^s \\ &= \frac{1}{(1+|\alpha|)^s} \geq \frac{1}{(1+|\alpha|)^d}, \end{aligned}$$

which suffices. ∎

We will now cover the non-polynomial case of Proposition 1, that is, the one with $rs < 0$. Recalling the definition of $m_f(\tau)$ in (12) with $K = \mathbb{C}$ and with $|\cdot|$ being the standard absolute value, and taking for example $r = 2$ and $s = -1$, we obtain, for large τ,

$$m_f(\tau) = \frac{|\tau|^2/|\tau - \alpha|}{|\tau|^2} = |\tau - \alpha|^{-1}.$$

Therefore unfortunately $\lim_{\tau\to\infty} m_f(\tau) = 0$, so m cannot be bounded away from 0 as in Lemma 2.

So we redefine m_f in this case. Our motivation comes from the calculation

$$\begin{aligned} H(f(\tau))^{[K:\mathbb{Q}]} &= \prod_{v\in M_K} |\tau - \alpha|_v^{-sd_v} \prod_{v\in M_K} \max\{1, |\tau|_v^r |\tau-\alpha|_v^s\}^{d_v} \\ &= \prod_{v\in M_K} \max\{|\tau|_v^r, |\tau-\alpha|_v^{-s}\}^{d_v} \end{aligned} \tag{13}$$

where we have applied the product formula for the number field K if $\tau \neq \alpha$.

Now if we assume $r \geq 0$ and set $t = -s \geq 0$, then the new definition

$$m_f(\tau) = \frac{\max\{|\tau|^r, |\tau-\alpha|^t\}}{\max\{1, |\tau|^d\}} \quad \text{with} \quad d = \deg f = \max\{r, t\} \tag{14}$$

will do the trick.

LEMMA 3. *Suppose $r \ge 0 \ge s = -t$ and if $t > 0$ let $\varepsilon = 1 - r/t$. Let $|\cdot|$ be an ultrametric absolute value on a field K. Then for any $\tau \in K$,*

$$m_f(\tau) \ge \begin{cases} \max\{1, |\alpha|\}^{-\varepsilon d} & \text{if } r < t \text{ and } |\alpha| = |\tau| > 1, \\ \max\{1, |\alpha|^{-1}\}^{-d} & \text{otherwise.} \end{cases}$$

Proof. As in the proof of Lemma 1 the proof is just a study of different cases.

Assume $|\tau| \ne |\alpha|$. Then one has $|\tau - \alpha| = \max\{|\tau|, |\alpha|\}$ and hence

$$m(\tau) = \frac{\max\{|\tau|^r, |\tau|^t, |\alpha|^t\}}{\max\{1, |\tau|^d\}}.$$

If $|\tau| \ge 1$ then clearly $m(\tau) \ge 1$ so let $|\tau| < 1$. Then

$$m(\tau) \ge |\alpha|^t \ge \frac{1}{\max\{1, |\alpha|^{-1}\}^t} \ge \frac{1}{\max\{1, |\alpha|^{-1}\}^d}$$

as desired.

Now assume $|\tau| = |\alpha|$. Here

$$m(\tau) \ge \frac{|\alpha|^r}{\max\{1, |\alpha|^d\}}.$$

If $|\alpha| \le 1$ then

$$m(\tau) \ge |\alpha|^r \ge \frac{1}{\max\{1, |\alpha|^{-1}\}^d}$$

as above.

Finally, if $|\alpha| > 1$ then $m(\tau) \ge |\alpha|^{r-d}$. The assertion is obvious if $r = d$, so assume $r < d = t$; then one has

$$m(\tau) \ge |\alpha|^{-\varepsilon d} = \frac{1}{\max\{1, |\alpha|\}^{\varepsilon d}}. \ \blacksquare$$

LEMMA 4. *Suppose $r \ge 0 \ge s = -t$, and if $t > 0$ let $\varepsilon = 1 - r/t$. Let $|\cdot|$ be an absolute value on a field K. Then for any $\tau \in K$,*

$$m_f(\tau) \ge \begin{cases} 2^{-d} \max\{1, |\alpha|^{-1}\}^{-d} \max\{1, |\alpha|\}^{-\varepsilon d} & \text{if } r < t, \\ 2^{-d} \max\{1, |\alpha|^{-1}\}^{-d} & \text{otherwise.} \end{cases}$$

Proof. We will show the slightly stronger statement

$$m(\tau) \ge \begin{cases} 2^{-d} \max\{1, |\alpha|\}^{-\varepsilon d} & \text{if } r < t,\ 1/2 \le |\tau/\alpha| \le 2,\ |\alpha| \ge 1, \\ 2^{-d} \max\{1, |\alpha|^{-1}\}^{-d} & \text{otherwise.} \end{cases} \tag{15}$$

First assume that $|\tau/\alpha| < 1/2$ or $|\tau/\alpha| > 2$. Then $\frac{1}{2}\max\{|\tau|, |\alpha|\} \le |\tau - \alpha|$, and therefore,

$$m(\tau) \ge 2^{-t} \frac{\max\{|\tau|^r, |\tau|^t, |\alpha|^t\}}{\max\{1, |\tau|^d\}}.$$

If $|\tau| \geq 1$ then $m(\tau) \geq 2^{-t} \geq 2^{-d}$ and we are done. Assume that $|\tau| < 1$. Then

$$m(\tau) \geq 2^{-t}|\alpha|^t \geq 2^{-d} \max\{1, |\alpha|^{-1}\}^{-d},$$

which implies (15).

Now assume that $1/2 \leq |\tau/\alpha| \leq 2$. If $|\tau| \leq |\alpha|$ then

$$m(\tau) \geq \frac{|\tau|^r}{\max\{1, |\alpha|^d\}} \geq 2^{-r} \frac{|\alpha|^r}{\max\{1, |\alpha|^d\}},$$

and if $|\tau| > |\alpha|$ then

$$m(\tau) \geq 2^{-d} \frac{|\alpha|^r}{\max\{1, |\alpha|^d\}};$$

so in both cases we have $m(\tau) \geq 2^{-d}|\alpha|^r \max\{1, |\alpha|\}^{-d}$. If $|\alpha| < 1$ then

$$m(\tau) \geq 2^{-d}|\alpha|^r \geq 2^{-d} \max\{1, |\alpha|^{-1}\}^{-d}$$

as above. Suppose $|\alpha| \geq 1$. Then $m(\tau) \geq 2^{-d}|\alpha|^{r-d}$. If $r = d$ our assertion is obvious. Finally, if $r < d$ then $t = d$ and we have

$$m(\tau) \geq 2^{-d}|\alpha|^{-\varepsilon d} = 2^{-d} \max\{1, |\alpha|\}^{-\varepsilon d}. \ \blacksquare$$

Proof of Proposition 1. One may assume $r \geq 0$ (apply property (i) if necessary). Let K be any number field containing α and τ, and take any $v \in M_K$.

If $s \geq 0$ then Lemmas 1 and 2 together imply

$$\frac{\max\{1, |\tau|_v^r |\tau - \alpha|_v^s\}}{\max\{1, |\tau|_v^d\}} \geq \frac{1}{\delta_v(2)^d \max\{1, |\alpha|_v\}^d}.$$

Now in the case $rs \geq 0$ the assertion of the proposition is proved by raising to the d_vth power, taking the product over all elements of M_K and extracting the $[K : \mathbb{Q}]$th root.

If $s < 0$ set $t = -s$ and if $t \neq 0$ set $\varepsilon = 1 - r/t$; then Lemmas 3 and 4 together imply

$$\frac{\max\{|\tau|_v^r, |\tau - \alpha|_v^t\}}{\max\{1, |\tau|_v^d\}} \geq \begin{cases} \delta_v(2)^{-d} \max\{1, |\alpha|_v\}^{-\varepsilon d} \max\{1, |\alpha|_v^{-1}\}^{-d} & \text{if } r < t, \\ \delta_v(2)^{-d} \max\{1, |\alpha|_v^{-1}\}^{-d} & \text{if } r \geq t. \end{cases}$$

Recall (13); now in the case $rs \leq 0$ the assertion is proved by raising to the d_vth power, taking the product over all elements of M_K and extracting the $[K : \mathbb{Q}]$th root. $\blacksquare$

We now turn to the proof of Proposition 2. It follows the same idea as the proof of Proposition 1; that is, each factor in the height is estimated separately. Again the following two lemmas hold in a more general context where K is any field with an absolute value $|\cdot|$. For $\theta \in (0, 1)$ and $\beta, z \in K$

define

$$(16) \qquad \widehat{m}_{\theta\beta}(z) = \frac{\max\{1, |z|^{\theta/(1-\theta)}|z+\beta|\}}{\max\{1, |z|^{1/(1-\theta)}\}}.$$

The subscripts θ and β will be omitted if the context makes it clear what is meant.

As in the previous section we shall estimate the finite part of the height function first.

LEMMA 5. *Let $|\cdot|$ be an ultrametric absolute value on a field K. Then for any $z \in K$,*

$$\widehat{m}_{\theta\beta}(z) \geq \frac{1}{\max\{1, |\beta|\}^{1/(1-\theta)}}.$$

Proof. The case where $|z| \leq 1$ is trivial because then $\widehat{m}(z) \geq 1$. If $|z| > 1$ we split into three cases. First if $|\beta| < |z|$ then $\widehat{m}(z) = 1$. Next if $|\beta| = |z|$ then $\widehat{m}(z) = \max\{|z|^{-1/(1-\theta)}, |1 + \beta/z|\} \geq |z|^{-1/(1-\theta)} = |\beta|^{-1/(1-\theta)}$. And finally, if $|\beta| > |z|$ then $\widehat{m}(z) = |\beta|/|z| > 1$. ■

Finally, we will give an estimate of the factors in the infinite part of the height function in the next lemma.

LEMMA 6. *Let $|\cdot|$ be an absolute value on a field K. Then for any $z \in K$,*

$$\widehat{m}_{\theta\beta}(z) \geq \frac{1-\theta}{2} \frac{1}{\max\{1, |\beta|\}^{1/(1-\theta)}}.$$

Proof. We split up the proof into two parts, the first one being when

$$(17) \qquad \frac{1-\theta}{2} |z|^{1/(1-\theta)} \max\{1, |\beta|\}^{-1/(1-\theta)} \leq 1.$$

But then one has

$$\widehat{m}(z) \geq \frac{1}{\max\{1, |z|\}^{1/(1-\theta)}} \geq \frac{1-\theta}{2} \frac{1}{\max\{1, |\beta|\}^{1/(1-\theta)}},$$

which is just the assertion. Now assume that (17) does not hold. This is equivalent to saying that $|z| > \phi\mu$ with

$$\phi = \left(\frac{2}{1-\theta}\right)^{1-\theta} \quad \text{and} \quad \mu = \max\{1, |\beta|\}.$$

Obviously one has $|z| > 1$ and therefore

$$\widehat{m}(z) \geq \frac{|z|^{\theta/(1-\theta)}|z+\beta|}{|z|^{1/(1-\theta)}} = \frac{|z+\beta|}{|z|} \geq 1 - \frac{|\beta|}{|z|} \geq 1 - \frac{\mu}{|z|}$$
$$> 1 - \frac{1}{\phi} \geq \left(1 - \frac{1}{\phi}\right) \frac{1}{\mu^{1/(1-\theta)}}.$$

So the lemma is proven if the inequality

$$1-\left(\frac{1-\theta}{2}\right)^{1-\theta}=1-\frac{1}{\phi}\geq\frac{1-\theta}{2} \tag{18}$$

holds. Set $\xi=1-\theta$; then (18) is equivalent to

$$g(\xi)\leq 1 \quad \text{for all } \xi\in(0,1), \quad \text{where} \quad g(\xi)=\left(\frac{\xi}{2}\right)^{\xi}+\frac{\xi}{2}. \tag{19}$$

Note that defining $g(0)=1$ makes the map continuous on $[0,1]$ and

$$\frac{d^2g}{d\xi^2}=\left(\frac{\xi}{2}\right)^{\xi}\left((\log(\xi/2)+1)^2+\frac{1}{\xi}\right)>0$$

for $\xi\in(0,1)$. So g is convex, which implies (19) because $g(0)=g(1)=1$. ∎

Note that Lemma 6 is in some cases an improvement of Lemma 2. Indeed, recall the definition of f and assume $r\geq 0$ and $s>0$ so $d=r+s>0$. Set $\theta=r/d$ and $\xi=1-\theta=s/d$. Then Lemma 6 with $\beta=-\alpha$, in conjunction with (12), (16), implies

$$m_f(z)=\widehat{m}_{\theta\alpha}(z)^{d(1-\theta)}\geq\left(\frac{1-\theta}{2}\right)^{d(1-\theta)}\frac{1}{\max\{1,|\alpha|\}^d}$$

and the latter is larger than $(2\max\{1,|\alpha|\})^{-d}$ if and only if $(\xi/2)^{\xi}>1/2$, that is, $\xi<1/2$.

We now prove Proposition 2. Let K be a number field containing β, τ and let $v\in M_K$. The case $m=0$ follows directly from property (v) of the height function so one may assume $\theta=m/n>0$. First consider $n>m$; then

$$\frac{\max\{1,|\tau^n+\beta\tau^m|_v\}}{\max\{1,|\tau|_v^n\}}=\frac{\max\{1,|z|_v^{\theta/(1-\theta)}|z+\beta|_v\}}{\max\{1,|z|_v^{1/(1-\theta)}\}}$$

for $z=\tau^{n-m}$. Now apply Lemmas 5 and 6, raise to the d_vth power, take the product over all elements of M_K and then the $[K:\mathbb{Q}]$th root to prove the first inequality of (11). The second inequality follows immediately from the fact that $1/(1-\theta)\leq n$.

Finally, if $n=m$ and $\beta\neq -1$ then standard height properties imply

$$\frac{H(P(\tau))}{H(\tau)^n}\geq\frac{1}{H(1+\beta)}\geq\frac{1}{2H(\beta)}. \ ∎$$

To prove Proposition 3 we will apply diophantine approximation to Proposition 2. This idea came up in a private correspondence of Bombieri–Masser–Zannier in June 2002.

Set $\theta=m/n$. The case $m=0$ can be dismissed as trivial so we may assume $\theta\in(0,1)$. We would like to apply diophantine approximation to θ to create a new polynomial with small degree.

Recall that for any $Q > 1$ there exists a pair $p, q \in \mathbb{Z}$ such that

$$|\theta q - p| \leq 1/Q \quad \text{and} \quad 0 < q < Q.$$

For a reference see [Cas57, p. 1]. Because $\theta \in (0, 1)$ one has

$$\theta q - p \leq 1/Q \quad \text{so} \quad p \geq \theta q - 1/Q > \theta q - 1 > -1$$

and thus $p \geq 0$; and furthermore

$$p - q\theta \leq 1/Q \quad \text{so} \quad p \leq 1/Q + q\theta < 1 + q\theta < 1 + q,$$

which implies $p \leq q$.

Now choose any $u \in \overline{\mathbb{Q}}^*$ with $u^q = \tau^n$. Set $k = mq - np$ and choose a $\beta \in \overline{\mathbb{Q}}^*$ with $\beta^q = \zeta\tau^k$. Note that $(\beta u^p)^q = \beta^q u^{pq} = \zeta\tau^{k+np} = \zeta\tau^{mq}$, so $\tau^m = \eta\beta u^p$ for some root of unity η. Now

$$P(\tau) = \tau^n + \zeta\tau^m = u^q + \xi\beta u^p$$

for a root of unity $\xi = \zeta\eta$. If $p = q$ and $\xi\beta = -1$ then $P(\tau) = 0$ so $\tau^{n-m} = -\zeta$ so τ is a root of unity and the required result follows trivially. If $p < q$ or $\xi\beta \neq -1$ we can apply Proposition 2 to get

$$\begin{aligned}\frac{H(P(\tau))}{H(\tau)^n} &= \frac{H(u^q + \xi\beta u^p)}{H(u)^q} \geq \frac{1}{2qH(\xi\beta)^q} \geq \frac{1}{2QH(\beta)^q} \\ &= \frac{1}{2QH(\tau)^{|k|}} \geq \frac{1}{2QH(\tau)^{n/Q}}\end{aligned}$$

because $|k| = n|\theta q - p|$. Therefore one has

$$H(P(\tau)) \geq \frac{H(\tau^n)^{1-1/Q}}{2Q} \tag{20}$$

for each $Q > 1$. Because the lower bound is continuous, $Q = 1$ is also allowable. Optimization of the right-hand side leads to the choice

$$Q_0 = \max\{1, \log H(\tau^n)\} \geq 1.$$

Inserting Q_0 into (20) gives the bound

$$\begin{aligned}H(P(\tau)) &\geq \begin{cases} H(\tau^n)(2e \log H(\tau^n))^{-1} & \text{if } H(\tau^n) > e, \\ 1/2 & \text{otherwise,} \end{cases} \\ &\geq \frac{H(\tau^n)}{2e \max\{1, \log H(\tau^n)\}}\end{aligned}$$

as desired. ∎

To prove the Corollary we note that the bound $h(\sigma + \sigma^\phi) - h(\sigma) \leq \log 2$ has already been shown in Section 2. For the corresponding lower bound we note that the case $\phi = 1$ is covered by elementary height properties; so assume $\phi < 1$. Choose $m, n \in \mathbb{Z}$ with $\phi = m/n$; then there is $\tau \in \overline{\mathbb{Q}}^*$ and ζ a root of unity with $\sigma = \tau^n$ and $\sigma^\phi = \zeta\tau^m$. We apply Proposition 3 to get

$$h(\sigma + \sigma^\phi) - h(\sigma) \geq -1 - \log 2 - \log \max\{1, h(\sigma)\},$$

thus concluding the proof. ∎

4. Proofs of Theorems 1 and 2. The proof of Theorem 1 is a simple task with the help of Proposition 1. Let for example $x^r y^s = 1$ with $r, s \in \mathbb{Z}$ not both zero. Apply Proposition 1 to $\pm 1 = x^r(x-\alpha)^s$ and $\pm 1 = y^s(y-\alpha)^r$ using $H(\pm 1) = 1$. At least one of the two cases satisfies the conditions for $e(f) = \deg f$ and the other will always work with $e(f) \leq 2\deg f$. For example if $|r| \geq |s|$ then $H(x) \leq 2H(\alpha)$. ∎

We now prove Theorem 2. Let for example $x^r y^s = 1$ with $r, s \in \mathbb{Z}$ not both zero and assume $r \geq 0$ and $H(x) \geq H(y)$. If necessary apply Proposition 1 as in the proof of Theorem 1 to reduce to the case $|r| < |s|$ and $rs < 0$. Set $n = |s|$ and $m = |r|$; then there exists $\tau \in \overline{\mathbb{Q}}^*$ with $x = \tau^n$ and $y = \zeta\tau^m$ for some root of unity ζ. Hence we have $P(\tau) = \alpha$ with $P = T^n + \zeta T^m$. Now Proposition 3 implies

$$H(\alpha) \geq \varphi(H(x)) \quad \text{with} \quad \varphi(z) = \frac{z}{2e\max\{1, \log z\}}, \tag{21}$$

φ being understood as a continuous map on $[1, \infty)$. Note that φ is increasing, which can be easily verified by restricting it to $[1, e]$ and $[e, \infty)$. Theorem 2 follows once we have shown that

$$\varphi(z_0) \geq H(\alpha) \quad \text{with} \quad z_0 = 14H(\alpha)\log(3H(\alpha)). \tag{22}$$

Indeed, (21) combined with (22) leads to $\varphi(z_0) \geq \varphi(H(x))$ and further to $z_0 \geq H(x)$ because φ is increasing.

Since $z_0 > e$ the inequality (22) is equivalent to

$$\frac{7}{e}\log(3w) - \log(14w) - \log\log(3w) \geq 0 \quad \text{with} \quad w = H(\alpha), \tag{23}$$

which certainly holds for $w = 1$. The derivative of the left-hand side of (23) with respect to w is

$$\frac{1}{w}\left(\frac{7}{e} - 1 - \frac{1}{\log(3w)}\right) \geq \frac{1}{w}\left(\frac{7}{e} - 2\right)$$

if $w \geq 1$. The right-hand side is positive for every $w \geq 1$ so we may conclude that (23) holds for every α, thus completing the proof. ∎

5. Proof of Theorem 3. Choose a large integer q, and define

$$x = \left(\frac{q}{q-1}\right)^n, \quad y = -\left(\frac{q}{q-1}\right)^{n-1} \quad \text{with} \quad n = [q\log q],$$

so that $H(x) = q^n > q^{n-1} = H(y)$. Then $x + y = \alpha$ with $\alpha = q^{n-1}/(q-1)^n$. Now $H(\alpha) = \max\{q^{n-1}, (q-1)^n\}$ so

$$\lim_{q\to\infty} \frac{\log H(\alpha)}{n\log q} = \lim_{q\to\infty} \max\left\{\frac{n-1}{n}, \frac{\log(q-1)}{\log q}\right\} = 1 \tag{24}$$

and one clearly has

$$\lim_{q\to\infty} \frac{n \log q}{q(\log q)^2} = 1. \tag{25}$$

Multiply (24) and (25) to get $\log H(\alpha) = q(\log q)^2\kappa(q)$ with $\lim_{q\to\infty}\kappa(q) = 1$. Taking the logarithm gives $\log\log H(\alpha) = \log q + 2\log\log q + \log\kappa(q)$, which leads us to

$$\lim_{q\to\infty} \frac{\log\log H(\alpha)}{\log q} = 1. \tag{26}$$

Now we want to show that the quotient $H(x)/H(\alpha)$ is large, so we will evaluate the limit of

$$q^{-1}\frac{H(x)}{H(\alpha)} = \min\{1, q^{-1}(1-q^{-1})^{-n}\} \tag{27}$$

as $q \to \infty$. Now

$$\begin{aligned} &\log(q^{-1}(1-q^{-1})^{-n}) \\ &\qquad = -\log q - n\log(1-q^{-1}) = -\log q + n(q^{-1} + O(q^{-2})) \\ &\qquad = -\log q + (q\log q + O(1))(q^{-1} + O(q^{-2})) = O\left(\frac{\log q}{q}\right) \end{aligned}$$

as $q \to \infty$. By inserting this expression into (27) one obtains

$$\lim_{q\to\infty} q^{-1}\frac{H(x)}{H(\alpha)} = 1. \tag{28}$$

Finally, by combining (24)–(26), and (28) we conclude that

$$\lim_{q\to\infty} \frac{H(x)}{\frac{H(\alpha)\log H(\alpha)}{(\log\log H(\alpha))^2}} = 1. \ \blacksquare$$

6. Proof of Theorem 4. If ζ is a root of unity, then $H(1+\zeta) \le 2$ with equality for example if $\zeta = 1$. We first prove the following variant of a special case of Theorem 5; it will also be used in the proof of Theorem 4.

Lemma 7. *If $\zeta \ne 1$ is a root of unity, then*

$$H(1+\zeta) \le \sqrt{2\sqrt{3}} = 1.8612\ldots.$$

Proof. Let K be a number field of degree D containing ζ. Multiply the product formula $\prod_{v\in M_K} |1-\zeta|_v^{d_v} = 1$ with the definition of the height and note that ζ is an algebraic integer to get

$$H(1+\zeta)^D \le \min\Big\{\prod_{v\mid\infty} \max\{1, |1+\zeta|_v\}^{d_v}, \prod_{v\mid\infty} \max\{|1-\zeta|_v, |1-\zeta^2|_v\}^{d_v}\Big\}.$$

Let Δ_1 be the set of infinite primes v with $|1-\zeta|_v \ge 1$, and let Δ_2 be all other infinite primes. Recall that infinite primes correspond to embeddings of K into $\mathbb{C}$ up to conjugation. If $v \in \Delta_1$, then elementary geometry gives $|1+\zeta|_v \le \sqrt{3}$, with the right-hand side replaced by 2 if we allow $v \in \Delta_2$. Similarly if $v \in \Delta_2$, then $\max\{|1-\zeta|_v, |1-\zeta^2|_v\} \le \sqrt{3}$; and if $\sqrt{3}$ is replaced by 2, then the inequality holds for $v \in \Delta_1$. Define $\delta_i = \sum_{v\in\Delta_i} d_v/D$; then $\delta_1+\delta_2 = 1$ and so

$$H(1+\zeta) \le \min\{\sqrt{3}^{\delta_1}2^{\delta_2}, 2^{\delta_1}\sqrt{3}^{\delta_2}\} = \sqrt{3}(2\sqrt{3}^{-1})^{\min\{\delta_1,1-\delta_1\}} \le \sqrt{2\sqrt{3}}. \ ■$$

The next lemma will take care of some special cases of Theorem 4.

LEMMA 8. *Let* $\alpha = n^\phi \ge 2$ *with* $n \in \mathbb{N}$, ϕ *a positive rational, and let* $x, y \in \overline{\mathbb{Q}}^*$ *with* $x+y=\alpha$.

(i) *If* $r, s \in \mathbb{Z}$ *are not both zero with* $rs \ge 0$ *and* $x^r y^s = 1$, *then*

$$\max\{H(x), H(y)\} \le \frac{3}{2}H(\alpha).$$

(ii) *If* $y = \zeta x$ *for some root of unity* ζ, *then*

$$\max\{H(x), H(y)\} \le \sqrt{2\sqrt{3}}\,H(\alpha).$$

Proof. We can and will assume $H(x) \ge H(y)$. Fix a number field K with $x, y \in K$. For part (i) assume $r \ge 0$, and if $r = 0$ assume $s > 0$. Apply Lemmas 1 and 2 with $f = T^r(T-\alpha)^s$ and $\tau = x$ to conclude

$$\prod_{v\in M_K} \frac{1}{\max\{1,|x|_v\}^{d_v d}} \ge \prod_{v\mid\infty} \frac{1}{(1+|\alpha|_v)^{d_v d}} \prod_{v\nmid\infty} \frac{1}{\max\{1,|\alpha|_v\}^{d_v d}}$$

with $d = r+s$. For a finite prime v we have $|\alpha|_v \le 1$ and therefore $H(x) \le 1+\alpha \le \frac{3}{2}H(\alpha)$. Now for part (ii): We have $x(1+\zeta) = \alpha$ and by hypothesis $\zeta \ne -1$; elementary height properties lead to $H(x) = H((1+\zeta)^{-1}\alpha) \le H((1+\zeta)^{-1})H(\alpha) = H(1+\zeta)H(\alpha) \le \sqrt{2\sqrt{3}}\,H(\alpha)$ by Lemma 7 if $\zeta \ne 1$. So now assume $\zeta = 1$; then $x = y = \alpha/2$ and so

$$\begin{aligned} H(x)^{[K:\mathbb{Q}]} &= \prod_{v\in M_K}\max\{1,|\alpha/2|_v^{d_v}\} = \prod_{v\mid\infty}|\alpha/2|_v^{d_v}\prod_{v\nmid\infty}\max\{1,|\alpha/2|_v^{d_v}\} \\ &\le (\alpha/2)^{[K:\mathbb{Q}]}\prod_{v\nmid\infty}|2|_v^{-d_v}, \end{aligned}$$

which implies $H(x) \le H(\alpha)$. ■

LEMMA 9. *Let* $\alpha = n^\phi \ge 2$ *with* $n \in \mathbb{N}$, ϕ *a positive rational, and let* $x, y \in \overline{\mathbb{Q}}^*$ *with* $x+y=\alpha$ *and* $x^r = y^t$ *where* $0 < r < t$ *are rational integers. Define* $\lambda = t/r$ *and let* K *be a number field containing* x, y. *Then* x *is an algebraic integer and furthermore*:

(i) *If* v *is a finite prime of* K *with* $|x|_v < 1$, *then* $|x|_v = |y|_v^\lambda = |\alpha|_v^\lambda$.
(ii) *One has*

(29) $$H(x)=\prod_{\substack{v\nmid\infty\\|x|_v<1}}\max\{1,|\alpha|_v^{-1}\}^{d_v\lambda/[K:\mathbb{Q}]}\le\alpha^{\lambda}.$$

(iii) *Let $\varepsilon\in[0,1)$ and $\delta\in(1,2]$ be such that*

(30) $$\lambda\ge 1+\frac{\log 2}{\log\alpha}(1+\varepsilon)\quad\text{and}\quad(\delta-1)(1-\delta^{-1})^{\frac{\log\alpha}{(1+\varepsilon)\log 2}}\alpha\ge 1.$$

Then

(31) $$|x|_v\le\delta|\alpha|_v\quad\text{for all } v\mid\infty\quad\text{and}\quad H(x)\le\delta H(\alpha).$$

Proof. Note that x is an algebraic integer. Indeed, α is an algebraic integer and x is a zero of the monic polynomial $(-1)^t(\alpha-T)^t-(-1)^tT^r\in\mathcal{O}_K[T]$.

Let v be as in part (i); then $x^r=y^t$ implies $|y|_v>|x|_v$ and so because $x+y=\alpha$ we have $|\alpha|_v=|y|_v=|x|_v^{r/t}<1$.

For part (ii) recall the definition (14) of $m=m_f$ with f defined as in Proposition 1 with $s=-t$. For any finite prime $v\in M_K$ one has

(32) $$m(x)=\max\{|x|_v^r,|x-\alpha|_v^t\}=|x|_v^r\le 1.$$

If on the other hand v is an infinite prime one has $|\alpha|_v=|n|_v^{\phi}=\alpha\ge 2$ so clearly $|x|_v\ge 1$ and therefore

(33) $$m(x)=|x|_v^{r-t}.$$

Using (32), (33) and applying the product formula, we conclude that

(34) $$\prod_{v\in M_K}m(x)^{d_v}=\prod_{v\nmid\infty}|x|_v^{d_vr}\prod_{v\mid\infty}|x|_v^{d_v(r-t)}=\prod_{v\nmid\infty}|x|_v^{d_vt}=\prod_{\substack{v\nmid\infty\\|x|_v<1}}|x|_v^{d_vt}.$$

Then by applying part (i) to (34) and using (13) with $f(x)=\pm1$ we get

$$\frac{1}{H(x)^{[K:\mathbb{Q}]t}}=\prod_{v\in M_K}m(x)^{d_v}=\prod_{\substack{v\nmid\infty\\|x|_v<1}}|\alpha|_v^{d_v\lambda t}=\prod_{\substack{v\nmid\infty\\|x|_v<1}}\max\{1,|\alpha|_v^{-1}\}^{-d_v\lambda t}$$
$$\ge H(\alpha^{-1})^{-[K:\mathbb{Q}]\lambda t},$$

from which (ii) follows at once.

To prove part (iii) assume (30) holds. Note that the first statement of (31) implies the second because x is an algebraic integer. We will prove the first inequality of (31) by contradiction. Assume $|x|_v>\delta|\alpha|_v$ for some $v\mid\infty$. Then $|x-\alpha|_v>|x|_v(1-\delta^{-1})$, and so $|x|_v^r=|x-\alpha|_v^t>(|x|_v(1-\delta^{-1}))^t$. The first inequality in (30) implies

$$\delta\alpha<|x|_v<(1-\delta^{-1})^{-\lambda/(\lambda-1)}\le(1-\delta^{-1})^{-(1+\frac{\log\alpha}{(1+\varepsilon)\log 2})},$$

which contradicts the second inequality in (30). ∎

We can now prove Theorem 4. Let for example $x^r y^s = 1$ with $r, s \in \mathbb{Z}$ not both zero and assume $r \geq 0$ and $H(x) \geq H(y)$. Fix a number field K with $x, y \in K$. We begin by proving the inequality (6). If $rs \geq 0$ or $r = -s$ or y is a root of unity, then Lemma 8 implies the assertion. So assume $rs < 0$ and $-s \neq r$ and y is not a root of unity. Then $H(y)^{-s} = H(x)^r \geq H(y)^r > 1$ by Kronecker's Theorem, and therefore $-s > r$.

For brevity we set $t = -s$ and $\lambda = t/r$. If $\lambda \leq 1 + \log 2/\log \alpha$ then (29) in Lemma 9 gives

$$H(x) \leq \alpha^\lambda \leq \alpha^{1+\log 2/\log \alpha} = 2H(\alpha). \tag{35}$$

On the other hand, if $\lambda > 1+\log 2/\log \alpha$, there exists an $\varepsilon > 0$ such that the first inequality in (30) holds. Now the second inequality in (30) holds strictly when $\delta = 2$, and so it must continue to hold for some $\delta < 2$. Hence the first inequality of (31) holds with some $\delta < 2$ and therefore $H(x) < 2H(\alpha)$.

Finally, we prove that $\max\{H(x), H(y)\} = 2H(\alpha)$ if and only if α is a rational power of 2 and $x = 2\alpha$ or $y = 2\alpha$. The "if" part is trivial. For the "only if" part assume $H(x) = 2H(\alpha) \geq H(y)$. As above we use Lemma 8 to reduce to the case $rs < 0$ and $t = -s > r$. We have already showed that $\lambda > 1 + \log 2/\log \alpha$ implies $H(x) < 2H(\alpha)$. But if $\lambda < 1 + \log 2/\log \alpha$ then (35) also implies $H(x) < 2H(\alpha)$. So we must have

$$\lambda = 1 + \log 2/\log \alpha \tag{36}$$

and thus α is a rational power of 2. By (36) the choice $\varepsilon = 0$, $\delta = 2$ satisfies the hypothesis of Lemma 9(iii). We conclude that $|x|_v \leq 2|\alpha|_v$ for infinite primes v. As α, x are algebraic integers we even have $|x|_v = |2\alpha|_v$ for all infinite primes v. Note that (36) implies

$$\alpha^\lambda = 2\alpha. \tag{37}$$

If v is a finite prime with $|2|_v < 1$ then $|x|_v < 1$; indeed, we must have equality in (29). So Lemma 9(i) gives $|x|_v = |\alpha|_v^\lambda$, and by (37) we conclude that $|x|_v = |2\alpha|_v$. But this last equality holds for any finite prime: indeed, if $|2|_v = 1$ then $|x|_v = |\alpha|_v = 1$ by Lemma 9(i). Hence $|x|_v = |2\alpha|_v$ for all primes, finite or infinite; therefore $x = 2\alpha\xi$ for a root of unity ξ. Let v be an infinite prime. Then the equality $|x|_v^r = |x-\alpha|_v^t$ and (37) imply $|2\xi-1|_v = 1$. For any $z, w \in K$ we have the equality $|z + w|_v^2 + |z - w|_v^2 = 2|z|_v^2 + 2|w|_v^2$; take $z = \xi - 1$ and $w = \xi$ to conclude that $\xi = 1$. Thus $x = 2\alpha$. ■

7. Proof of Theorem 5.

We start with an elementary estimate.

LEMMA 10. *Let* $\phi \in \mathbb{N}$ *and* $1 + \frac{2}{3}\phi^{-1} \leq \lambda \leq 1 + \frac{4}{3}\phi^{-1}$. *Then*

$$\frac{1}{2}\max\{1, \lambda(1+\phi^{-1})^{-1}\}\log(2^{2\phi+1} + 2\max\{1, (\lambda/2)^{1/(1-\lambda)}\}) \leq \log(1.98 \cdot 2^\phi). \tag{38}$$

Proof. Let $g(\lambda)$ be the left-hand side of (38). The map $\lambda \mapsto (\lambda/2)^{1/(1-\lambda)}$ decreases for $1 < \lambda < 4$. If $\phi = 1$ the lemma follows easily by considering the cases $\lambda \le 2$ and $\lambda > 2$. We will therefore assume $\phi \ge 2$. Since $(1 + 1/w)^w$ increases for $w \ge 1$ we conclude that

$$(\lambda/2)^{1/(1-\lambda)} \le \frac{2^{3\phi/2}}{\left(1+\frac{2}{3\phi}\right)^{3\phi/2}} \le \frac{3^3}{2^6}\cdot 2^{3\phi/2}.$$

If x and y are positive then $\log(x+y) \le x/y + \log y$, thus

$$\begin{aligned} g(\lambda) &\le \frac{1}{2}\max\left\{1, \frac{\lambda}{1+\phi^{-1}}\right\}\log\left(\frac{3^3}{2^6}\cdot 2^{3\phi/2+1} + 2^{2\phi+1}\right) \\ &\le \max\left\{1, \frac{\lambda}{1+\phi^{-1}}\right\}\left(\frac{3^3}{2^8} + \frac{1}{2}\log 2^{2\phi+1}\right). \end{aligned}$$

If $\lambda \le 1 + \phi^{-1}$, then

$$g(\lambda) \le \frac{3^3}{2^8} + \frac{1}{2}\log 2 + \log 2^{\phi}$$

and we are done. On the other hand, if $\lambda > 1 + \phi^{-1}$, then

$$\begin{aligned} g(\lambda) &\le \frac{\phi + \frac{4}{3}}{\phi+1}\left(\frac{3^3}{2^8} + \frac{1}{2}\log 2^{2\phi+1}\right) = \frac{\phi\left(\frac{5}{6}\log 2 + \frac{3^3}{2^8}\right) + \frac{3^2}{2^6} + \frac{2}{3}\log 2}{\phi+1} + \log 2^{\phi} \\ &< \frac{5}{6}\log 2 + \frac{3^3}{2^8} + \log 2^{\phi}. \ \blacksquare \end{aligned}$$

We proceed as follows: let x and $y = \alpha - x$ be multiplicatively dependent with $H(y) \le H(x) < 2H(\alpha)$, and let $P \in \mathbb{Z}[T]$ be the minimal polynomial of x. Then we will show $P(2\alpha) \ne 0$. With the help of the finite primes lying above 2 we will even find a lower bound for $|P(2\alpha)|$ in terms of $H(x)$. More precisely:

LEMMA 11. *Let $\alpha = 2^{\phi}$ for $\phi \in \mathbb{N}$ and let $x, y \in \overline{\mathbb{Q}}^*$ with $x + y = \alpha$ and $x^r = y^t$ where $0 < r < t$ are rational integers. Define $\lambda = t/r$ and $k(w) = 2\alpha^2 + 2w^{2/\lambda} - w^2$. If $H(x) < 2H(\alpha)$, then*

$$\log H(x) \le \frac{1}{2}\max\{1, \lambda(1+\phi^{-1})^{-1}\}\log\sup_{w\ge 1} k(w). \tag{39}$$

Proof. Fix a finite Galois extension $K/\mathbb{Q}$ with Galois group G such that $x \in K$. Let $P \in \mathbb{Z}[T]$ be the minimal polynomial of x; then because x is an algebraic integer we have

$$P^{[K:\mathbb{Q}(x)]} = \prod_{\sigma\in G}(T - \sigma x). \tag{40}$$

Let v be any finite prime of M_K extending the 2-adic absolute value, i.e. $v \mid 2$.

Then

$$|P(2\alpha)|_v^{[K:\mathbb{Q}(x)]} = \prod_{\sigma\in G} |2\alpha - \sigma x|_v \le \prod_{\sigma\in G} \max\{|2|_v^{1+\phi}, |\sigma x|_v\}.$$

Recall from Lemma 9(i) that if $v' \nmid \infty$ then $|x|_{v'} < 1$ implies $|x|_{v'} = |\alpha|_{v'}^{\lambda}$. Let g be the number of finite primes of M_K lying over 2 and g' be the number of finite primes v' with $|2|_{v'}, |x|_{v'} < 1$. Now G acts transitively on the set of all finite primes lying over 2 ([Lan94, Proposition 11, p. 12]), therefore each stabilizer has cardinality $[K:\mathbb{Q}]/g$, so

$$\begin{aligned} |P(2\alpha)|_v^{[K:\mathbb{Q}(x)]} &\le \prod_{\sigma\in G} \max\{|2|_{\sigma^{-1}v}^{1+\phi}, |x|_{\sigma^{-1}v}\} = \prod_{v'|2} \max\{|2|_{v'}^{1+\phi}, |x|_{v'}\}^{[K:\mathbb{Q}]/g} \\ &= \prod_{\substack{v'|2 \\ |x|_{v'}<1}} \max\{|2|_{v'}^{1+\phi}, |\alpha|_{v'}^{\lambda}\}^{[K:\mathbb{Q}]/g} = |2|_v^{[K:\mathbb{Q}](g'/g)\min\{1+\phi,\phi\lambda\}}. \end{aligned}$$

Recall that $\alpha \in \mathbb{Q}$, so if $P(2\alpha) = 0$ then $x = 2\alpha$. In this case $H(x) = 2H(\alpha)$, which contradicts our hypothesis. We conclude that $P(2\alpha) \ne 0$. Apply $P(2\alpha) \in \mathbb{Z}$ and the product formula to see

$$\begin{aligned} (41) \qquad |P(2\alpha)|^{[K:\mathbb{Q}]} &= \prod_{v|\infty} |P(2\alpha)|_v^{d_v} = \prod_{v\nmid\infty} |P(2\alpha)|_v^{-d_v} \ge \prod_{v|2} |P(2\alpha)|_v^{-d_v} \\ &\ge 2^{[K:\mathbb{Q}](g'/g)\min\{1+\phi,\phi\lambda\}\deg P}. \end{aligned}$$

Because $K/\mathbb{Q}$ is Galois we have $d_v g = [K:\mathbb{Q}]$ for any $v\,|\,2$. So Lemma 9(ii) yields

$$H(x) = \prod_{\substack{v\nmid\infty \\ |x|_v<1,\, |2|_v<1}} \max\{1, |\alpha|_v^{-1}\}^{d_v\lambda/[K:\mathbb{Q}]} = 2^{(g'/g)\phi\lambda}.$$

This equality inserted into (41) gives

$$(42) \qquad |P(2\alpha)|^{[K:\mathbb{Q}]} \ge H(x)^{[K:\mathbb{Q}]\min\{1,\lambda^{-1}(1+\phi^{-1})\}\deg P}.$$

We continue by bounding the left-hand side of (42) from above. Let v be an infinite prime. Recall that for $z_1, z_2 \in K$ we have $|z_1+z_2|_v^2 + |z_1-z_2|_v^2 = 2|z_1|_v^2 + 2|z_2|_v^2$. Let $\sigma \in G$, take $z_1 = \sigma\alpha$, $z_2 = \sigma\alpha - \sigma x$ in the previous equality and recall $|\sigma\alpha - \sigma x|_v = |\sigma x|_v^{1/\lambda}$ to conclude

$$(43) \quad |2\alpha - \sigma x|_v^2 = |2\sigma\alpha - \sigma x|_v^2 = 2|\sigma\alpha|_v^2 + 2|\sigma x|_v^{2/\lambda} - |\sigma x|_v^2 = k(|\sigma x|_v).$$

Note that $|\sigma x|_v \ge 1$; indeed, if $|\sigma x|_v < 1$ then $|\sigma y|_v = |\sigma x|_v^{1/\lambda} < 1$ and so $|\alpha|_v < 2$, a contradiction. Apply (43) to (40) to get

$$|P(2\alpha)|^{[K:\mathbb{Q}(x)]} = \prod_{\sigma\in G} |2\alpha - \sigma x|_v = \prod_{\sigma\in G} k(|\sigma x|_v)^{1/2} \le (\sup_{w\ge 1} k(w))^{[K:\mathbb{Q}]/2}.$$

Combine the previous upper bound with the lower bound in (42) to conclude the proof. ∎

We can now prove Theorem 5. Assume $H(y) \leq H(x) < 2H(\alpha)$ and let $r, s \in \mathbb{Z}$ not both zero with $r \geq 0$ such that $x^r y^s = 1$. With the help of Lemma 8 we reduce to the case $t = -s > r > 0$ as we did at the beginning of the proof of Theorem 4. Define $\lambda = t/r$; we split up the argument into cases.

The first two cases $\lambda < 1 + \frac{2}{3}\phi^{-1}$ and $\lambda > 1 + \frac{4}{3}\phi^{-1}$ are effectively covered by Lemma 9. Indeed, part (ii) applied to the first case gives $H(x) \leq \alpha^\lambda \leq 2^{2/3}H(\alpha)$. For the second case set $\varepsilon = 1/3$ and $\delta = 1.9$. This choice clearly satisfies the right-hand inequality of (30). Because

$$(\delta - 1)(1 - \delta^{-1})^{\phi/(1+\varepsilon)}\alpha = \frac{9}{10}\left(2\left(\frac{9}{19}\right)^{3/4}\right)^{\phi} \geq 1$$

the left-hand inequality of (30) holds as well. Thus $H(x) \leq 1.9H(\alpha)$.

Now assume $1 + \frac{2}{3}\phi^{-1} \leq \lambda \leq 1 + \frac{4}{3}\phi^{-1}$. Let k be the function defined in Lemma 11. Elementary calculus yields

$$\sup_{w\geq 1} k(w) = k(w_0) \quad \text{with} \quad w_0 = \max\{1, (\lambda/2)^{\lambda/(2(1-\lambda))}\}.$$

Apply the previous inequality to Lemma 11 and obtain the bound

$$\begin{aligned}\log H(x) &\leq \frac{1}{2}\max\{1, \lambda(1+\phi^{-1})^{-1}\}\log k(w_0)\\ &\leq \frac{1}{2}\max\{1, \lambda(1+\phi^{-1})^{-1}\}\log(2\alpha^2 + 2\max\{1, (\lambda/2)^{1/(1-\lambda)}\}).\end{aligned}$$

Now Lemma 10 implies $H(x) \leq 1.98H(\alpha)$. ∎

8. Proof of Theorem 6. The next lemma proves the first inequality in Theorem 6.

LEMMA 12. *Let K be a number field with* $\operatorname{rank} \mathcal{O}_K^* = 1$ *and $\alpha \in \mathbb{Q}^*$; then $\# S_K(\alpha) \leq 292$.*

Proof. If $\alpha \notin \mathbb{Z}$ then $S_K(\alpha)$ is empty so assume $\alpha \in \mathbb{Z} \setminus \{0\}$. Let ω be the number of roots of unity in K, η a fundamental unit, R the regulator, and D the degree of K.

If $(x, y) \in S_K(\alpha)$ there exist $r, s \in \mathbb{Z}$ not both zero such that $x^r y^s = 1$; we shall furthermore assume that $r \geq 0$ and $H(x) \geq H(y)$ (so we will have to multiply the number of solutions under this hypothesis by 2 to get a bound for the total number of solutions). If $H(y) = 1$ then y is a root of unity, hence in this case we can choose $r = 0$ and $s > 0$. So in all cases one may assume $|s| \geq r$.

First assume $\alpha \neq \pm 1$. Let Δ denote the set of all infinite primes v for which $|x|_v \geq 1$ and let $\delta \in [0, 1]$ with $\delta D = \sum_{v\in\Delta} d_v$. Note that in the case

$s < 0$ one has $\delta = 1$ because $|\alpha| \geq 2$. Since x is a unit the height is given by

$$H(x) = \prod_{v \in \Delta} |x|_v^{d_v/D}. \tag{44}$$

Let $v \in \Delta$; if $|x|_v < |\alpha|_v/2$, then $|x|_v^{-r/s} = |\alpha - x|_v \geq |\alpha|_v - |x|_v > |x|_v$, which leads to $r/s < -1$ so $|r/s| > 1$, contradicting $|r| \leq |s|$. We conclude that $|x|_v \geq |\alpha|_v/2 = \max\{1, |\alpha|_v\}/2$ for all $v \in \Delta$. So by (44),

$$H(x) \geq 2^{-\delta} H(\alpha)^{\delta}. \tag{45}$$

We now deduce a corresponding upper bound for $H(x)$. If $s \geq 0$, then Lemma 2 with $f = T^r(T - \alpha)^s$ and $\tau = x$ applied to (44) leads to

$$H(x) \leq 2^{\delta} H(\alpha)^{\delta}. \tag{46}$$

If on the other hand $s < 0$, then (46) also holds by Theorem 4 because $\delta = 1$.

With the bounds (45) and (46) we can apply a gap principle. There exists a unique $a \in \mathbb{Z}$ and a root of unity ζ such that $x = \zeta\eta^a$. Apply height functorial properties and the bounds to see that $|a|$ lies in an interval of length $\delta \log 4/\log H(\eta)$. Hence there are at most $2(\log 4/\log H(\eta) + 1)$ possibilities for a. Clearly this estimate remains valid for $\alpha = \pm 1$ because in this case Theorem 1 implies $0 \leq \log H(x) \leq \log 2$. We also note that $R = D \log H(\eta)$ and therefore

$$\# S_K(\alpha) \leq 4\omega\left(\frac{D \log 4}{R} + 1\right). \tag{47}$$

Elementary considerations lead to $D \leq 4$ and $\omega \leq 12$. Now a result of Friedman ([Fri89, Theorem B]) which states $R/\omega \geq 0.09058$ completes the proof. ■

To prove Theorem 6 let $(x, y) \in S_{\mathcal{F}}(\alpha)$. The proof splits up into two cases:

(i) There exist $n \in \mathbb{Z}$ and a root of unity $\zeta \in \mathcal{F}$ such that $x = \zeta y^n$ with $-2 \leq n \leq 2$ or $y = \zeta x^n$ with $n = 0, \pm 2$.

(ii) Otherwise.

First assume case (i). Elementary arguments show that there are 24 roots of unity ζ in $\mathcal{F}$. For each such ζ, substituting $y = \alpha - x$ in (i) gives eight polynomial equations in x of degree at most 3; thus the number of x is at most $24 \cdot 8 \cdot 3 = 576$.

Now assume that case (i) does not hold. Set $K = \mathbb{Q}(x) = \mathbb{Q}(x, y)$ and $D = [K : \mathbb{Q}]$. Because x and y are not roots of unity we have $\operatorname{rank} \mathcal{O}_K^* = 1$. Let η be a fundamental unit of K. We claim that $H(\eta)^D \leq 4$ will complete the proof. Indeed, assuming this inequality, a well known argument bounds the number of units with degree d and height at most $4^{1/d}$ by $2d \prod_{k=1}^{d-1} \left(2 \cdot \binom{d}{k} \cdot 4 + 1\right)$. Take the sum over this expression for $2 \leq d \leq 4$;

thus there are at most 430706 possibilities for η. There are six roots of unity in $\mathcal{F}$ such that one of these generates the group of roots of unity in K. Let ζ be such a root of unity; then $K = \mathbb{Q}(\eta, \zeta)$. Now the assertion follows from Lemma 12 applied to the field K.

We will now show $H(\eta)^D \le 4$. There are $a, b \in \mathbb{Z}$ and roots of unity ζ, ξ such that $x = \zeta\eta^a$ and $y = \xi\eta^b$. Let σ_1, σ_2 be two distinct non-conjugate embeddings of K into $\mathbb{C}$. These correspond to the two infinite primes in M_K. Define $d_i = 1$ if $\sigma_i(K) \subset \mathbb{R}$ and $d_i = 2$ otherwise. We may assume $d_1 \ge d_2$. Now $|\sigma_i(\eta)| \ne 1$ for $i = 1, 2$ by Kronecker's Theorem, so by replacing η with η^{-1} if necessary we may assume $|\sigma_1(\eta)| > 1$. Let l_i be a logarithm of $\sigma_i(\eta)$. Note that $D \log H(\eta) = d_1 \log |\sigma_1(\eta)| = d_1 \operatorname{Re}(l_1)$, hence it suffices to show $\operatorname{Re}(l_1) \le \log 2$. The equality $|\sigma_1(\eta)|^{d_1} |\sigma_2(\eta)|^{d_2} = 1$ implies

$$d_1 \operatorname{Re}(l_1) + d_2 \operatorname{Re}(l_2) = 0. \tag{48}$$

Because $\alpha \in \mathbb{Q}$ one has $\sigma_1(x) - \sigma_2(x) = \sigma_2(y) - \sigma_1(y)$. Apply (48) to get

$$\begin{aligned} &|e^{q(a)|a| \operatorname{Re}(l_1) + \gamma_1 i} - e^{-q(-a)|a| \operatorname{Re}(l_1) + \gamma_2 i}| \\ &\qquad = |e^{q(b)|b| \operatorname{Re}(l_1) + \gamma_3 i} - e^{-q(-b)|b| \operatorname{Re}(l_1) + \gamma_4 i}| \end{aligned} \tag{49}$$

where the γ_i are real numbers and

$$q(w) = \begin{cases} 1 & \text{if } w \ge 0, \\ d_1/d_2 & \text{if } w < 0, \end{cases}$$

is an integer. Now define $k = q(b)|b| - q(a)|a| \in \mathbb{Z}$; then $k \ne 0$ because otherwise we would be in case (i). So first assume $k \ge 1$. Apply the triangle inequality to (49) and use $\operatorname{Re}(l_1) \ge 0$ to conclude that

$$e^{q(a)|a| \operatorname{Re}(l_1)} + 1 \ge e^{(q(a)|a| + k) \operatorname{Re}(l_1)} - 1. \tag{50}$$

Note that we have $|a| \ge 1$, or else we would be back in case (i). Now (50) and $q(a) \ge 1$ imply

$$e^{\operatorname{Re}(l_1)} - 2e^{-\operatorname{Re}(l_1)} - 1 \le 0. \tag{51}$$

The left-hand side of (51) increases in $\operatorname{Re}(l_1)$. Substitute $\log 2$ for $\operatorname{Re}(l_1)$ to get the bound $\operatorname{Re}(l_1) \le \log 2$. Now assume $k \le -1$; then similar arguments to those above involving the triangle inequality and this time $|b| \ge 1$ also lead to (51) and thus again $\operatorname{Re}(l_1) \le \log 2$. ∎

References

[BS96] F. Beukers and H. P. Schlickewei, *The equation $x + y = 1$ in finitely generated groups*, Acta Arith. 78 (1996), 189–199.

[BMZ99] E. Bombieri, D. Masser and U. Zannier, *Intersecting a curve with algebraic subgroups of multiplicative groups*, Int. Math. Res. Not. 1999, no. 20, 1119–1140.

[Cas57] J. W. S. Cassels, *An Introduction to Diophantine Approximation*, Cambridge Univ. Press, 1957.

[CZ00] P. B. Cohen and U. Zannier, *Multiplicative dependence and bounded height, an example*, in: Algebraic Number Theory and Diophantine Approximation (Graz, 1998), de Gruyter, 2000, 93–101.

[Eve84] J. H. Evertse, *On equations in S-units and the Thue–Mahler equation*, Invent. Math. 75 (1984), 561–584.

[Fri89] E. Friedman, *Analytic formulas for the regulator of a number field*, ibid. 98 (1989), 599–622.

[HS00] M. Hindry and J. H. Silverman, *Diophantine Geometry. An Introduction*, Springer, 2000.

[Lan83] S. Lang, *Fundamentals of Diophantine Geometry*, Springer, 1983.

[Lan94] —, *Algebraic Number Theory*, Springer, 1994.

Mathematisches Institut
Universität Basel
Rheinsprung 21
CH-4051 Basel, Switzerland
E-mail: habegger@math-lab.unibas.ch

Received on 8.2.2005
and in revised form on 16.6.2005 (4931)