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An asymptotic formula related to the divisors of the quaternary quadratic form

by

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1. Introduction. The quadratic forms

$$\mathcal{G}(m_1, m_2) := m_1^2 + m_2^2,$$
$$\mathcal{G}(m_1, m_2, m_3) := m_1^2 + m_2^2 + m_3^2,$$
$$\mathcal{G}(m_1, m_2, m_3, m_4) := m_1^2 + m_2^2 + m_3^2 + m_4^2, \quad \ldots$$

are important in number theory. They have been studied by using different methods.

Let $d(n)$, $\Lambda(n)$ and $\mu(n)$ stand for the Dirichlet divisor function, the von Mangoldt function and the Möbius function respectively. In 2000, Gang Yu [Y] studied the binary quadratic form above and obtained

$$\sum_{1\le m_1,m_2\le x} d(m_1^2+m_2^2) = A_1x^2\log x + A_2x^2 + O(x^{3/2+\epsilon}). \tag{1.1}$$

Also in 2000, C. Calderón and M. J. de Velasco [CV] studied the divisors of the quadratic form $m_1^2+m_2^2+m_3^2$ and proved the asymptotic formula

$$\sum_{1\le m_1,m_2,m_3\le x} d(m_1^2+m_2^2+m_3^2) = \frac{8\zeta(3)}{5\zeta(4)}x^3\log x + O(x^3). \tag{1.2}$$

The error term in (1.2) was improved to $O(x^{8/3})$ by Ruting Guo and Wengguang Zhai [GZ] with the help of the circle method.

In 2009, Friedlander and Iwaniec [FI] studied the number of prime vectors among integer lattice points in the 3-dimensional ball. They proved that the number $\pi_3(x)$ of points $(m_1, m_2, m_3) \in \mathbb{Z}^3$ with

$$m_1^2+m_2^2+m_3^2 = p \le x \tag{1.3}$$

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satisfies

$$\pi_3(x) \sim \frac{4\pi}{3}\frac{x^{3/2}}{\log x}, \tag{1.4}$$

which can be viewed as a generalization of the prime number theorem. The asymptotic formula (1.4) is proved by using Gauss's formula for the function $r_3(p)$ and the properties of $L(1,\chi_p)$, where $r_3(p)$ denotes the number of ways p can be written as a sum of three squares, and $L(1,\chi_p)$ is the Dirichlet L-function with the Kronecker symbol $\chi_p(n) = \left(\frac{-4p}{n}\right)$.

In this paper, we study the quaternary quadratic form

$$\mathcal{G}(m_1,m_2,m_3,m_4) := m_1^2+m_2^2+m_3^2+m_4^2$$

and give some estimates by generalizing Guo–Zhai's method. Our main results are as follows.

THEOREM 1.1. *Define*

$$S(x) := \sum_{1\le m_1,m_2,m_3,m_4\le x} d(m_1^2+m_2^2+m_3^2+m_4^2).$$

Then for $x \ge 2$, we have

$$S(x) = 2K_1L_1x^4\log x + (K_1L_2+K_2L_1)x^4 + O(x^{7/2+\epsilon}), \tag{1.5}$$

where

$$K_1 := \sum_{q=1}^{\infty} q^{-5} \sum_{\substack{0\le a<q\\(a,q)=1}} G^4(a,0,q), \quad G(a,0,q) = \sum_{r=1}^{q} e(ar^2/q),$$

$$K_2 := \sum_{q=1}^{\infty} \frac{-2\log q + 2\gamma}{q^5} \sum_{\substack{0\le a<q\\(a,q)=1}} G^4(a,0,q),$$

$$L_1 := \int_{-\infty}^{\infty} \mathcal{I}_1(\lambda)\,d\lambda, \quad L_2 := \int_{-\infty}^{\infty} \mathcal{I}_2(\lambda)\,d\lambda,$$

$$\mathcal{I}_1(\lambda) := \Big(\int_0^1 e(u^2\lambda)\,du\Big)^4 \int_0^4 e(-u\lambda)\,du,$$

$$\mathcal{I}_2(\lambda) := \Big(\int_0^1 e(u^2\lambda)\,du\Big)^4 \int_0^4 e(-u\lambda)\log u\,du.$$

THEOREM 1.2. *Define*

$$\pi_\Lambda(x) := \sum_{m_1^2+m_2^2+m_3^2+m_4^2\le x} \Lambda(m_1^2+m_2^2+m_3^2+m_4^2).$$

Then for any fixed constant $A > 0$, *we have*

$$\pi_A(x) = 16K_3L_3x^2 + O(x^2 \log^{-A} x) \quad (x \geq 2), \tag{1.6}$$

where

$$K_3 := \sum_{q=1}^{\infty} \frac{1}{q^4\varphi(q)} \sum_{\substack{0\leq a<q \\ (a,q)=1}} G^4(a,0,q)C_q(-a),$$

$$L_3 := \int_{-\infty}^{\infty} \mathcal{I}_3(\lambda)\,d\lambda,$$

$$\mathcal{I}_3(\lambda) := \Big(\int_0^1 e(u^2\lambda)\,du\Big)^4 \int_0^1 e(-u\lambda)\,du.$$

Notation. As usual, the letter ϵ denotes a positive constant which can be arbitrarily small. $C_q(r)$ denotes the Ramanujan sum. Finally, $G(a,b,q)$ denotes the quadratic Gauss sum

$$G(a,b,q) = \sum_{r=1}^{q} e\Big(\frac{ar^2+br}{q}\Big), \quad \text{where} \quad e(t) := e^{2\pi it}.$$

2. Outline of the circle method. In this paper, x is a large positive integer. In order to apply the circle method, we assume

$$\log x < P < x, \quad 2P^2 < Q, \quad Q > x^{1+\epsilon}, \quad PQ < x^2. \tag{2.1}$$

By Dirichlet's lemma on rational approximation, each $\alpha \in [-1/Q, 1-1/Q]$ may be written in the form

$$\alpha = a/q + \lambda, \quad |\lambda| \leq 1/qQ, \tag{2.2}$$

for some integers a, q with $1 \leq a \leq q \leq Q$ and $(a,q) = 1$. We denote by $\mathcal{M}(a,q)$ the set of α satisfying (2.2), and define the *major arcs* $\mathcal{M}$ and *minor arcs* $C(\mathcal{M})$ as follows:

$$\mathcal{M} = \bigcup_{q\leq P} \bigcup_{\substack{0<a<q \\ (a,q)=1}} \mathcal{M}(a,q), \qquad C(\mathcal{M}) = [-1/Q, 1-1/Q] \setminus \mathcal{M}. \tag{2.3}$$

Let

$$S_1(\alpha; y) := \sum_{1\leq m\leq y} e(m^2\alpha), \qquad S_2(\alpha; y) := \sum_{1\leq n\leq y} d(n)e(n\alpha). \tag{2.4}$$

By (2.4) and the well-known identity

$$\int_0^1 e(u\alpha)\,d\alpha = \begin{cases} 1 & \text{if } u = 0, \\ 0 & \text{if } u \in \mathbb{Z}, u \neq 0, \end{cases} \tag{2.5}$$

we have

$$S(x) := \sum_{1\le m_1,m_2,m_3,m_4\le x} d(m_1^2+m_2^2+m_3^2+m_4^2) = \int_0^1 S_1^4(\alpha;x)S_2(-\alpha;4x^2)\,d\alpha = S_1(x)+S_2(x), \tag{2.6}$$

where

$$S_1(x) := \int_{\mathcal{M}} S_1^4(\alpha;x)S_2(-\alpha;4x^2)\,d\alpha,$$
$$S_2(x) := \int_{C(\mathcal{M})} S_1^4(\alpha;x)S_2(-\alpha;4x^2)\,d\alpha.$$

The problem is now reduced to evaluating $S_1(x)$ and giving an upper bound of $S_2(x)$.

3. Some lemmas. We need some classical results. Lemma 3.1 can be found in [H] and Lemmas 3.2 and 3.3 in [PP].

LEMMA 3.1. *Suppose $q\in\mathbb{N}$, $a,b\in\mathbb{Z}$, $q\ge 3$ and $(a,q)=1$. Then*

$$G(a,b,q)\ll\sqrt{q}.$$

LEMMA 3.2. *Suppose $f(\cdot)$ is a real-valued continuously differentiable function on $[t_1,t_2]$ such that $|f'(t)|\gg\Delta>0$ for all $t\in[t_1,t_2]$. Then*

$$\int_{t_1}^{t_2} e(f(t))\,dt\ll 1/\Delta.$$

LEMMA 3.3. *Suppose $f(\cdot)$ is a real-valued twice continuously differentiable function on $[t_1,t_2]$ such that $|f''(t)|\gg\Delta>0$ for all $t\in[t_1,t_2]$. Then*

$$\int_{t_1}^{t_2} e(f(t))\,dt\ll 1/\sqrt{\Delta}.$$

4. Estimating $S_1(\alpha;x)$. The estimation of $S_1(\alpha;x)$ is similar to that in [GZ, Lemmas 4.1 and 5.1], and leads to:

LEMMA 4.1. *Suppose $\alpha=a/q+\lambda\in\mathcal{M}$ with $0\le a<q\le P$, $(a,q)=1,|\lambda|\le 1/PQ$ and $PQ\le x^2$, $Q>x^{1+\epsilon}$. Then*

$$S_1(\alpha;x)=\frac{G(a,0,q)}{q}x\int_0^1 e(u^2x^2\lambda)\,du+O(\sqrt{q}\log(q+1)). \tag{4.1}$$

LEMMA 4.2. *Suppose* $\alpha = a/q + \lambda \in \mathcal{C}(M)$ *with* $1 \le a \le q$, $(a,q) = 1$, $|\lambda| \le 1/qQ$ *and* $P < q \le Q$. *Then*

$$S_1(\alpha; x) \ll xP^{-1/2} + Q^{1/2}\log^{1/2} x. \tag{4.2}$$

5. Estimating $S_2(-\alpha; 4x^2)$ on the major arcs. Suppose $\alpha = a/q + z \in \mathcal{M}$ with $0 \le a < q \le P$, $(a,q) = 1$ and $|z| \le 1/qQ$. Using some results of [GZ, Section 7], we have

$$\sum_{1\le n\le 4x^2} d(n)e(-n\alpha) = J_1 + J_2,$$

where

$$\begin{aligned} J_1 &= \frac{2x^2\log x}{q}\int_0^4 e(-ux^2\lambda)\,du + \frac{x^2}{q}\int_0^4 e(-ux^2\lambda)\log u\,du \\ &\quad + \frac{-2\log q + 2\gamma}{q}x^2\int_0^4 e(-ux^2\lambda)\,du, \\ J_2 &\ll x^\epsilon(q^{1/2}x^2Q^{-1} + q^{2/3}x^{2/3}). \end{aligned}$$

Thus we get the following lemma.

LEMMA 5.1. *Suppose* $\alpha = a/q + \lambda \in \mathcal{M}$ *with* $PQ \le x^2$ *and* $Q > x^{1+\epsilon}$. *Then*

$$\begin{aligned} S_2(-\alpha; 4x^2) &= \frac{2x^2\log x}{q}\int_0^4 e(-ux^2\lambda)\,du + \frac{x^2}{q}\int_0^4 e(-ux^2\lambda)\log u\,du \\ &\quad + \frac{-2\log q + 2\gamma}{q}x^2\int_0^4 e(-ux^2\lambda)\,du + O(q^{1/2}x^{2+\epsilon}Q^{-1} + q^{2/3}x^{2/3+\varepsilon}). \end{aligned}$$

6. Proof of Theorem 1.1. We first treat the integral on the major arcs. We have

$$\begin{aligned} &\int_{\mathcal{M}} S_1^4(\alpha; x)S_2(-\alpha; 4x^2)\,d\alpha \\ &\qquad = \sum_{1\le q\le P}\ \sum_{\substack{0\le a<q\\(a,q)=1}}\ \int_{a/q-1/qQ}^{a/q+1/qQ} S_1^4(\alpha; x)S_2(-\alpha; 4x^2)\,d\alpha. \end{aligned} \tag{6.1}$$

Suppose $\alpha = a/q + \lambda \in \mathcal{M}$. From Lemmas 4.1 and 5.1 we get

$$
\begin{aligned}
(6.2)\qquad & S_1^4(\alpha;x)S_2(-\alpha;4x^2)\\
&= 2x^6\log x\frac{G^4(a,0,q)}{q^5}\Big(\int_0^1 e(u^2x^2\lambda)\,du\Big)^4\int_0^4 e(-ux^2\lambda)\,du\\
&\quad + x^6\frac{G^4(a,0,q)}{q^5}\Big(\int_0^1 e(u^2x^2\lambda)\,du\Big)^4\int_0^4 e(-ux^2\lambda)\log u\,du\\
&\quad + x^6\frac{G^4(a,0,q)(-2\log q+2\gamma)}{q^5}\Big(\int_0^1 e(u^2x^2\lambda)\,du\Big)^4\int_0^4 e(-ux^2\lambda)\,du\\
&\quad + O(x^{6+\epsilon}q^{-3/2}Q^{-1}+x^{14/3+\epsilon}q^{-4/3}+x^{5+\epsilon}q^{-2}).
\end{aligned}
$$

Thus

$$
\begin{aligned}
(6.3)\qquad & \int_{a/q-1/qQ}^{a/q+1/qQ} S_1^4(\alpha;x)S_2(-\alpha;4x^2)\,d\alpha\\
&= 2x^6\log x\frac{G^4(a,0,q)}{q^5}\int_{-1/qQ}^{1/qQ}\Big(\int_0^1 e(u^2x^2\lambda)\,du\Big)^4\int_0^4 e(-ux^2\lambda)\,du\,d\lambda\\
&\quad + x^6\frac{G^4(a,0,q)}{q^5}\int_{-1/qQ}^{1/qQ}\Big(\int_0^1 e(u^2x^2\lambda)\,du\Big)^4\int_0^4 e(-ux^2\lambda)\log u\,du\,d\lambda\\
&\quad + x^6\frac{G^4(a,0,q)(-2\log q+2\gamma)}{q^5}\\
&\qquad\qquad \times\int_{-1/qQ}^{1/qQ}\Big(\int_0^1 e(u^2x^2\lambda)\,du\Big)^4\int_0^4 e(-ux^2\lambda)\,du\,d\lambda\\
&\quad + O(x^{6+\epsilon}q^{-5/2}Q^{-2}+x^{14/3+\epsilon}q^{-7/3}Q^{-1}+x^{5+\epsilon}q^{-3}Q^{-1})\\
&= 2x^4\log x\frac{G^4(a,0,q)}{q^5}\int_{-x^2/qQ}^{x^2/qQ}\Big(\int_0^1 e(u^2\lambda)\,du\Big)^4\int_0^4 e(-u\lambda)\,du\,d\lambda\\
&\quad + x^4\frac{G^4(a,0,q)}{q^5}\int_{-x^2/qQ}^{x^2/qQ}\Big(\int_0^1 e(u^2x^2\lambda)\,du\Big)^4\int_0^4 e(-ux^2\lambda)\log u\,du\,d\lambda\\
&\quad + x^4\frac{G^4(a,0,q)(-2\log q+2\gamma)}{q^5}\\
&\qquad\qquad \times\int_{-x^2/qQ}^{x^2/qQ}\Big(\int_0^1 e(u^2x^2\lambda)\,du\Big)^4\int_0^4 e(-ux^2\lambda)\,du\,d\lambda\\
&\quad + O(x^{6+\epsilon}q^{-5/2}Q^{-2}+x^{14/3+\epsilon}q^{-7/3}Q^{-1}+x^{5+\epsilon}q^{-3}Q^{-1})
\end{aligned}
$$

$$= 2x^4 \log x \frac{G^4(a,0,q)}{q^5} \int_{-x^2/qQ}^{x^2/qQ} \mathcal{H}_1(z)\,dz + x^4 \frac{G^4(a,0,q)}{q^5} \int_{-x^2/qQ}^{x^2/qQ} \mathcal{H}_2(z)\,dz$$

$$+ x^4 \frac{G^4(a,0,q)(-2\log q + 2\gamma)}{q^5} \int_{-x^2/qQ}^{x^2/qQ} \mathcal{H}_1(z)\,dz$$

$$+ O(x^{6+\epsilon} q^{-5/2} Q^{-2} + x^{14/3+\epsilon} q^{-7/3} Q^{-1} + x^{5+\epsilon} q^{-3} Q^{-1}),$$

where $\mathcal{H}_1(\lambda)$ and $\mathcal{H}_2(\lambda)$ were defined in Theorem 1.1.

We can choose P and Q to satisfy $x^2/PQ > 3$. So we first give upper bounds of $\mathcal{H}_1(\lambda)$ and $\mathcal{H}_2(\lambda)$ for $|\lambda| > 3$. Using Lemmas 3.2 and 3.3, we get

$$\text{(6.4)} \qquad \begin{aligned} G_\lambda(y) &:= \int_0^y e(-u\lambda)\,du \ll 1/|\lambda| \quad (y > 0), \\ &\int_0^1 e(u^2\lambda)\,du \ll 1/|\lambda|^{1/2}. \end{aligned}$$

By partial summation and (6.4) we have

$$\text{(6.5)} \qquad \begin{aligned} \int_0^1 e(-u\lambda)\log u\,du &= \int_0^{1/|\lambda|} e(-u\lambda)\log u\,du + \int_{1/|\lambda|}^1 e(-u\lambda)\log u\,du \\ &= \int_0^{1/|\lambda|} e(-u\lambda)\log u\,du + \int_{1/|\lambda|}^1 \log u\,dG_\lambda(u) \\ &= \int_0^{1/|\lambda|} e(-u\lambda)\log u\,du + G_\lambda(u)\log u\Big|_{1/|\lambda|}^1 - \int_{1/|\lambda|}^1 G_\lambda(u)u^{-1}\,du \\ &\ll |\lambda|^{-1}\log|\lambda|. \end{aligned}$$

Hence we get

$$\mathcal{H}_1(\lambda) \ll |\lambda|^{-3}, \quad \mathcal{H}_2(\lambda) \ll |\lambda|^{-3}\log|\lambda| \quad (|\lambda| \geq 3),$$

and for $U \geq 2$ we have

$$\text{(6.6)} \qquad \begin{aligned} \int_{|\lambda|>U} \mathcal{H}_1(\lambda)\,d\lambda &\ll \int_{|\lambda|>U} z^{-3}\,d\lambda \ll U^{-2}, \\ \int_{|\lambda|>U} \mathcal{H}_2(\lambda)\,d\lambda &\ll \int_{|\lambda|>U} z^{-3}\log\lambda\,d\lambda \ll U^{-2}\log U, \end{aligned}$$

which means that the integrals $\int_{-\infty}^{\infty} \mathcal{H}_1(\lambda)\,d\lambda$ and $\int_{-\infty}^{\infty} \mathcal{H}_2(\lambda)\,d\lambda$ converge. Taking $U = x^2/qQ$ in (6.6), we get

$$\int_{|\lambda|>x^2/qQ} \mathcal{H}_1(\lambda)\,dz \ll \int_{|\lambda|>x^2/qQ} \lambda^{-3}\,d\lambda \ll x^{-4}q^2Q^2,$$

$$\int_{|\lambda|>x^2/qQ} \mathcal{H}_2(\lambda)\,dz \ll \int_{|\lambda|>x^2/qQ} \lambda^{-3}\log\lambda\,d\lambda \ll x^{-4}(\log x)q^2Q^2.$$

Inserting the above two estimates into (6.3) we have

$$(6.7)\quad \int_{a/q-1/qQ}^{a/q+1/qQ} S_1^4(\alpha;x)S_2(-\alpha;4x^2)\,d\alpha$$
$$= 2x^4\log x\,\frac{G^4(a,0,q)}{q^5}\int_{-x^2/qQ}^{x^2/qQ} \mathcal{H}_1(\lambda)\,dz + x^4\frac{G^4(a,0,q)}{q^5}\int_{-x^2/qQ}^{x^2/qQ} \mathcal{H}_2(\lambda)\,dz$$
$$+ x^4\frac{G^4(a,0,q)(-2\log q+2\gamma)}{q^5}\int_{-x^2/qQ}^{x^2/qQ} \mathcal{H}_1(\lambda)\,dz$$
$$+ O(Q^2q^{-1}\log x + x^{6+\epsilon}q^{-5/2}Q^{-2} + x^{14/3+\epsilon}q^{-7/3}Q^{-1} + x^{5+\epsilon}q^{-3}Q^{-1}).$$

Combining (6.1) and (6.7) we get

$$(6.8)\quad \int_{\mathcal{M}} S_1^4(\alpha;x)S_2(-\alpha;4x^2)\,d\alpha$$
$$= 2x^4\log x\sum_{1\le q\le P} q^{-5}\sum_{\substack{0\le a<q\\(a,q)=1}} G^4(a,0,q)\int_{-\infty}^{\infty} \mathcal{H}_1(\lambda)\,dz$$
$$+ x^4\sum_{1\le q\le P} q^{-5}\sum_{\substack{0\le a<q\\(a,q)=1}} G^4(a,0,q)\int_{-\infty}^{\infty} \mathcal{H}_2(\lambda)\,dz$$
$$+ x^4\sum_{1\le q\le P} \frac{-2\log q+2\gamma}{q^5}\sum_{\substack{0\le a<q\\(a,q)=1}} G^4(a,0,q)\int_{-\infty}^{\infty} \mathcal{H}_1(\lambda)\,dz$$
$$+ O(Q^2P\log x + x^{6+\epsilon}Q^{-2} + x^{14/3+\epsilon}Q^{-1} + x^{5+\epsilon}Q^{-1})$$
$$= 2K_1L_1x^4\log x + (K_1L_2+K_2L_1)x^4$$
$$+ O(x^4P^{-1}\log P + Q^2P\log x + x^{6+\epsilon}Q^{-2} + x^{14/3+\epsilon}Q^{-1} + x^{5+\epsilon}Q^{-1}).$$

We take $P = x^{1/2}/12$ and $Q = 3x^{3/2}$ and insert them into (6.8), to get

$$(6.9)\quad \int_{\mathcal{M}} S_1^4(\alpha;x)S_2(-\alpha;4x^2)\,d\alpha$$
$$= 2K_1L_1x^4\log x + (K_1L_2+K_2L_1)x^4 + O(x^{7/2+\epsilon}),$$

where K_1, K_2 and L_1, L_2 were defined in Theorem 1.1.

Now we study the integral on the minor arcs. We have

$$(6.10)\qquad \int\limits_{C(\mathcal{M})} S_1^4(\alpha;x)S_2(-\alpha;4x^2)\,d\alpha$$
$$\ll \max_{\alpha\in C(\mathcal{M})} |S_1(\alpha;x)|^2 \int\limits_0^1 |S_1(\alpha;x)|^2|S_2(-\alpha;4x^2)|\,d\alpha$$
$$\ll \max_{\alpha\in C(\mathcal{M})} |S_1(\alpha;x)|^2 \Big(\int\limits_0^1 |S_1(\alpha;x)|^4\,d\alpha\Big)^{1/2}\Big(\int\limits_0^1 |S_2(-\alpha;4x^2)|^2\,d\alpha\Big)^{1/2}$$
$$\ll \max_{\alpha\in C(\mathcal{M})} |S_1(\alpha;x)|^2 \Big(\sum_{\substack{m_1^2+m_2^2=m_3^2+m_4^2\\ 1\le m_1,m_2,m_3,m_4\le x}} 1\Big)^{1/2}\Big(\sum_{1\le n\le 4x^2} d(n)\Big)^{1/2}$$
$$\ll \max_{\alpha\in C(\mathcal{M})} |S_1(\alpha;x)|^2 \Big(\sum_{\substack{m_1^2-m_3^2=m_4^2-m_2^2\\ 1\le m_1,m_2,m_3,m_4\le x}} 1\Big)^{1/2}\Big(\sum_{1\le n\le 4x^2} d(n)\Big)^{1/2}$$
$$\ll \max_{\alpha\in C(\mathcal{M})} |S_1(\alpha;x)|^2 \Big(\sum_{n\le 2x^2} d^2(n)\Big)^{1/2}\Big(\sum_{1\le n\le 4x^2} d(n)\Big)^{1/2}$$
$$\ll \max_{\alpha\in C(\mathcal{M})} |S_1(\alpha;x)|^2 x^2\log^2 x \ll x^{7/2+\epsilon},$$

where we used Lemma 4.2 and the well-known estimates

$$\sum_{n\le x} d^2(n) \ll x\log^3 x,\qquad \sum_{n\le x} d(n)\ll x\log x.$$

From (2.4), (6.9) and (6.10) the proof of Theorem 1.1 is complete.

7. Proof of Theorem 1.2. The proof of Theorem 1.2 is easier than the proof of Theorem 1.1.

Suppose P_1 and Q_1 are two large real numbers to be determined later, which satisfy

$$\log\sqrt{x} < P_1 < \sqrt{x},\quad 2P_1^2 < Q_1,\quad Q_1 > x^{1/2+\epsilon},\quad P_1Q_1 < x.$$

Each $\alpha\in[-1/Q, 1-1/Q]$ may be written in the form

$$(7.1)\qquad \alpha = a/q+\lambda,\quad |\lambda|\le 1/qQ,$$

for some integers a, q with $1\le a\le q\le Q$ and $(a,q)=1$. We denote by $\mathcal{M}'(a,q)$ the set of α satisfying (7.1), and define the major arcs $\mathcal{M}'$ and minor arcs $C(\mathcal{M}')$ as follows:

$$\mathcal{M}' := \bigcup_{1\le q\le P_1}\ \bigcup_{\substack{0\le a<q\\ (a,q)=1}} \mathcal{M}'(a,q),\quad C(\mathcal{M}') := [-1/Q_1, 1-1/Q_1]\setminus \mathcal{M}'.$$

Let

$$S_3(\alpha;y) := \sum_{|m|\le y} e(m^2\alpha), \quad S_4(\alpha;y) := \sum_{1\le n\le y} \Lambda(n)e(n\alpha). \tag{7.2}$$

It is easily seen that

$$S_3(\alpha;y) = 2S_1(\alpha;y)+1. \tag{7.3}$$

By (2.4) and (7.3), we have

$$\begin{aligned}\pi_\Lambda(x) &= \int_0^1 S_3^4(\alpha;\sqrt{x})S_4(-\alpha;x)\,d\alpha \\ &= 16\int_0^1 S_1^4(\alpha;\sqrt{x})S_4(-\alpha;x)\,d\alpha + O(x^{3/2}\log x).\end{aligned} \tag{7.4}$$

Suppose $\alpha = a/q+\lambda \in \mathcal{M}'$ with $0\le a<q\le P_1$, $(a,q)=1$, $|\lambda|\le 1/P_1Q_1$ and $P_1Q_1\le x$, $Q_1 > x^{1/2+\epsilon}$. In much the same way as for Lemma 4.1, but more easily, we obtain

$$S_1(\alpha;\sqrt{x}) = \frac{G(a,0,q)}{q}\sqrt{x}\int_0^1 e(u^2x\lambda)\,du + O\big((1+x|\lambda|)\sqrt{q}\log(q+1)\big). \tag{7.5}$$

For $S_4(-\alpha;x)$, similar to [PP, (6.21)] we have

$$S_4(-\alpha;x) = x\frac{C_q(-a)}{\varphi(q)}\int_0^1 e(-ux\lambda)\,du + O(xe^{-c\sqrt{\log x}}), \tag{7.6}$$

where $c>0$ is an absolute positive constant and $C_q(r)$ is the Ramanujan sum. From (7.5) and (7.6) we get, as in the proof of Theorem 1.1,

$$\begin{aligned}&\int_{\mathcal{M}'} S_1^4(\alpha;\sqrt{x})S_4(-\alpha;x)\,d\alpha \\ &= x^2\sum_{q=1}^{\infty}\frac{1}{q^4\varphi(q)}\sum_{\substack{0\le a<q\\(a,q)=1}} G^4(a,0,q)C_q(-a) \\ &\qquad\times\int_{-\infty}^{\infty}\Big(\int_0^1 e(u^2\lambda)\,du\Big)^4\int_0^1 e(-u\lambda)\,du\,dz \\ &\quad + O(x^2P_1^{-1}+Q_1^2P_1+x^{3+\epsilon}Q_1^{-1}e^{-c\sqrt{\log x}}+x^{7/2+\epsilon}Q_1^{-2}) \\ &= K_3L_3x^2 + O(x^2P_1^{-1}+Q_1^2P_1+x^{3+\epsilon}Q_1^{-1}e^{-c\sqrt{\log x}}+x^{7/2+\epsilon}Q_1^{-2}),\end{aligned} \tag{7.7}$$

where K_3 and L_3 were defined in Theorem 1.2.

Now consider the integral on $C(\mathcal{M}')$. According to Dirichlet's lemma, each $\alpha\in C(\mathcal{M}')$ can be written as $\alpha = a/q+\lambda$ with $1\le a\le q$, $(a,q)=1$,

$|z| \le 1/qQ_1$ and $P_1 < q \le Q_1$. Lemma 4.2 still holds. So we have

$$S_1(\alpha;\sqrt{x}) \ll \sqrt{x}\,P_1^{-1/2} + Q_1^{1/2}\log^{1/2} x \ll \sqrt{x}\,P_1^{-1/2}.$$

Hence similar to (6.10) we have

$$\begin{aligned}(7.8)\qquad &\int_{C(\mathcal{M}')} S_1^4(\alpha;\sqrt{x})S_4(-\alpha;x)\,d\alpha\\ &\ll \max_{\alpha\in C(\mathcal{M}')}|S_1(\alpha;\sqrt{x})|^2\int_0^1|S_1(\alpha;\sqrt{x})|^2|S_4(-\alpha;x)|\,d\alpha\\ &\ll \max_{\alpha\in C(\mathcal{M}')}|S_1(\alpha;\sqrt{x})|^2\Big(\int_0^1|S_1(\alpha;\sqrt{x})|^4\,d\alpha\Big)^{1/2}\Big(\int_0^1|S_4(-\alpha;x|^2\,d\alpha\Big)^{1/2}\\ &\ll \max_{\alpha\in C(\mathcal{M}')}|S_1(\alpha;\sqrt{x})|^2\Big(\sum_{n\le x}d^2(n)\Big)^{1/2}\Big(\sum_{1\le n\le x}\Lambda^2(n)\Big)^{1/2}\\ &\ll \max_{\alpha\in C(\mathcal{M}')}|S_1(\alpha;\sqrt{x})|^2 x\log^2 x \ll x^2P_1^{-1}\log^2 x.\end{aligned}$$

Now take $P_1 = \log^{A+2} x$ and $Q_1 = x\log^{-8A-8} x$. Combining (7.4), (7.7) and (7.8) we have

$$\pi_\Lambda(x) = \int_0^1 S_3^4(\alpha;\sqrt{x})S_4(-\alpha;x)\,d\alpha = 16K_3L_3x^2 + O(x^2\log^{-A} x).$$

Then the proof of Theorem 1.2 is complete.

8. Remark. Apart from the above results, we can find many similar results for

$$\sum_{m_1,m_2,m_3,m_4} f(m_1^2+m_2^2+m_3^2+m_4^2)$$

and

$$\sum_{m_1,m_2,m_3,m_4} f(m_1^2+m_2^2+m_3^2+m_4^2)g_1(m_1)g_2(m_2)g_3(m_3)g_4(m_4),$$

where f, g_1, g_2, g_3, g_4 are arithmetic functions which have good value distribution in residue classes to large moduli.

Here are some results which can be proved by similar methods:

$$S_{\mathbb{N}}(x;\mu) := \sum_{1\le m_1,m_2,m_3,m_4\le x} \mu(m_1^2+m_2^2+m_3^2+m_4^2) \ll x^4\log^{-A} x,$$

$$S_{\mathbb{Z}}(x;\mu) := \sum_{m_1^2+m_2^2+m_3^2+m_4^2\le x} \mu(m_1^2+m_2^2+m_3^2+m_4^2) \ll x^2\log^{-A} x,$$

$$S_{\mathbb{P}}(x;\mu) := \sum_{1\le p_1,p_2,p_3,p_4\le x} \mu(p_1^2+p_2^2+p_3^2+p_4^2) \ll x^4\log^{-A} x,$$

$$S_{\mathbb{N}}(x;d) := \sum_{1\le p_1,p_2,p_3,p_4\le x} d(p_1^2+p_2^2+p_3^2+p_4^2) \sim c_0x^4\log^{-3} x,$$

$$S_{\mathbb{N}}(x;d) := \sum_{1\le m_1,m_2,m_3,m_4\le x} d(m_1^2+m_2^2+m_3^2+m_4^2)d(m_1)d(m_2)d(m_3)d(m_4) \sim c_1x^4\log^5 x.$$

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