ACTA ARITHMETICA
166.2 (2014)

The spt-crank for overpartitions

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1. Introduction. Here we consider Ramanujan type congruences for various spt type functions and combinatorial interpretations of them in terms of rank and crank type functions. We recall the spt function began with Andrews in [2] defining $\mathrm{spt}(n)$ as the number of smallest parts in the partitions of n. In the same paper he proved the following congruences.

Theorem 1.1. *For $n \geq 0$ we have*

$$\mathrm{spt}(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$

$$\mathrm{spt}(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$

$$\mathrm{spt}(13n+6) \equiv 0 \pmod{13}. \tag{1.3}$$

These congruences are reminiscent of the Ramanujan congruences for the partition function. The proof of Theorem 1.1 relied on relating the spt function to the second moment of the rank function for partitions. With this, $\mathrm{spt}(n)$ could be expressed in terms of rank differences. Formulas for the required rank differences are found in [9] and [25].

We recall an overpartition of n is a partition of n in which the first occurrence of a part may be overlined. In [14] Bringmann, Lovejoy, and Osburn defined $\overline{\mathrm{spt}}(n)$ as the number of smallest parts in the overpartitions of n. Additionally they defined $\overline{\mathrm{spt}}_1(n)$ to be the number of smallest parts in the overpartitions of n with smallest part odd, and $\overline{\mathrm{spt}}_2(n)$ to be the number of smallest parts in the overpartitions of n with smallest even. We alter these definitions to only include the overpartitions of n where the smallest part is not overlined. This simply means the count of smallest parts here is half of the count of smallest parts in [14] and in other articles. This does not have any effect on congruences unless the modulus is even. We illustrate this change with an example.

2010 *Mathematics Subject Classification*: Primary 11P83, 05A17; Secondary 11P82, 05A19, 33P15, 11F03.

Key words and phrases: number theory, Andrews' spt-function, congruences, partitions, overpartitions, smallest parts function, Bailey's Lemma, Bailey pairs.

DOI: 10.4064/aa166-2-3

The overpartitions of 4 are 4, $\overline{4}$, $3+1$, $\overline{3}+1$, $3+\overline{1}$, $\overline{3}+\overline{1}$, $2+2$, $\overline{2}+2$, $2+1+1$, $\overline{2}+1+1$, $2+\overline{1}+1$, $\overline{2}+\overline{1}+1$, $1+1+1+1$, and $\overline{1}+1+1+1$, and so $\overline{\mathrm{spt}}(4)=13$, $\overline{\mathrm{spt}}_1(4)=10$, and $\overline{\mathrm{spt}}_2(4)=3$.

Bringmann, Lovejoy, and Osburn [14] proved the following congruences for their new spt functions.

Theorem 1.2. *For $n\geq 0$ we have*

$$\overline{\mathrm{spt}}(3n)\equiv 0 \pmod{3}, \tag{1.4}$$
$$\overline{\mathrm{spt}}_1(3n)\equiv 0 \pmod{3}, \tag{1.5}$$
$$\overline{\mathrm{spt}}_1(5n)\equiv 0 \pmod{5}, \tag{1.6}$$
$$\overline{\mathrm{spt}}_2(3n)\equiv 0 \pmod{3}, \tag{1.7}$$
$$\overline{\mathrm{spt}}_2(3n+1)\equiv 0 \pmod{3}, \tag{1.8}$$
$$\overline{\mathrm{spt}}_2(5n+3)\equiv 0 \pmod{5}. \tag{1.9}$$

The proof of these congruences relied on expressing these functions in terms of the second moments of certain rank and crank functions which relate to quasi-modular forms. We will give another proof of these congruences, which gives their new combinatorial interpretations. We describe this method shortly.

In [1] Ahlgren, Bringmann, and Lovejoy defined $\mathrm{M2spt}(n)$ to be the number of smallest parts in the partitions of n without repeated odd parts and with smallest part even. One congruence they proved for $\mathrm{M2spt}(n)$ is that for any prime $\ell\geq 3$, any integer $m\geq 1$, and n such that $\left(\frac{-n}{\ell}\right)=1$, we have

$$\mathrm{M2spt}\left(\frac{\ell^{2m}n+1}{8}\right)\equiv 0 \pmod{\ell^m}.$$

However none of the current known congruences for $\mathrm{M2spt}(n)$ appear to be of the form of the congruences we have mentioned for $\mathrm{spt}(n)$ and $\overline{\mathrm{spt}}(n)$, rather they are congruences related to certain Hecke operators. One of the results of this paper will be to prove such congruences by giving combinatorial refinements.

We will prove the following congruences for $\mathrm{M2spt}(n)$.

Theorem 1.3. *For $n\geq 0$ we have*

$$\mathrm{M2spt}(3n+1)\equiv 0 \pmod{3}, \tag{1.10}$$
$$\mathrm{M2spt}(5n+1)\equiv 0 \pmod{5}, \tag{1.11}$$
$$\mathrm{M2spt}(5n+3)\equiv 0 \pmod{5}. \tag{1.12}$$

Also we will determine the parity of $\overline{\mathrm{spt}}(n)$, $\overline{\mathrm{spt}}_1(n)$, and $\overline{\mathrm{spt}}_2(n)$.

Theorem 1.4. *For $n\geq 1$ we have* $\overline{\mathrm{spt}}(n)\equiv 1 \pmod{2}$ *if and only if n is a square or twice a square,* $\overline{\mathrm{spt}}_1(n)\equiv 1 \pmod{2}$ *if and only if n is an odd* *square, and* $\overline{\mathrm{spt}}_2(n) \equiv 1 \pmod 2$ *if and only if* n *is an even square or twice a square.*

In Theorem 1.4 it is important to note that we are using the convention of not counting the smallest parts of overpartitions when the smallest part is overlined. Otherwise $\overline{\mathrm{spt}}(n)$, $\overline{\mathrm{spt}}_1(n)$, and $\overline{\mathrm{spt}}_2(n)$ are trivially always even and instead these congruences tell when they are 0 or 2 modulo 4. The method we use to prove these parity results gives a combinatorial explanation as well; however, if one works modulo 2 just with the single variable generating functions listed below, the parity follows immediately upon noticing the generating functions reduce to certain sum of divisors generating functions.

The generating functions for the spt functions are given as follows; these are special cases of a general SPT function due to Bringmann, Lovejoy, and Osburn [15, Section 7],

$$\mathrm{SPT}(d,e;q) = \frac{(-dq;q)_\infty(-eq;q)_\infty}{(deq;q)_\infty(q;q)_\infty}\sum_{n=1}^{\infty}\frac{q^n(q;q)_n(deq;q)_n}{(1-q^n)^2(-dq;q)_n(-eq;q)_n}.$$

The case $d=0$, $e=0$ gives a generating function for $\mathrm{spt}(n)$,

$$\sum_{n=1}^{\infty}\mathrm{spt}(n)q^n = \sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2(q^{n+1};q)_\infty}.$$

The case $d=1$, $e=0$ gives a generating function for $\overline{\mathrm{spt}}(n)$,

$$\sum_{n=1}^{\infty}\overline{\mathrm{spt}}(n)q^n = \sum_{n=1}^{\infty}\frac{q^n(-q^{n+1};q)_\infty}{(1-q^n)^2(q^{n+1};q)_\infty}. \tag{1.13}$$

The case $d=1$, $e=1/q$, $q=q^2$ gives a generating function for $\overline{\mathrm{spt}}_2(n)$,

$$\sum_{n=1}^{\infty}\overline{\mathrm{spt}}_2(n)q^n = \sum_{n=1}^{\infty}\frac{q^{2n}(-q^{2n+1};q)_\infty}{(1-q^{2n})^2(q^{2n+1};q)_\infty}. \tag{1.14}$$

Similar to the $\overline{\mathrm{spt}}(n)$ and $\overline{\mathrm{spt}}_2(n)$ we see a generating function for $\overline{\mathrm{spt}}_1(n)$ is

$$\sum_{n=1}^{\infty}\overline{\mathrm{spt}}_1(n)q^n = \sum_{n=0}^{\infty}\frac{q^{2n+1}(-q^{2n+2};q)_\infty}{(1-q^{2n+1})^2(q^{2n+2};q)_\infty}. \tag{1.15}$$

The case $d=0$, $e=1/q$, $q=q^2$ gives a generating function for $\mathrm{M2spt}(n)$,

$$\sum_{n=1}^{\infty}\mathrm{M2spt}(n)q^n = \sum_{n=1}^{\infty}\frac{q^{2n}(-q^{2n+1};q^2)_\infty}{(1-q^{2n})^2(q^{2n+2};q^2)_\infty}. \tag{1.16}$$

Here we are using the product notation,

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), \quad (a_1,\ldots,a_j;q)_\infty = (a_1;q)_\infty\cdots(a_j;q)_\infty,$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad (a_1,\dots,a_j;q)_n = (a_1;q)_n\cdots(a_j;q)_n.$$

We note that the three special cases of $\mathrm{SPT}(d,e;q)$ described above are quasimock theta functions (see [15, p. 240] for a definition).

Andrews, Garvan, and Liang [7] found combinatorial interpretations of the congruences modulo 5 and 7 in Theorem 1.1 in terms of weighted counts of special vector partitions called *S-partitions*. This was done by adding an extra variable to the generating function of the spt function. In particular they defined

$$\mathrm{S}(z,q) = \sum_{n=1}^{\infty}\frac{q^n(q^{n+1};q)_\infty}{(zq^n;q)_\infty(z^{-1}q^n;q)_\infty} = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N_{\mathrm{S}}(m,n)z^mq^n.$$

One then finds the congruences in (1.1) and (1.2) follow by showing the coefficients of q^{5n+4} in $\mathrm{S}(\zeta_5,q)$ and q^{7n+5} in $\mathrm{S}(\zeta_7,q)$ are zero, where ζ_5 is a primitive fifth root of unity and ζ_7 is a primitive seventh root of unity. This is the approach we take to prove the congruences for $\overline{\mathrm{spt}}(n)$, $\overline{\mathrm{spt}}_1(n)$, $\overline{\mathrm{spt}}_2(n)$, and $\mathrm{M2spt}(n)$, and their combinatorial refinements.

In the next section we give two-variable generalizations of the generating functions (1.13)–(1.16), introduce various ranks and cranks, and state numerous identities for these functions. At the end of the next section we describe the plan for the remainder of the paper.

2. Statement of results and preliminaries. In this paper we give alternate proofs of the congruences in Theorem 1.2 and prove the congruences of Theorems 1.3 and 1.4 as well as giving combinatorial interpretations. We consider two-variable generalizations of the generating functions from the introduction. We set

$$\overline{\mathrm{S}}(z,q) = \sum_{n=1}^{\infty}\frac{q^n(-q^{n+1};q)_\infty(q^{n+1};q)_\infty}{(zq^n;q)_\infty(z^{-1}q^n;q)_\infty} = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N_{\overline{\mathrm{S}}}(m,n)z^mq^n,$$

$$\overline{\mathrm{S}}_2(z,q) = \sum_{n=1}^{\infty}\frac{q^{2n}(-q^{2n+1};q)_\infty(q^{2n+1};q)_\infty}{(zq^{2n};q)_\infty(z^{-1}q^{2n};q)_\infty} = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N_{\overline{\mathrm{S}}_2}(m,n)z^mq^n,$$

$$\overline{\mathrm{S}}_1(z,q) = \sum_{n=0}^{\infty}\frac{q^{2n+1}(-q^{2n+2};q)_\infty(q^{2n+2};q)_\infty}{(zq^{2n+1};q)_\infty(z^{-1}q^{2n+1};q)_\infty} = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N_{\overline{\mathrm{S}}_1}(m,n)z^mq^n,$$

$$\mathrm{S2}(z,q) = \sum_{n=1}^{\infty}\frac{q^{2n}(q^{2n+2};q^2)_\infty(-q^{2n+1};q^2)_\infty}{(zq^{2n};q^2)_\infty(z^{-1}q^{2n};q^2)_\infty} = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N_{\mathrm{S2}}(m,n)z^mq^n.$$

In each two-variable generating function we set $z=1$ to recover the generating functions from the introduction. We see that $\overline{\mathrm{S}}(1,q)$ is the generating function for $\overline{\mathrm{spt}}(n)$, $\overline{\mathrm{S}}_2(1,q)$ is the generating function for $\overline{\mathrm{spt}}_2(n)$, $\overline{\mathrm{S}}_1(1,q)$ is the generating function for $\overline{\mathrm{spt}}_1(n)$, and $\mathrm{S2}(1,q)$ is the generating function for $\mathrm{M2spt}(n)$.

Furthermore we define

$$N_{\overline{\mathrm{S}}}(k,t,n) = \sum_{m\equiv k \pmod t} N_{\overline{\mathrm{S}}}(m,n)$$

so

$$\overline{\mathrm{spt}}(n) = \sum_{m=-\infty}^{\infty} N_{\overline{\mathrm{S}}}(m,n) = \sum_{k=0}^{r-1} N_{\overline{\mathrm{S}}}(k,r,n)$$

for any positive integer r. We similarly define

$$N_{\overline{\mathrm{S}}_2}(k,t,n) = \sum_{m\equiv k \pmod t} N_{\overline{\mathrm{S}}_2}(m,n),$$

$$N_{\overline{\mathrm{S}}_1}(k,t,n) = \sum_{m\equiv k \pmod t} N_{\overline{\mathrm{S}}_1}(m,n),$$

$$N_{\mathrm{S2}}(k,t,n) = \sum_{m\equiv k \pmod t} N_{\mathrm{S2}}(m,n).$$

We use these series to give another proof of the spt congruences.

First we consider the congruence in (1.4) of Theorem 1.2. With ζ_3 a primitive third root of unity, we have

$$\overline{\mathrm{S}}(\zeta_3,q) = \sum_{n=1}^{\infty}\big(N_{\overline{\mathrm{S}}}(0,3,n) + N_{\overline{\mathrm{S}}}(1,3,n)\zeta_3 + N_{\overline{\mathrm{S}}}(2,3,n)\zeta_3^2\big)q^n.$$

The minimal polynomial for ζ_3 is $1+x+x^2$, and so if

$$N_{\overline{\mathrm{S}}}(0,3,n) + N_{\overline{\mathrm{S}}}(1,3,n)\zeta_3 + N_{\overline{\mathrm{S}}}(2,3,n)\zeta_3^2 = 0$$

then

$$N_{\overline{\mathrm{S}}}(0,3,3n) = N_{\overline{\mathrm{S}}}(1,3,3n) = N_{\overline{\mathrm{S}}}(2,3,3n). \tag{2.1}$$

But if (2.1) holds, then

$$\overline{\mathrm{spt}}(3n) = 3N_{\overline{\mathrm{S}}}(k,3,3n) \quad \text{for } k=0,1,2$$

and so clearly $\overline{\mathrm{spt}}(3n) \equiv 0 \pmod 3$. That is, if we show the coefficient of q^{3n} in $\overline{\mathrm{S}}(\zeta_3,q)$ to be zero, then we have proved the first congruence in Theorem 1.2, and the stronger result (2.1).

In the same fashion, the congruences (1.5) and (1.6) will follow by showing the coefficients of q^{3n} in $\overline{\mathrm{S}}_1(\zeta_3,q)$ and the coefficients of q^{5n} in $\overline{\mathrm{S}}_1(\zeta_5,q)$ are zero. The congruences (1.7)–(1.9) will follow by showing the coefficients of q^{3n} and q^{3n+1} in $\overline{\mathrm{S}}_2(\zeta_3,q)$ and the coefficients of q^{5n+3} in $\overline{\mathrm{S}}_2(\zeta_5,q)$ are zero. The congruences in Theorem 1.3 will follow by showing the coefficients of q^{3n+1} in $\mathrm{S2}(\zeta_3,q)$ and the coefficients of q^{5n+1} and q^{5n+3} in $\mathrm{S2}(\zeta_5,q)$ are zero.

To this end, we will express the series $\overline{\mathrm{S}}(z,q)$, $\overline{\mathrm{S}}_1(z,q)$, $\overline{\mathrm{S}}_2(z,q)$, $\mathrm{S2}(z,q)$ as the difference of the generating functions for certain ranks and cranks. In [7] Andrews, the first author, and Liang found that $\mathrm{S}(z,q)$ could be expressed in terms of the difference of the rank and crank of a partition. We recall that the *rank* of a partition is the largest part minus the number of parts. The *crank* of a partition is the largest part if there are no ones and otherwise is the number of parts larger than the number of ones minus the number of ones.

As in [14], for an overpartition π of n we define the *residual crank* of π to be the crank of the subpartition of π consisting of the nonoverlined parts of π. We let $\overline{M}(m,n)$ denote the number of overpartitions of n with this residual crank equal to m. The generating function for $\overline{M}(m,n)$ is then given by

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{M}(m,n)z^mq^n=\frac{(-q;q)_\infty(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}.$$

Of course this interpretation is not quite correct, as $(q;q)_\infty/(zq,z^{-1}q;q)_\infty$ does not agree at q^1 for the crank of the partition consisting of a single one. Thus the interpretation of this residual crank is not quite correct for overpartitions whose nonoverlined parts consist of a single one.

As in [14] and elsewhere, for an overpartition π of n we define the *Dyson rank* of π to be the largest part minus the number of parts of π. Let $\overline{N}(m,n)$ denote the number of overpartitions of n with Dyson rank equal to m. As in [22, Proposition 1.1 and proof of Proposition 3.2], the generating function for $\overline{N}(m,n)$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{N}(m,n)z^mq^n&=\sum_{n=0}^{\infty}\frac{(-1;q)_nq^{n(n+1)/2}}{(zq;q)_n(z^{-1}q;q)_n}\\&=\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^nq^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}\right).\end{aligned}$$

The second equality is obtained by Watson's transformation.

We define another residual crank as follows. For a partition π of n with distinct odd parts we take the crank of the partition $\pi_e/2$ obtained by taking the subpartition π_e, of the even parts of π, and halving each part of π_e. We let $M2(m,n)$ denote the number of partitions π of n with distinct odd parts and such that the partition $\pi_e/2$ has crank m. Then the generating function for $M2(m,n)$ is given by

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}M2(m,n)z^mq^n=\frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(zq^2;q^2)_\infty(z^{-1}q^2;q^2)_\infty}.$$

Again this interpretation is not quite correct, here it fails for partitions with distinct odd parts whose only even parts are a single two.

We recall the M_2-*rank* of a partition π without repeated odd parts is given by

$$M_2\text{-rank}(\pi) = \left\lceil \frac{l(\pi)}{2} \right\rceil - \#(\pi),$$

where $l(\pi)$ is the largest part of π and $\#(\pi)$ is the number of parts of π. The M_2-rank was introduced by Berkovich and the first author [10]. We let $N2(m,n)$ denote the number of partitions of n with distinct odd parts and M_2-rank m. By Lovejoy and Osburn [24] the generating function for $N2(m,n)$ is given by

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N2(m,n)z^m q^n = \sum_{n=0}^{\infty} q^{n^2} \frac{(-q;q^2)_n}{(zq^2;q^2)_n(z^{-1}q^2;q^2)_n}.$$

We set

$$\overline{N}(k,t,n) = \sum_{m\equiv k \,(\mathrm{mod}\, t)} \overline{N}(m,n), \qquad N2(k,t,n) = \sum_{m\equiv k \,(\mathrm{mod}\, t)} N2(m,n),$$

$$\overline{M}(k,t,n) = \sum_{m\equiv k \,(\mathrm{mod}\, t)} \overline{M}(m,n), \qquad M2(k,t,n) = \sum_{m\equiv k \,(\mathrm{mod}\, t)} M2(m,n).$$

We see $\overline{N}(-m,n) = \overline{N}(m,n)$ and so $\overline{N}(k,t,n) = \overline{N}(t-k,t,n)$. Similarly we have $\overline{M}(k,t,n) = \overline{M}(t-k,t,n)$, $N2(k,t,n) = N2(t-k,t,n)$, and $M2(k,t,n) = M2(t-k,t,n)$.

We will show the following:

Theorem 2.1.

$$(1-z)(1-z^{-1})\overline{\mathrm{S}}(z,q) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} (\overline{N}(m,n) - \overline{M}(m,n))z^m q^n.$$

Theorem 2.2.

$$(1-z)(1-z^{-1})\mathrm{S2}(z,q) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} (N2(m,n) - M2(m,n))z^m q^n.$$

Theorem 2.3.

$$\begin{aligned}(1-z)(1-z^{-1})\overline{\mathrm{S}}_2(z,q) = &\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} \left(\frac{\overline{N}(m,n)}{2} - \overline{M}(m,n)\right) z^m q^n \\ &+ \frac{(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^n}{(1-zq^n)(1-z^{-1}q^n)}\right).\end{aligned}$$

THEOREM 2.4.

$$(1-z)(1-z^{-1})\overline{\mathrm{S}}_1(z,q) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\frac{\overline{N}(m,n)}{2}z^m q^n - \frac{(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^n}{(1-zq^n)(1-z^{-1}q^n)}\right).$$

In [23] Lovejoy and Osburn determined formulas for the differences of $\overline{N}(s,\ell,\ell n+d)$ for $\ell=3,5$ and in [24] they did the same for $N2(s,\ell,\ell n+d)$. From these difference formulas, we know the 3-dissection and 5-dissection for the generating functions of $\overline{N}(m,n)$ and $N2(m,n)$. In particular, we will have the following.

THEOREM 2.5.

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{N}(m,n)\zeta_3^m q^n = \overline{N}_{0,3}(q^3)+q\overline{N}_{1,3}(q^3)+q^2\overline{N}_{2,3}(q^3)$$

where

$$\overline{N}_{0,3}(q) = \frac{(q^3;q^3)_\infty^4(q^2;q^2)_\infty}{(q;q)_\infty^2(q^6;q^6)_\infty^2}, \tag{2.2}$$

$$\overline{N}_{1,3}(q) = 2\frac{(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q;q)_\infty}. \tag{2.3}$$

THEOREM 2.6.

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{N}(m,n)\zeta_5^m q^n = \overline{N}_{0,5}(q^5)+q\overline{N}_{1,5}(q^5)+q^2\overline{N}_{2,5}(q^5)+q^3\overline{N}_{3,5}(q^5)+q^4\overline{N}_{4,5}(q^5)$$

where

$$\overline{N}_{0,5}(q) = \frac{(q^4,q^6;q^{10})_\infty(q^5;q^5)_\infty^2}{(q^2,q^3;q^5)_\infty^2(q^{10};q^{10})_\infty}+2(\zeta_5+\zeta_5^{-1})q\frac{(q^{10};q^{10})_\infty}{(q^3,q^4,q^6,q^7;q^{10})_\infty}, \tag{2.4}$$

$$\overline{N}_{3,5}(q) = \frac{2(1-\zeta_5-\zeta_5^{-1})(q^{10};q^{10})_\infty}{(q^2,q^3;q^5)_\infty}. \tag{2.5}$$

THEOREM 2.7.

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}N2(m,n)\zeta_3^m q^n = N2_{0,3}(q^3)+qN2_{1,3}(q^3)+q^2N2_{2,3}(q^3)$$

where

$$N2_{1,3}(q) = \frac{(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty(q^3;q^3)_\infty(q^{12};q^{12})_\infty}. \tag{2.6}$$

THEOREM 2.8.

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N2(m,n)\zeta_5^m q^n = N2_{0,5}(q^5) + qN2_{1,5}(q^5) + q^2N2_{2,5}(q^5) + q^3N2_{3,5}(q^5) + q^4N2_{4,5}(q^5)$$

where

$$N2_{1,5}(q) = \frac{(-q^5, q^{10}; q^{10})_\infty}{(q^2, q^8; q^{10})_\infty}, \tag{2.7}$$

$$N2_{3,5}(q) = (\zeta_5 + \zeta_5^4)\frac{(-q^5, q^{10}; q^{10})_\infty}{(q^4, q^6; q^{10})_\infty}. \tag{2.8}$$

The terms $\overline{N}_{2,3}(q)$, $\overline{N}_{1,5}(q)$, $\overline{N}_{2,5}(q)$, $\overline{N}_{4,5}(q)$, $N2_{0,3}(q)$, $N2_{2,3}(q)$, $N2_{0,5}(q)$, $N2_{2,5}(q)$, and $N2_{4,5}(q)$ are also products and series in q and follow from the difference formulas of Lovejoy and Obsurn [23], [24]. However, we will not need them here.

We will determine dissections for the cranks and other series. In particular, we will prove the following.

THEOREM 2.9.

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} \overline{M}(m,n)\zeta_3^m q^n = \overline{M}_{0,3}(q^3) + q\overline{M}_{1,3}(q^3) + q^2\overline{M}_{2,3}(q^3)$$

where

$$\overline{M}_{0,3}(q) = \frac{(q^3;q^3)_\infty^4 (q^2;q^2)_\infty}{(q;q)_\infty^2 (q^6;q^6)_\infty^2}, \tag{2.9}$$

$$\overline{M}_{1,3}(q) = -\frac{(q^6;q^6)_\infty (q^3;q^3)_\infty}{(q;q)_\infty}, \tag{2.10}$$

$$\overline{M}_{2,3}(q) = -2\frac{(q^6;q^6)_\infty^4}{(q^3;q^3)_\infty^2 (q^2;q^2)_\infty}. \tag{2.11}$$

THEOREM 2.10.

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} \overline{M}(m,n)\zeta_5^m q^n = \overline{M}_{0,5}(q^5) + q\overline{M}_{1,5}(q^5) + q^2\overline{M}_{2,5}(q^5) + q^3\overline{M}_{3,5}(q^5) + q^4\overline{M}_{4,5}(q^5)$$

where

$$\overline{M}_{0,5}(q) = \frac{(q^4, q^6, q^{10}; q^{10})_\infty}{(q, q^4; q^5)_\infty (q^2, q^8; q^{10})_\infty} - q(\zeta_5 + \zeta_5^4)\frac{(q^2, q^8, q^{10}; q^{10})_\infty}{(q^2, q^3; q^5)_\infty (q^4, q^6; q^{10})_\infty}, \tag{2.12}$$

$$\overline{M}_{1,5}(q) = (\zeta_5+\zeta_5^4)\frac{(q^4,q^6,q^{10};q^{10})_\infty}{(q^2,q^3;q^5)_\infty(q^2,q^8;q^{10})_\infty}, \tag{2.13}$$

$$\overline{M}_{2,5}(q) = -\frac{(q^{10};q^{10})_\infty}{(q,q^4;q^5)_\infty}, \tag{2.14}$$

$$\overline{M}_{3,5}(q) = -(\zeta_5+\zeta_5^4)\frac{(q^{10};q^{10})_\infty}{(q^2,q^3;q^5)_\infty}, \tag{2.15}$$

$$\overline{M}_{4,5}(q) = -\frac{(q^2,q^8,q^{10};q^{10})_\infty}{(q,q^4;q^5)_\infty(q^4,q^6;q^{10})_\infty}. \tag{2.16}$$

Theorem 2.11.

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M2(m,n)\zeta_3^m q^n = \overline{M}_{0,3}(q^3) + q\overline{M}_{1,3}(q^3) + q^2\overline{M}_{2,3}(q^3)$$

where

$$M2_{0,3}(q) = \frac{(q^6;q^6)_\infty^{10}(q^4;q^4)_\infty(q;q)_\infty}{(q^{12};q^{12})_\infty^4(q^3;q^3)_\infty^4(q^2;q^2)_\infty^3}, \tag{2.17}$$

$$M2_{1,3}(q) = \frac{(q^6;q^6)_\infty^4}{(q^{12};q^{12})_\infty(q^3;q^3)_\infty(q^2;q^2)_\infty}, \tag{2.18}$$

$$M2_{2,3}(q) = -2\frac{(q^{12};q^{12})_\infty^2(q^3;q^3)_\infty^2(q^2;q^2)_\infty}{(q^6;q^6)_\infty^2(q^4;q^4)_\infty(q;q)_\infty}. \tag{2.19}$$

Theorem 2.12.

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M2(m,n)\zeta_5^m q^n = M2_{0,5}(q^5) + qM2_{1,5}(q^5) + q^2M2_{2,5}(q^5) + q^3M2_{3,5}(q^5) + q^4M2_{4,5}(q^5)$$

where

$$M2_{0,5}(q) = \frac{(-q^3,-q^5,-q^7,q^{10};q^{10})_\infty}{(-q,q^4,q^6,-q^9;q^{10})_\infty}, \tag{2.20}$$

$$M2_{1,5}(q) = \frac{(-q^5,q^{10};q^{10})_\infty}{(q^2,q^8;q^{10})_\infty}, \tag{2.21}$$

$$M2_{2,5}(q) = (\zeta_5+\zeta_5^4)\frac{(q^2,-q^3,-q^5,-q^7,q^8,q^{10};q^{10})_\infty}{(-q,q^4,q^4,q^6,q^6,-q^9;q^{10})_\infty} - \frac{(-q,q^4,-q^5,q^6,-q^9,q^{10};q^{10})_\infty}{(q^2,q^2,-q^3,-q^7,q^8,q^8;q^{10})_\infty}, \tag{2.22}$$

$$M2_{3,5}(q) = (\zeta_5+\zeta_5^4)\frac{(-q^5,q^{10};q^{10})_\infty}{(q^4,q^6;q^{10})_\infty}, \tag{2.23}$$

$$M2_{4,5}(q) = -(\zeta_5+\zeta_5^4)\frac{(-q,-q^5,-q^9,q^{10};q^{10})_\infty}{(q^2,-q^3,-q^7,q^8;q^{10})_\infty}. \tag{2.24}$$

THEOREM 2.13.

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(1-\zeta_3)(1-\zeta_3^{-1})(-1)^n q^n}{(1-\zeta_3 q^n)(1-\zeta_3^{-1}q^n)}\right) = A_0(q^3)+qA_1(q^3)+q^2A_2(q^3)$$

where

$$A_0(q) = \frac{(q^3;q^3)_\infty^4 (q^2;q^2)_\infty}{2(q;q)_\infty^2 (q^6;q^6)_\infty^2}, \tag{2.25}$$

$$A_1(q) = -2\frac{(q^6;q^6)_\infty (q^3;q^3)_\infty}{(q;q)_\infty}, \tag{2.26}$$

$$A_2(q) = 2\frac{(q^6;q^6)_\infty^4}{(q^3;q^3)_\infty^2 (q^2;q^2)_\infty}. \tag{2.27}$$

THEOREM 2.14.

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(1-\zeta_5)(1-\zeta_5^{-1})(-1)^n q^n}{(1-\zeta_5 q^n)(1-\zeta_5^{-1}q^n)}\right) = B_0(q^5)+qB_1(q^5)+q^2B_2(q^5)+q^3B_3(q^5)+q^4B_4(q^5)$$

where

$$B_0(q) = \frac{(q^5;q^5)_\infty^2 (q^4,q^6;q^{10})_\infty}{2(q^{10};q^{10})_\infty (q^2,q^3;q^5)_\infty^2} + (\zeta_5+\zeta_5^{-1})\frac{q(q^{10};q^{10})_\infty}{(q^3,q^4,q^6,q^7;q^{10})_\infty}, \tag{2.28}$$

$$B_1(q) = (\zeta_5+\zeta_5^{-1}-1)\frac{(q^4,q^6,q^{10};q^{10})_\infty}{(q^2,q^8;q^{10})_\infty^2 (q^3,q^7;q^{10})_\infty}, \tag{2.29}$$

$$B_2(q) = (1-2\zeta_5-2\zeta_5^{-1})\frac{(q^{10};q^{10})_\infty}{(q,q^9;q^{10})_\infty (q^4,q^6;q^{10})_\infty}, \tag{2.30}$$

$$B_3(q) = -\frac{(q^{10};q^{10})_\infty}{(q^2,q^3;q^5)_\infty}, \tag{2.31}$$

$$B_4(q) = (\zeta_5+\zeta_5^{-1})\frac{(q^2,q^8,q^{10};q^{10})_\infty}{(q,q^9;q^{10})_\infty (q^4,q^6;q^{10})_\infty^2}. \tag{2.32}$$

With these dissections, we need only match up the appropriate terms for each congruence. The congruence for $\overline{\mathrm{spt}}(3n)$ of Theorem 1.2 follows from using (2.2) and (2.9) to get

$$\overline{N}_{0,3} - \overline{M}_{0,3} = 0,$$

which along with Theorem 2.1 shows that the coefficients of q^{3n} in $\overline{\mathrm{S}}(\zeta_3,q)$ are zero.

The congruences for $\overline{\mathrm{spt}}_1(3n)$ and $\overline{\mathrm{spt}}_1(5n)$ follow from using (2.2), (2.25), (2.4), and (2.28) to get

$$\overline{N}_{0,3}/2 - A_0 = 0, \qquad \overline{N}_{0,5}/2 - B_0 = 0,$$

and then applying Theorem 2.4.

The congruences for $\overline{\mathrm{spt}}_2(3n)$, $\overline{\mathrm{spt}}_2(3n+1)$, and $\overline{\mathrm{spt}}_2(5n+3)$ follow from using (2.2), (2.25), (2.9), (2.3), (2.26), (2.10), (2.5), (2.31), and (2.15) to get

$$\frac{\overline{N}_{0,3}}{2} + A_0 - \overline{M}_{0,3} = 0, \qquad \frac{\overline{N}_{1,3}}{2} + A_1 - \overline{M}_{1,3} = 0, \qquad \frac{\overline{N}_{3,5}}{2} + B_3 - \overline{M}_{3,5} = 0,$$

and an application of Theorem 2.3.

The congruences for M2spt$(3n+1)$, M2spt$(5n+1)$, and M2spt$(5n+3)$ follow from using (2.6), (2.18), (2.7), (2.21), (2.8), and (2.23) to get

$$N2_{1,3} - M2_{1,3} = 0, \qquad N2_{1,5} - M2_{1,5} = 0, \qquad N2_{3,5} - M2_{3,5} = 0,$$

and applying Theorem 2.2.

For Theorem 1.4 we have to do a little better. In particular we will prove the following.

THEOREM 2.15.

$$\begin{aligned}
\overline{\mathrm{S}}(i,q) &= \sum_{n=1}^{\infty} q^{n^2} - \sum_{n=1}^{\infty} (-1)^n q^{2n^2}, \\
\overline{\mathrm{S}}_1(i,q) &= \sum_{n=1}^{\infty} q^{(2n-1)^2}, \\
\overline{\mathrm{S}}_2(i,q) &= \sum_{n=1}^{\infty} q^{(2n)^2} - \sum_{n=1}^{\infty} (-1)^n q^{2n^2}.
\end{aligned}$$

Considering just $\overline{\mathrm{spt}}_1(n)$, we have

$$\begin{aligned}
\sum_{n=1} q^{(2n-1)^2} &= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_{\overline{\mathrm{S}}_1}(m,n) i^m q^n \\
&= \sum_{n=0}^{\infty} \big(N_{\overline{\mathrm{S}}_1}(0,4,n) - N_{\overline{\mathrm{S}}_1}(2,4,n) + i(N_{\overline{\mathrm{S}}_1}(1,4,n) - N_{\overline{\mathrm{S}}_1}(3,4,n)) \big) q^n \\
&= \sum_{n=0}^{\infty} (N_{\overline{\mathrm{S}}_1}(0,4,n) - N_{\overline{\mathrm{S}}_1}(2,4,n)) q^n.
\end{aligned}$$

But

$$\begin{aligned}
\overline{\mathrm{spt}}_1(n) &= N_{\overline{\mathrm{S}}_1}(0,4,n) + N_{\overline{\mathrm{S}}_1}(1,4,n) + N_{\overline{\mathrm{S}}_1}(2,4,n) + N_{\overline{\mathrm{S}}_1}(3,4,n) \\
&= N_{\overline{\mathrm{S}}_1}(0,4,n) + 2N_{\overline{\mathrm{S}}_1}(1,4,n) + N_{\overline{\mathrm{S}}_1}(2,4,n)
\end{aligned}$$

and so we see that

$$\overline{\mathrm{spt}}_1(n) \equiv 1 \pmod{2}$$

if and only if n is an odd square. The parity of $\overline{\mathrm{spt}}(n)$ and $\overline{\mathrm{spt}}_2(n)$ follows in the same fashion.

In [7] Andrews, the first author, and Liang also showed that $N_{\mathrm{S}}(m,n)$, the coefficients in $\mathrm{S}(z,q)$, are nonnegative. The same phenomenon occurs here.

THEOREM 2.16. *For all m and n the coefficients $N_{\overline{\mathrm{S}}}(m,n)$, $N_{\overline{\mathrm{S}}_1}(m,n)$, and $N_{\overline{\mathrm{S}}_2}(m,n)$ are nonnegative.*

In Section 3 we give combinatorial interpretations of the series $\overline{\mathrm{S}}(z,q)$, $\overline{\mathrm{S}}_1(z,q)$, $\overline{\mathrm{S}}_2(z,q)$, and $\mathrm{S2}(z,q)$ in terms of weighted vector partitions and then prove Theorem 2.16. For $\overline{\mathrm{S}}(z,q)$, $\overline{\mathrm{S}}_1(z,q)$ and $\overline{\mathrm{S}}_2(z,q)$ we define the spt-crank in terms of marked overpartitions (see (3.9)). In Theorem 3.8 we give a combinatorial interpretation of each of the spt-overpartition congruences in Theorem 1.2 in terms of marked overpartitions. In Section 4 we prove the theorems on expressing $\overline{\mathrm{S}}(z,q)$, $\overline{\mathrm{S}}_1(z,q)$, $\overline{\mathrm{S}}_2(z,q)$, and $\mathrm{S2}(z,q)$ in terms of the difference between a rank and crank. In Section 5 we prove the various dissections. In Section 6 we conclude with remarks on the nonnegativity of the coefficients of $\mathrm{S2}(z,q)$; a recent result by Andrews, Chan, Kim, and Osburn [5] on the first moments for the rank and crank of overpartitions; and the remaining spt function of [15].

3. Combinatorial interpretations. In this section we provide combinatorial interpretations of the coefficients in the series $\overline{\mathrm{S}}(z,q)$, $\overline{\mathrm{S}}_1(z,q)$, $\overline{\mathrm{S}}_2(z,q)$, and $\mathrm{S2}(z,q)$. For all four series we provide an interpretation in terms of certain vector partitions with four components. For the three series $\overline{\mathrm{S}}(z,q)$, $\overline{\mathrm{S}}_1(z,q)$, and $\overline{\mathrm{S}}_2(z,q)$ we give two additional interpretations—one in terms of pairs of partitions, and the other in terms of marked overpartitions. This latter interpretation will give interpretations of the congruences for overpartitions directly in terms of the overpartitions themselves.

3.1. Vector partitions and $\overline{\mathrm{S}}$-partitions. The coefficients in the series $\overline{\mathrm{S}}(z,q)$, $\overline{\mathrm{S}}_1(z,q)$, $\overline{\mathrm{S}}_2(z,q)$, and $\mathrm{S2}(z,q)$ can be interpreted in terms of cranks of vector partitions. This can be done with vectors with four components, each a partition with certain restrictions.

We let $\overline{V} = \mathcal{D}\times\mathcal{P}\times\mathcal{P}\times\mathcal{D}$, where $\mathcal{P}$ denotes the set of all partitions and $\mathcal{D}$ denotes the set of all partitions into distinct parts. For a partition π we let $s(\pi)$ denote the smallest part of π (with the convention that the empty partition has smallest part ∞), $\#(\pi)$ the number of parts in π, and $|\pi|$ the sum of the parts of π. For $\vec{\pi} = (\pi_1,\pi_2,\pi_3,\pi_4)\in\overline{V}$, we define the weight $\omega(\vec{\pi}) = (-1)^{\#(\pi_1)-1}$, the $\mathrm{crank}(\vec{\pi}) = \#(\pi_2)-\#(\pi_3)$, and the norm $|\vec{\pi}| = |\pi_1|+|\pi_2|+|\pi_3|+|\pi_4|$. We say $\vec{\pi}$ is a *vector partition* of n if $|\vec{\pi}| = n$.

We then let

$$\overline{\mathrm{S}} = \{(\pi_1,\pi_2,\pi_3,\pi_4) \in \overline{V} : 1 \le s(\pi_1) < \infty,\ s(\pi_1) \le s(\pi_2),\ s(\pi_1) \le s(\pi_3),\ s(\pi_1) < s(\pi_4)\}.$$

Also, $\overline{\mathrm{S}}_1$ and $\overline{\mathrm{S}}_2$ denote the subsets of $\overline{\mathrm{S}}$ with $s(\pi_1)$ odd and even, respectively.

We then see that the number of vector partitions of n in $\overline{\mathrm{S}}$ with crank m counted according to the weight ω is exactly $N_{\overline{\mathrm{S}}}(m,n)$. Similarly the number of vector partitions of n in $\overline{\mathrm{S}}_1$ with crank m counted according to the weight ω is $N_{\overline{\mathrm{S}}_1}(m,n)$, and the number of vector partitions of n in $\overline{\mathrm{S}}_2(m,n)$ with crank m counted according to the weight ω is $N_{\overline{\mathrm{S}}_2}(m,n)$.

We let $n_o(\pi)$ and $n_e(\pi)$ denote the number of odd and even parts, respectively, of π. We let

$$\mathrm{S2} = \{(\pi_1,\pi_2,\pi_3,\pi_4) \in \overline{\mathrm{S}} : n_o(\pi_1) = 0,\ n_o(\pi_2) = 0,\ n_o(\pi_3) = 0,\ n_e(\pi_4) = 0\}.$$

Then $N_{\mathrm{S2}}(m,n)$ is the number of vector partitions of n from S2 with crank m counted according to the weight ω.

For each of the four spt functions, we give an example to illustrate a congruence.

Example 3.1 ($n = 3$). The four overpartitions of 3 with smallest part not overlined are 3, $2+1$, $\overline{2}+1$, and $1+1+1$. We then have $\overline{\mathrm{spt}}(3) = 6$. There are eight vector partitions of 3 from $\overline{\mathrm{S}}$. These vector partitions along with their weights and cranks are given as follows:

$\overline{\mathrm{S}}$-vector partition	Weight	Crank	mod 3
$[1,-,-,2]$	1	0	0
$[1,-,1+1,-]$	1	-2	1
$[1,-,2,-]$	1	-1	2
$[1,1,1,-]$	1	0	0
$[1,1+1,-,-]$	1	2	2
$[1,2,-,-]$	1	1	1
$[1+2,-,-]$	-1	0	0
$[3,-,-,-]$	1	0	0

Here we have used $-$ to indicate the empty partition. We see that

$$N_{\overline{\mathrm{S}}}(0,3,3) = N_{\overline{\mathrm{S}}}(1,3,3) = N_{\overline{\mathrm{S}}}(2,3,3) = 2 = \tfrac{1}{3}\overline{\mathrm{spt}}(3).$$

We note that the overpartitions of 3 all have smallest part odd, so that $\overline{\mathrm{spt}}_1(3) = 6$. Moreover, all vector partitions of 3 from $\overline{\mathrm{S}}$ are also from $\overline{\mathrm{S}}_1$. Thus we also have

$$N_{\overline{\mathrm{S}}_1}(0,3,3) = N_{\overline{\mathrm{S}}_1}(1,3,3) = N_{\overline{\mathrm{S}}_1}(2,3,3) = 2 = \tfrac{1}{3}\overline{\mathrm{spt}}_1(3).$$

EXAMPLE 3.2 ($n = 4$). The two overpartitions of 4 with smallest part even and not overlined are 4 and 2+2, so $\overline{\mathrm{spt}}_2(4) = 3$. There are three vector partitions of 4 from $\overline{\mathrm{S}}_2$:

$\overline{\mathrm{S}}_2$-vector partition	Weight	Crank	mod 3
$[2, 2, -, -]$	1	1	1
$[2, -, 2, -]$	1	-1	2
$[4, -, -, -]$	1	0	0

We see that $N_{\overline{\mathrm{S}}_2}(0, 3, 4) = N_{\overline{\mathrm{S}}_2}(1, 3, 4) = N_{\overline{\mathrm{S}}_2}(2, 3, 4) = 1 = \frac{1}{3}\overline{\mathrm{spt}}_2(4)$.

EXAMPLE 3.3 ($n = 6$). The three partitions of 6 without repeated odd parts and with smallest part even are 6, 4+2, 2+2+2, so that $\mathrm{M2spt}(6) = 5$. There are seven vector partitions of 6 from S2:

S2-vector partition	Weight	Crank	mod 5
$[2, -, 4, -]$	1	-1	4
$[2, -, 2 + 2, -]$	1	-2	3
$[2, 2, 2, -]$	1	0	0
$[2, 4, -, -]$	1	1	1
$[2, 2 + 2, -, -]$	1	2	2
$[4 + 2, -, -, -]$	-1	0	0
$[6, -, -, -]$	1	0	0

We see that

$$\begin{aligned} N_{\mathrm{S2}}(0, 5, 6) &= N_{\mathrm{S2}}(1, 5, 6) = N_{\mathrm{S2}}(2, 5, 6) = N_{\mathrm{S2}}(3, 5, 6) = N_{\mathrm{S2}}(4, 5, 6) \\ &= 1 = \tfrac{1}{5}\,\mathrm{M2spt}(6). \end{aligned}$$

3.2. $\overline{\mathrm{SP}}$-partition pairs. In this section we prove that for all m, n,

$$N_{\overline{\mathrm{S}}}(m, n) \geq 0, \tag{3.1}$$

and provide a combinatorial interpretation in terms of partition pairs.

3.2.1. *Proof of nonnegativity.* We compute

$$\begin{aligned} \overline{\mathrm{S}}(z, q) &= \sum_{n=1}^{\infty} \sum_{m} N_{\overline{\mathrm{S}}}(m, n) z^m q^n \\ &= \sum_{n=1}^{\infty} \frac{q^n (q^{n+1}; q)_\infty}{(zq^n; q)_\infty (z^{-1}q^n; q)_\infty} (-q^{n+1}; q)_\infty \\ &= \sum_{n=1}^{\infty} \frac{q^n (q^{2n}; q)_\infty}{(zq^n; q)_\infty (z^{-1}q^n; q)_\infty} \, \frac{(q^{2n+2}; q^2)_\infty}{(q^{2n}; q)_\infty} \\ &= \sum_{n=1}^{\infty} q^n \sum_{k=0}^{\infty} \frac{(z^{-1}q^n)^k}{(zq^{n+k}; q)_\infty (q)_k} \, \frac{1}{(1 - q^{2n})} \, \frac{1}{(q^{2n+1}; q^2)_\infty} \end{aligned} \tag{3.2}$$

by [11, Prop. 4.1]. The inequality (3.1) clearly follows. Replacing n by $2n+1$ and $2n$ in the second line of (3.2) gives $N_{\overline{\mathrm{S}}_1}(m,n) \geq 0$ and $N_{\overline{\mathrm{S}}_2}(m,n) \geq 0$, respectively.

3.2.2. *The* $\overline{\text{sptcrank}}$ *in terms of partition pairs.* We define

$$\overline{\mathrm{SP}} = \{\vec{\lambda} = (\lambda_1,\lambda_2) \in \mathcal{P}\times\mathcal{P} : 0 < s(\lambda_1) \leq s(\lambda_2) \text{ and all parts of } \lambda_2 \text{ that are } \geq 2s(\lambda_1)+1 \text{ are odd}\}.$$

First we show that

$$\overline{\mathrm{spt}}(n) = \sum_{\substack{\vec{\lambda}\in\overline{\mathrm{SP}} \\ |\vec{\lambda}|=|\lambda_1|+|\lambda_2|=n}} 1. \tag{3.3}$$

Indeed,

$$\begin{aligned}
\sum_{n=1}^{\infty} \overline{\mathrm{spt}}(n) q^n &= \sum_{n=1}^{\infty} \frac{q^n(-q^{n+1};q)_\infty}{(1-q^n)^2(q^{n+1};q)_\infty} \\
&= \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2(q^{n+1};q)_\infty}\,\frac{(q^{2n+2};q^2)_\infty}{(q^{n+1};q)_\infty} \\
&= \sum_{n=1}^{\infty} \frac{q^n}{(q^n;q)_\infty}\,\frac{1}{(1-q^n)(1-q^{n+1})\cdots(1-q^{2n})(q^{2n+1};q^2)_\infty} \\
&= \sum_{n=1}^{\infty} \sum_{\substack{\lambda_1\in\mathcal{P} \\ s(\lambda_1)=n}} q^{|\lambda_1|} \sum_{\substack{\lambda_2\in\mathcal{P} \\ s(\lambda_2)\geq n \\ \text{all parts in } \lambda_2 \geq 2n+1 \text{ are odd}}} q^{|\lambda_2|} \\
&= \sum_{n=1}^{\infty} \sum_{\substack{\vec{\lambda}=(\lambda_1,\lambda_2)\in\overline{\mathrm{SP}} \\ s(\lambda_1)=n}} q^{|\vec{\lambda}|} = \sum_{\vec{\lambda}\in\overline{\mathrm{SP}}} q^{|\vec{\lambda}|},
\end{aligned}$$

and (3.3) follows.

We let $\overline{\mathrm{SP}}_1$ be the set of $\vec{\lambda} = (\lambda_1,\lambda_2) \in \overline{\mathrm{SP}}$ with $s(\lambda_1)$ odd and let $\overline{\mathrm{SP}}_2$ be the set of $\vec{\lambda} = (\lambda_1,\lambda_2) \in \overline{\mathrm{SP}}$ with $s(\lambda_1)$ even. Then in the same fashion we have

$$\overline{\mathrm{spt}}_1(n) = \sum_{\substack{\vec{\lambda}\in\overline{\mathrm{SP}}_1 \\ |\vec{\lambda}|=n}} 1, \qquad \overline{\mathrm{spt}}_2(n) = \sum_{\substack{\vec{\lambda}\in\overline{\mathrm{SP}}_2 \\ |\vec{\lambda}|=n}} 1.$$

Next we define a $\overline{\text{crank}}$ of partition pairs $\vec{\lambda} = (\lambda_1,\lambda_2) \in \overline{\mathrm{SP}}$ by interpreting the coefficient of $z^m q^n$ in (3.2). For $\vec{\lambda} = (\lambda_1,\lambda_2) \in \overline{\mathrm{SP}}$ let

$$k(\vec{\lambda}) = \#\text{ of parts } j \text{ in } \lambda_2 \text{ such that } s(\lambda_1) \le j \le 2s(\lambda_1) - 1,$$

and

$$\overline{\operatorname{crank}}(\vec{\lambda}) = \begin{cases} (\#\text{ of parts} \ge s(\lambda_1) + k \text{ of } \lambda_1) - k & \text{if } k > 0, \\ (\#\text{ of parts of } \lambda_1) - 1 & \text{if } k = 0, \end{cases} \tag{3.4}$$

where $k = k(\vec{\lambda})$. We have

Theorem 3.4.

$$N_{\overline{\mathrm{S}}}(m,n) = \#\ \textit{of}\ \vec{\lambda} = (\lambda_1,\lambda_2) \in \overline{\mathrm{SP}}\ \textit{with}\ |\vec{\lambda}| = n\ \textit{and}\ \overline{\operatorname{crank}}(\vec{\lambda}) = m, \tag{3.5}$$

$$N_{\overline{\mathrm{S}}_1}(m,n) = \#\ \textit{of}\ \vec{\lambda} = (\lambda_1,\lambda_2) \in \overline{\mathrm{SP}}_1\ \textit{with}\ |\vec{\lambda}| = n\ \textit{and}\ \overline{\operatorname{crank}}(\vec{\lambda}) = m, \tag{3.6}$$

$$N_{\overline{\mathrm{S}}_2}(m,n) = \#\ \textit{of}\ \vec{\lambda} = (\lambda_1,\lambda_2) \in \overline{\mathrm{SP}}_2\ \textit{with}\ |\vec{\lambda}| = n\ \textit{and}\ \overline{\operatorname{crank}}(\vec{\lambda}) = m. \tag{3.7}$$

Proof. From (3.2) we have

$$\begin{aligned}
\overline{\mathrm{S}}(z,q) &= \sum_{n=1}^{\infty} \sum_{m} N_{\overline{\mathrm{S}}}(m,n) z^m q^n \\
&= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{q^n}{(1-q^n)\cdots(1-q^{n+k-1})(zq^{n+k};q)_\infty} \\
&\qquad\qquad \times z^{-k} q^{nk} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{1}{(1-q^{2n})} \frac{1}{(q^{2n+1};q^2)_\infty} \\
&= \sum_{n=1}^{\infty} \frac{q^n}{(zq^n;q)_\infty} \frac{1}{(1-q^{2n})} \frac{1}{(q^{2n+1};q^2)_\infty} \\
&\quad + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{q^n}{(1-q^n)\cdots(1-q^{n+k-1})(zq^{n+k};q)_\infty} \\
&\qquad\qquad \times z^{-k} q^{nk} \begin{bmatrix} n+k-1 \\ k \end{bmatrix} \frac{1}{(1-q^{2n})} 1(q^{2n+1};q^2)_\infty.
\end{aligned}$$

We note that the q-binomial coefficient $\left[\begin{smallmatrix} n+k-1 \\ k \end{smallmatrix}\right]$ is the generating function for partitions into parts $\le n-1$ with number of parts $\le k$. Thus we see that $q^{nk}\left[\begin{smallmatrix} n+k-1 \\ k \end{smallmatrix}\right]$ is the generating function for partitions into exactly k parts j, where $n \le j \le 2n-1$. Hence

$$\begin{aligned}
\overline{\mathrm{S}}(z,q) &= \sum_{n=1}^{\infty} \sum_{\substack{\vec{\lambda} = (\lambda_1,\lambda_2) \in \overline{\mathrm{SP}} \\ s(\lambda_1) = n,\ k(\vec{\lambda}) = 0}} z^{\#(\lambda_1)-1} q^{|\vec{\lambda}|} + \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \sum_{\substack{\vec{\lambda} = (\lambda_1,\lambda_2) \in \overline{\mathrm{SP}} \\ s(\lambda_1) = n,\ k(\vec{\lambda}) = k}} z^{\overline{\operatorname{crank}}(\vec{\lambda})} q^{|\vec{\lambda}|} \\
&= \sum_{\vec{\lambda} \in \overline{\mathrm{SP}}} z^{\overline{\operatorname{crank}}(\vec{\lambda})} q^{|\vec{\lambda}|}.
\end{aligned}$$

The result (3.5) follows. The results (3.6) and (3.7) follow in a similar fashion. ■

3.2.3. *Examples.* We illustrate our combinatorial interpretation of $\overline{\mathrm{spt}}$, $\overline{\mathrm{spt}}_1$, $\overline{\mathrm{spt}}_2$ congruences in terms of the $\overline{\mathrm{crank}}$ of $\overline{\mathrm{SP}}$-partition pairs.

Example 3.5 ($n = 3$). The overpartitions of 3 with smallest parts not overlined are 3, $2+1$, $\overline{2}+1$, $1+1+1$, so that $\overline{\mathrm{spt}}(3) = 6$. There are six $\overline{\mathrm{SP}}$-partition pairs of 3:

$\overline{\mathrm{SP}}$-partition pair	k	$\overline{\mathrm{crank}}$	mod 3
$[3,-]$	0	0	0
$[2+1,-]$	0	1	1
$[1+1+1,-]$	0	2	2
$[1+1,1]$	1	-1	2
$[1,1+1]$	2	-2	1
$[1,2]$	0	0	0

We see that $N_{\overline{\mathrm{S}}}(0,3,3) = N_{\overline{\mathrm{S}}}(1,3,3) = N_{\overline{\mathrm{S}}}(2,3,3) = 2 = \frac{1}{3}\overline{\mathrm{spt}}(3)$.

Example 3.6 ($n = 5$). There are ten overpartitions of 5 with smallest parts odd and not overlined:

$$\begin{array}{lllll} 5, & 4+1, & \overline{4}+1, & 3+1+1, & \overline{3}+1+1 \\ 2+2+1, & \overline{2}+2+1, & 2+1+1+1, & \overline{2}+1+1+1, & 1+1+1+1+1, \end{array}$$

so that $\overline{\mathrm{spt}}_1(5) = 20$. There are 20 $\overline{\mathrm{SP}}_1$-partition pairs of 5:

$\overline{\mathrm{SP}}_1$-partition pair	k	$\overline{\mathrm{crank}}$	mod 5
$[1,1+1+1+1]$	4	-4	1
$[1,2+1+1]$	2	-2	3
$[1,2+2]$	0	0	0
$[1,3+1]$	1	-1	4
$[1+1,1+1+1]$	3	-3	2
$[1+1,2+1]$	1	-1	4
$[1+1,3]$	0	1	1
$[1+1+1,1+1]$	2	-2	3
$[1+1+1,2]$	0	2	2
$[2+1,1+1]$	2	-2	3
$[2+1,2]$	0	1	1
$[1+1+1+1,1]$	1	-1	4
$[2+1+1,1]$	1	0	0
$[3+1,1]$	1	0	0
$[1+1+1+1+1,-]$	0	4	4

$\overline{\mathrm{SP}}_1$-partition pair	k	$\overline{\text{crank}}$	mod 5
$[2+1+1+1,-]$	0	3	3
$[2+2+1,-]$	0	2	2
$[3+1+1,-]$	0	2	2
$[4+1,-]$	0	1	1
$[5,-]$	0	0	0

We see that

$$\begin{aligned} N_{\overline{\mathrm{S}}_1}(0,5,5) &= N_{\overline{\mathrm{S}}_1}(1,5,5) = N_{\overline{\mathrm{S}}_1}(2,5,5) = N_{\overline{\mathrm{S}}_1}(3,5,5) = N_{\overline{\mathrm{S}}_1}(4,5,5) \\ &= 4 = \tfrac{1}{5}\overline{\mathrm{spt}}_1(5). \end{aligned}$$

EXAMPLE 3.7 ($n = 8$). There are nine overpartitions of 8 with smallest parts even and not overlined:

$$\begin{array}{lllll} 8, & 6+2, & \overline{6}+2, & 3+3+2, & \overline{3}+3+2, \\ 4+4, & 4+2+2, & \overline{4}+2+2, & 2+2+2+2, & \end{array}$$

so that $\overline{\mathrm{spt}}_2(8) = 15$. There are 15 $\overline{\mathrm{SP}}_2$-partition pairs of 8:

$\overline{\mathrm{SP}}_2$-partition pair	k	$\overline{\text{crank}}$	mod 5
$[2,2+2+2]$	3	-3	2
$[2,3+3]$	2	-2	3
$[2,4+2]$	1	-1	4
$[2+2,2+2]$	2	-2	3
$[2+2,4]$	0	1	1
$[4,4]$	1	-1	4
$[3+2,3]$	1	0	0
$[2+2+2,2]$	1	-1	4
$[4+2,2]$	1	0	0
$[2+2+2+2,-]$	0	3	3
$[3+3+2,-]$	0	2	2
$[4+2+2,-]$	0	2	2
$[4+4,-]$	0	1	1
$[6+2,-]$	0	1	1
$[8,-]$	0	0	0

We see that

$$\begin{aligned} N_{\overline{\mathrm{S}}_2}(0,5,8) &= N_{\overline{\mathrm{S}}_2}(1,5,8) = N_{\overline{\mathrm{S}}_2}(2,5,8) = N_{\overline{\mathrm{S}}_2}(3,5,8) = N_{\overline{\mathrm{S}}_2}(4,5,8) \\ &= 3 = \tfrac{1}{5}\overline{\mathrm{spt}}_2(8). \end{aligned}$$

3.3. SPT-crank for marked overpartitions. Andrews, Dyson, and Rhoades [6] defined a *marked partition* as a pair (λ, k) where λ is a partition and k is an integer identifying one of its smallest parts; that is, $k = 1, 2, \dots, \nu(\lambda)$, where $\nu(\lambda)$ is the number of smallest parts of λ. They asked for a statistic like the crank which would divide the relevant marked partitions into t equal classes for $t = 5, 7, 13$, thus explaining the congruences (1.1), (1.2), (1.3). This problem was solved by Chan, Ji and Zang [17] for the cases $t = 5, 7$. They defined an spt-crank for double-marked partitions and found a bijection between double-marked partitions and marked partitions. It is an open problem to define the spt-crank directly in terms of marked partitions. In this section we solve the analogous problem for overpartitions by defining a statistic on marked overpartitions.

3.3.1. *Definition of* $\overline{\text{sptcrank}}$ *for marked overpartitions.* We define a *marked overpartition* of n as a pair (π, j) where π is an overpartition of n in which the smallest part is not overlined, and j is an integer $1 \le j \le \nu(\pi)$, where as above $\nu(\pi)$ is the number of smallest parts of π. It is clear that

$$\overline{\text{spt}}(n) = \#\text{ of marked overpartitions } (\pi, j) \text{ of } n. \tag{3.8}$$

For example, there are six marked overpartitions of 3:

$$\begin{array}{lll} (\overline{2}+1,1), & (2+1,1), & (3,1), \\ (1+1+1,1), & (1+1+1,2), & (1+1+1,3), \end{array}$$

so that $\overline{\text{spt}}(3) = 6$.

To define the $\overline{\text{sptcrank}}$ of a marked overpartition we first need to define a function $k(m,n)$. For a positive integer m we write

$$m = b(m)2^{j(m)},$$

where $b(m)$ is odd and $j(m) \ge 0$. For integers m, n such that $m \ge n+1$, we define $j_0(m,n)$ to be the smallest nonnegative integer j_0 such that

$$b(m)2^{j_0} \ge n+1.$$

We define

$$k(m,n) = \begin{cases} 0 & \text{if } b(m) \ge 2n, \\ 2^{j(m)-j_0(m,n)} & \text{if } b(m)2^{j_0(m,n)} < 2n, \\ 0 & \text{if } b(m)2^{j_0(m,n)} = 2n. \end{cases}$$

Note that if $j_0(m,n) \ge 1$ then $b(m)2^{j_0(m,n)} \le 2n$, so that the function $k(m,n)$ is well-defined. For a partition $\pi : m_1 + m_2 + \cdots + m_a$ into distinct parts $m_1 > m_2 > \cdots > m_a \ge n+1$ we define the function

$$k(\pi, n) = \sum_{j=1}^{a} k(m_j, n) = \sum_{m \in \pi} k(m,n).$$

For a marked overpartition (π, j) we let π_1 be the partition formed by the nonoverlined parts of π, and π_2 be the partition (into distinct parts) formed by the overlined parts of π, so that $s(\pi_2) > s(\pi_1)$. We define a function

$$\overline{k}(\pi, j) = \nu(\pi_1) - j + k(\pi_2, s(\pi_1)).$$

Finally we can now define

$$(3.9)\qquad \overline{\mathrm{sptcrank}}(\pi, j) = \begin{cases} (\#\text{ of parts} \geq s(\pi_1) + \overline{k} \text{ of } \pi_1) - \overline{k} & \text{if } \overline{k} = \overline{k}(\pi, j) > 0, \\ (\#\text{ of parts of } \pi_1) - 1 & \text{if } \overline{k} = \overline{k}(\pi, j) = 0. \end{cases}$$

Here is our main theorem.

Theorem 3.8.

(i) *The residue of the* $\overline{\mathrm{sptcrank}}$ *modulo* 3 *divides the marked overpartitions of* $3n$ *into three equal classes.*

(ii) *The residue of the* $\overline{\mathrm{sptcrank}}$ *modulo* 3 *divides the marked overpartitions of* $3n$ *and of* $3n + 1$ *with smallest part even into three equal classes.*

(iii) *The residue of the* $\overline{\mathrm{sptcrank}}$ *modulo* 5 *divides the marked overpartitions of* $5n + 3$ *with smallest part even into five equal classes.*

(iv) *The residue of the* $\overline{\mathrm{sptcrank}}$ *modulo* 3 *divides the marked overpartitions of* $3n$ *with smallest part odd into three equal classes.*

(v) *The residue of the* $\overline{\mathrm{sptcrank}}$ *modulo* 5 *divides the marked overpartitions of* $5n$ *with smallest part odd into five equal classes.*

(vi) *The residue of the* $\overline{\mathrm{sptcrank}}$ *modulo* 4 *divides the marked overpartitions of* n *into four classes with two classes of equal size and the remaining two classes are of equal size unless* n *is a square or twice a square in which case the remaining two classes differ in size by exactly* 1.

(vii) *The residue of the* $\overline{\mathrm{sptcrank}}$ *modulo* 4 *divides the marked overpartitions of* n *with smallest part odd into four classes with two classes of equal size and the remaining two classes of equal size unless* n *is an odd square in which case the remaining two classes differ in size by exactly* 1.

(viii) *The residue of the* $\overline{\mathrm{sptcrank}}$ *modulo* 4 *divides the marked overpartitions of* n *with smallest part even into four classes with two classes of equal size and the remaining two classes are of equal size unless* n *is an even square or twice a square in which case the remaining two classes differ in size by exactly* 1.

3.3.2. *Examples*

Example 3.9 ($n = 3$). There are six marked overpartitions of 3 so that $\overline{\mathrm{spt}}(3) = 6$. Here we abbreviate $k(\pi_2, s(\pi_1))$ to just k.

(π, j)	π_1	π_2	$\nu(\pi_1)$	k	$\overline{k}$	$\overline{\text{sptcrank}}$	mod 3	mod 4
$(\overline{2}+1,1)$	1	2	1	0	0	0	0	0
$(1+1+1,1)$	$1+1+1$	–	3	0	2	-2	1	2
$(1+1+1,2)$	$1+1+1$	–	3	0	1	-1	2	3
$(1+1+1,3)$	$1+1+1$	–	3	0	0	2	2	2
$(2+1,1)$	$2+1$	–	1	0	0	1	1	1
$(3,1)$	3	–	1	0	0	0	0	0

We see that the residue of the $\overline{\text{sptcrank}}$ modulo 3 divides the marked overpartitions of 3 into three equal classes. This illustrates Theorem 3.8(i). We see that the residue of the $\overline{\text{sptcrank}}$ modulo 4 divides the marked overpartitions of 3 into four classes, two of which are both of size 2 and the other two are both of size 1. This illustrates Theorem 3.8(vi), noting that 3 is neither a square nor twice a square.

Example 3.10 ($n = 5$). There are 15 marked overpartitions of 8 with smallest part even so that $\overline{\text{spt}}_2(8) = 15$. Here we abbreviate $\nu = \nu(\pi_1)$ and $k = k(\pi_2, s(\pi_1))$.

(π, j)	π_1	π_2	ν	k	$\overline{k}$	$\overline{\text{sptcrank}}$	mod 4	mod 5
$(\overline{6}+2,1)$	2	6	1	2	2	-2	2	3
$(\overline{4}+2+2,1)$	$2+2$	4	2	0	1	-1	3	4
$(\overline{4}+2+2,2)$	$2+2$	4	2	0	0	1	1	1
$(\overline{3}+3+2,1)$	$3+2$	3	1	1	1	0	0	0
$(2+2+2+2,1)$	$2+2+2+2$	–	4	0	3	-3	1	2
$(2+2+2+2,2)$	$2+2+2+2$	–	4	0	2	-2	2	3
$(2+2+2+2,3)$	$2+2+2+2$	–	4	0	1	-1	3	4
$(2+2+2+2,4)$	$2+2+2+2$	–	4	0	0	3	3	3
$(3+3+2,1)$	$3+3+2$	–	1	0	0	2	2	2
$(4+2+2,1)$	$4+2+2$	–	2	0	1	0	0	0
$(4+2+2,2)$	$4+2+2$	–	2	0	0	2	2	2
$(6+2,1)$	$6+2$	–	1	0	0	1	1	1
$(4+4,1)$	$4+4$	–	2	0	1	-1	3	4
$(4+4,2)$	$4+4$	–	2	0	0	1	1	1
$(8,1)$	8	–	1	0	0	0	0	0

We see that the residue of the $\overline{\text{sptcrank}}$ modulo 5 divides the marked overpartitions of 8 with even smallest part into five equal classes. This illustrates Theorem 3.8(iii). We see that the residue of the $\overline{\text{sptcrank}}$ modulo 4 divides the marked overpartitions of 8 with even smallest part into four classes, two of which are both of size 4 and the other two are of sizes 3 and 4. This illustrates Theorem 3.8(viii), noting that 8 is twice a square.

EXAMPLE 3.11 ($n = 5$). There are 20 marked overpartitions of 5 with smallest part odd so that $\overline{\mathrm{spt}}_1(5) = 20$. Again we abbreviate $\nu = \nu(\pi_1)$ and $k = k(\pi_2, s(\pi_1))$.

(π, j)	π_1	π_2	ν	k	$\overline{k}$	$\overline{\mathrm{sptcrank}}$	mod 5
$(\overline{4}+1, 1)$	1	4	1	0	0	0	0
$(\overline{3}+1+1, 1)$	$1+1$	3	2	0	1	-1	4
$(\overline{3}+1+1, 2)$	$1+1$	3	2	0	0	1	1
$(\overline{2}+1+1+1, 1)$	$1+1+1$	2	3	0	2	-2	3
$(\overline{2}+1+1+1, 2)$	$1+1+1$	2	3	0	1	-1	4
$(\overline{2}+1+1+1, 3)$	$1+1+1$	2	3	0	0	2	2
$(\overline{2}+2+1, 1)$	$2+1$	2	1	0	0	1	1
$(1+1+1+1+1, 1)$	$1+1+1+1+1$	–	5	0	4	-4	1
$(1+1+1+1+1, 2)$	$1+1+1+1+1$	–	5	0	3	-3	2
$(1+1+1+1+1, 3)$	$1+1+1+1+1$	–	5	0	2	-2	3
$(1+1+1+1+1, 4)$	$1+1+1+1+1$	–	5	0	1	-1	4
$(1+1+1+1+1, 5)$	$1+1+1+1+1$	–	5	0	0	4	4
$(2+1+1+1, 1)$	$2+1+1+1$	–	3	0	2	-2	3
$(2+1+1+1, 2)$	$2+1+1+1$	–	3	0	1	0	0
$(2+1+1+1, 3)$	$2+1+1+1$	–	3	0	0	3	3
$(2+2+1, 1)$	$2+2+1$	–	1	0	0	2	2
$(3+1+1, 1)$	$3+1+1$	–	2	0	1	0	0
$(3+1+1, 2)$	$3+1+1$	–	2	0	0	2	2
$(4+1, 1)$	$4+1$	–	1	0	0	1	1
$(5, 1)$	5	–	1	0	0	0	0

We see that the residue of the $\overline{\mathrm{sptcrank}}$ modulo 5 divides the marked overpartitions of 5 with odd smallest part into five equal classes. This illustrates Theorem 3.8(v).

3.3.3. *Proof of the main result*

BIJECTION 3.12. *Let $\mathcal{M}$ denote the set of marked overpartitions. There is a weight-preserving bijection $\Phi : \mathcal{M} \to \overline{\mathrm{SP}}$ such that*

$$\overline{k}(\pi, j) = k(\vec{\lambda}), \tag{3.10}$$

$$\overline{\mathrm{sptcrank}}(\pi, j) = \overline{\mathrm{crank}}(\vec{\lambda}), \tag{3.11}$$

where $\vec{\lambda} = (\lambda_1, \lambda_2) = \Phi(\pi, j)$.

Once this theorem is proved, the main Theorem 3.8 will follow from Theorems 2.1, 2.3, 2.4, and 3.4 and the appropriate dissections listed in Section 2.

Before we can construct the bijection Φ, we need to extend Euler's Theorem that the number of partitions of n into distinct parts equals the number of partitions of n into odd parts. Let n be a nonnegative integer. Let $\mathcal{D}_n$ denote the set of partitions into distinct parts $\geq n+1$. Let $\mathcal{P}_n$ denote the set of partitions into parts $\geq n+1$ in which all parts $> 2n$ are odd. Then we have

THEOREM 3.13. *Let $n \geq 0$ and $\ell \geq 1$. Then the number of partitions of ℓ from $\mathcal{D}_n$ equals the number of partitions of ℓ from $\mathcal{P}_n$.*

REMARK 3.14. Euler's Theorem is the case $n = 0$.

Proof of Theorem 3.13. We compute

$$\begin{aligned}
1 + \sum_{\pi \in \mathcal{D}_n} q^{|\pi|} &= \prod_{j=n+1}^{\infty} (1+q^j) \\
&= \prod_{j=n+1}^{\infty} \frac{(1+q^j)(1-q^j)}{1-q^j} = \prod_{j=n+1}^{\infty} \frac{1-q^{2j}}{1-q^j} \\
&= \frac{1}{(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{2n})(q^{2n+1};q^2)_\infty} \\
&= 1 + \sum_{\pi \in \mathcal{P}_n} q^{|\pi|}.
\end{aligned}$$

The result follows by considering the coefficient of q^ℓ on both sides of this identity. ∎

We require a bijective proof of this theorem. Glaisher (see [27, p. 23]) has a well-known straightforward bijective proof of Euler's Theorem. We extend this in a natural way to obtain a bijective proof of our theorem.

BIJECTION 3.15. *Let $n \geq 1$. There is a weight-preserving bijection*

$$\Psi_n : \mathcal{D}_n \to \mathcal{P}_n, \qquad \Psi_n(\pi) = \lambda,$$

such that

$$k(\pi, n) = \#\ \textit{of parts} \leq 2n-1\ \textit{of}\ \lambda. \tag{3.12}$$

We define Ψ_n as follows. Let $\pi \in \mathcal{D}_n$. We describe the image of each part m of π. We note that $m \geq n+1$ and as before we write

$$m = b(m) 2^{j(m)},$$

where $b(m)$ is odd and $j(m) \geq 0$, we observe $j(m) \geq j_0(m,n)$. Let

$$m \longmapsto \overbrace{2^{j_0(m,n)}b(m), 2^{j_0(m,n)}b(m), \ldots, 2^{j_0(m,n)}b(m)}^{2^{j(m)-j_0(m,n)}\text{ times}}, \tag{3.13}$$

which preserves the weight since

$$m = (2^{j(m)-j_0(m,n)})(2^{j_0(m,n)}b(m)).$$

Recall that $j_0(m,n)$ is the smallest nonnegative integer j_0 such that

$$b(m)2^{j_0} \geq n+1;$$

we see that each image part is $\geq n+1$. If an image part $2^{j_0(m,n)}b(m)$ is even then $j_0(m,n) \geq 1$ and

$$2^{j_0(m,n)}b(m) \leq 2n,$$

as noted before, so that each even image part is $\leq 2n$. Also any odd image part satisfies $2^{j_0(m,n)}b(m) = b(m) \geq n+1$. This induces a well-defined map

$$\Psi_n : \mathcal{D}_n \to \mathcal{P}_n.$$

We show this map is onto. Let λ be a partition in $\mathcal{P}_n$. Let p be a part of λ and let μ_p denote its multiplicity. Then we write

$$p = b(p)2^{j_0(p,n)} \geq n+1,$$

and note $j(p) = j_0(p,n)$ since p is a part of λ and $\lambda \in \mathcal{P}_n$. Now we write μ_p in binary form:

$$\mu_p = \sum_a 2^{\mu_p(a)}.$$

This part p with multiplicity μ_p arises from a partition in $\mathcal{D}_n$ with parts $b(p)2^{j_0(p,n)+\mu_p(a)}$ under the action of Ψ_n. We see that Ψ_n is onto and Theorem 3.13 implies that it is a weight-preserving bijection.

Next we prove (3.12). We let

$$\widetilde{k} = \#\text{ of parts} \leq 2n-1 \text{ of } \lambda.$$

We note that if m is a part of π then as before

$$m = b(m)2^{j(m)} \geq n+1,$$

and $j(m) \geq j_0(m,n)$. Under the map Ψ_n the image of m is given by (3.13). This contributes $2^{j(m)-j_0(m,n)}$ to $\widetilde{k}$ provided $b(m)2^{j_0(m,n)} < 2n$, and (3.12) follows.

Example 3.16 ($n=3$). We illustrate the bijection Ψ_n when $n=3$. There are six partitions of 16 in $\mathcal{D}_3$, the set of partitions into distinct parts ≥ 4:

$$\begin{array}{llll}
7+5+4 & \to 7\cdot 2^0, 5\cdot 2^0, 1\cdot 2^2 & \to 7\cdot 2^0, 5\cdot 2^0, 1\cdot 2^2 & \to 7+5+4,\\
9+7 & \to 9\cdot 2^0, 7\cdot 2^0 & \to 9\cdot 2^0, 7\cdot 2^0 & \to 9+7,\\
10+6 & \to 5\cdot 2^1, 3\cdot 2^1 & \to 5\cdot 2^0, 5\cdot 2^0, 3\cdot 2^1 & \to 6+5+5,\\
11+5 & \to 11\cdot 2^0, 5\cdot 2^0 & \to 11\cdot 2^0, 5\cdot 2^0 & \to 11+5,\\
12+4 & \to 3\cdot 2^2, 1\cdot 2^2 & \to 3\cdot 2^1, 3\cdot 2^1, 1\cdot 2^2 & \to 6+6+4,\\
16 & \to 1\cdot 2^4 & \to 1\cdot 2^2, 1\cdot 2^2, 1\cdot 2^2, 1\cdot 2^2 & \to 4+4+4+4.
\end{array}$$

Each partition has been mapped into $\mathcal{P}_3$, the set of partitions with smallest part ≥ 4 and all parts > 6 odd.

We can now construct our weight-preserving bijection $\Phi : \mathcal{M} \to \overline{\mathrm{SP}}$. Suppose (π, j) is a marked overpartition with $1 \leq j \leq \nu(\pi)$. As described before, we let π_1 be the partition formed by the nonoverlined parts of π, and π_2 be the partition (into distinct parts) formed by the overlined parts of π, so that

$$s(\pi_2) > s(\pi_1) = s(\pi) = n.$$

We let

$$\pi_1 = (\overbrace{n, \ldots, n}^{\nu}, n_2, n_3, \ldots, n_a), \qquad \pi_2 = (m_1, m_2, \ldots, m_b),$$

where

$$n < n_2 \leq n_3 \leq \cdots \leq n_a, \qquad n < m_1 < m_2 < \cdots < m_b.$$

Define

$$\Phi(\pi, j) = \vec{\lambda} = (\lambda_1, \lambda_2),$$

where

$$\lambda_1 = (\overbrace{n, \ldots, n}^{j}, n_2, n_3, \ldots, n_a), \qquad \lambda_2 = (\overbrace{n, \ldots, n}^{\nu - j}, \Psi_n(\pi_2)). \tag{3.14}$$

The map Φ is clearly weight-preserving. We see that $s(\lambda_1) = n$ and $\lambda_1 \in \mathcal{P}$. In addition, $\Psi_n(\pi_2)$ is a partition into parts $\geq n+1$ with all parts $\geq 2n+1$ being odd so that $\lambda_2 \in \overline{\mathrm{SP}}$ and the map Φ is well-defined. By (3.3) and (3.8) we need only show that Φ is onto.

Let

$$\vec{\lambda} = (\lambda_1, \lambda_2) \in \overline{\mathrm{SP}}.$$

Let $n = s(\lambda_1)$ so that $\lambda_1, \lambda_2 \in \mathcal{P}$, $s(\lambda_2) \geq s(\lambda_1) = n$ and all parts $\geq 2n+1$ of λ_2 are odd. Let $j = \nu(\lambda_1)$, and let ℓ denote the number of parts of λ_2 that are equal to n, so that $j \geq 1$ and $\ell \geq 0$. Remove any parts of λ_2 equal to n to form the partition $\widetilde{\lambda}_2$ and add the parts removed from λ_2 to λ_1 to form the partition π_1. Now let $\pi_2 = \Psi_n^{-1}(\widetilde{\lambda}_2)$ so that π_2 is a partition into distinct parts $\geq n+1$. Form the partition π by overlining the parts of π_2 and adding them to π_1. We see that $(\pi, j) \in \mathcal{M}$, $1 \leq j \leq \nu(\pi) = j + \ell$ and

$$\Phi(\pi, j) = \vec{\lambda} = (\lambda_1, \lambda_2).$$

The map Φ is onto and hence a bijection.

Now we prove (3.10), (3.11). As before, we let

$$\Phi(\pi, j) = \vec{\lambda} = (\lambda_1, \lambda_2),$$

where λ_1, λ_2 are given in (3.14), so that $s(\lambda_1) = n$, and $1 \leq j \leq \nu(\pi) = \nu(\pi_1)$. Then

$$\begin{aligned} k(\vec{\lambda}) &= \nu - j + (\#\text{ of parts} \leq 2n-1 \text{ of } \Psi_n(\pi_2)) \\ &= \nu(\pi_1) - j + k(\pi_2, n) \quad \text{(by (3.12))} \\ &= \nu(\pi_1) - j + k(\pi_2, s(\pi_1)) = \overline{k}(\pi, j), \end{aligned}$$

which proves (3.10). Finally, from (3.4) we have

$$\overline{\text{crank}}(\vec{\lambda}) = \begin{cases} (\#\text{ of parts} \geq s(\pi_1) + \overline{k} \text{ of } \pi_1) - \overline{k} & \text{if } \overline{k} > 0, \\ (\#\text{ of parts of } \pi_1) - 1 & \text{if } \overline{k} = 0, \end{cases}$$

where $\overline{k} = \overline{k}(\pi, j)$, since $\overline{k}(\pi, j) = k(\vec{\lambda})$, and if $\overline{k} = \overline{k}(\pi, j) = 0$, then $\nu(\pi_1) = j$ and $k(\pi_2, s(\pi_1)) = 0$, in which case the number of parts of π_1 equals the number of parts of λ_1. Hence we have

$$\overline{\text{crank}}(\vec{\lambda}) = \overline{\text{sptcrank}}(\pi, j),$$

which is (3.11). This completes the proof of our main result.

4. Proofs of Theorems 2.1–2.4. These four proofs all follow the same method. The generating function for the rank series is rewritten using Watson's transformation and then the two-variable series matches the difference of a rank and crank by Bailey's Lemma.

We recall that a pair of sequences of functions, (α_n, β_n), forms a *Bailey pair* for (a, q) if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}.$$

The limiting case of Bailey's Lemma shows for a Bailey pair (α_n, β_n) that

$$\sum_{n=0}^{\infty} (\rho_1, \rho_2; q)_n \left(\frac{aq}{\rho_1\rho_2}\right)^n \beta_n = \frac{(aq/\rho_1, aq/\rho_2; q)_\infty}{(aq, aq/\rho_1\rho_2; q)_\infty} \sum_{n=0}^{\infty} \frac{(\rho_1, \rho_2; q)_n \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n}{(aq/\rho_1, aq/\rho_2; q)_n}.$$

Proof of Theorem 2.1. We use the Bailey pair E(1) of [30, p. 469] for $(1, q)$ given by

$$\alpha_n = \begin{cases} 1, & n = 0, \\ (-1)^n 2q^{n^2}, & n \geq 1, \end{cases} \qquad \beta_n = \frac{1}{(q^2;q^2)_n}.$$

Then by Bailey's Lemma we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(z, z^{-1}; q)_n q^n}{(-q, q; q)_n} &= \frac{(zq, z^{-1}q; q)_\infty}{(q, q; q)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(z, z^{-1}; q)_n (-1)^n 2q^{n^2+n}}{(zq, z^{-1}q; q)_n}\right) \\ &= \frac{(zq, z^{-1}q; q)_\infty}{(q, q; q)_\infty} \left(1 + 2\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}\right). \end{aligned}$$

But then

$$
\begin{aligned}
\overline{\mathrm{S}}(z,q) &= \sum_{n=1}^{\infty} \frac{q^n(-q^{n+1},q^{n+1};q)_\infty}{(zq^n,z^{-1}q^n;q)_\infty} \\
&= \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty}\sum_{n=0}^{\infty}\frac{(z,z^{-1};q)_n q^n}{(-q,q;q)_n} - \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{(-q,q,zq,z^{-1}q;q)_\infty}{(z,z^{-1},q,q;q)_\infty}\left(1+2\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}\right) \\
&\quad - \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{(-q;q)_\infty}{(1-z)(1-z^{-1})(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}\right) \\
&\quad - \frac{(q^2;q^2)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{1}{(1-z)(1-z^{-1})} \\
&\quad \times \left(\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{N}(m,n)z^m q^n - \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{M}(m,n)z^m q^n\right). \blacksquare
\end{aligned}
$$

Proof of Theorem 2.2. Before using a Bailey pair, we will apply a limiting case of Watson's transformation to the generating function of $N2(m,n)$. We recall that Watson's transformation gives

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(aq/bc,d,e;q)_n\left(\frac{aq}{de}\right)^n}{(q,aq/b,aq/c;q)_n} \\
&\quad = \frac{(aq/d,aq/e;q)_\infty}{(aq,aq/de;q)_\infty}\sum_{n=0}^{\infty}\frac{(a,\sqrt{a}q,-\sqrt{a}q,b,c,d,e;q)_n(aq)^{2n}(-1)^n q^{n(n-1)/2}}{(q,\sqrt{a},-\sqrt{a},aq/b,aq/c,aq/d,aq/e;q)_n(bcde)^n}.
\end{aligned}
$$

Applying this with $q\mapsto q^2$, $a=1$, $b=z$, $c=z^{-1}$, $d=-q$ and $e\to\infty$ we get

$$
\begin{aligned}
\sum_{n=0}^{\infty} q^{n^2}\frac{(-q;q^2)_n}{(zq^2;q^2)_n(z^{-1}q^2;q^2)_n} &= \lim_{e\to\infty}\sum_{n=0}^{\infty}\frac{(q^2,-q,e;q^2)_n(-1)^n e^{-n}q^n}{(q^2,z^{-1}q^2,zq^2;q^2)_n} \\
= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}&\left(1+\lim_{a\to1,\,e\to\infty}\sum_{n=1}^{\infty}\frac{(1-a)(-q^2,z,z^{-1},e;q^2)_n q^{n^2+2n}}{(1-\sqrt{a})(-1,z^{-1}q^2,zq^2;q^2)_n e^n}\right) \\
= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}&\left(1+\sum_{n=1}^{\infty}\frac{(1+q^{2n})(1-z)(1-z^{-1})(-1)^n q^{2n^2+n}}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right).
\end{aligned}
$$

Using one of the unlabeled Bailey pairs in [30, p. 468] we have, after replacing q by q^2, a Bailey pair for $(1, q^2)$ given by

$$\alpha_n = \begin{cases} 1, & n = 0, \\ (-1)^n q^{2n^2}(q^n + q^{-n}), & n \geq 1, \end{cases}$$
$$\beta_n = \frac{1}{(-q, q^2; q^2)_n}.$$

Then by Bailey's Lemma we deduce that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(z, z^{-1}; q^2)_n q^{2n}}{(-q, q^2; q^2)_n} \\
&\quad = \frac{(zq^2, z^{-1}q^2; q^2)_\infty}{(q^2, q^2; q^2)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(z, z^{-1}; q^2)_n (-1)^n q^{2n^2+2n}(q^n + q^{-n})}{(zq^2, z^{-1}q^2; q^2)_n}\right) \\
&\quad = \frac{(zq^2, z^{-1}q^2; q^2)_\infty}{(q^2, q^2; q^2)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{2n^2+2n}(q^n + q^{-n})}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right).
\end{aligned}$$

But then

$$\begin{aligned}
\mathrm{S2}(z, q) &= \sum_{n=1}^{\infty} \frac{q^{2n}(q^{2n+2}, -q^{2n+1}; q^2)_\infty}{(zq^{2n}, z^{-1}q^{2n}; q^2)_\infty} \\
&= \frac{(-q, q^2; q^2)_\infty}{(z, z^{-1}; q^2)_\infty} \sum_{n=0}^{\infty} \frac{(z, z^{-1}; q^2)_n q^{2n}}{(-q, q^2; q^2)_n} - \frac{(-q, q^2; q^2)_\infty}{(z, z^{-1}; q^2)_\infty} \\
&= \frac{(-q, q^2, zq^2, z^{-1}q^2; q^2)_\infty}{(z, z^{-1}, q^2, q^2; q^2)_\infty} \\
&\quad \times \left(1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{2n^2+2n}(q^n + q^{-n})}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right) - \frac{(-q, q^2; q^2)_\infty}{(z, z^{-1}; q^2)_\infty} \\
&= \frac{(-q; q^2)_\infty}{(1-z)(1-z^{-1})(q^2; q^2)_\infty} \\
&\quad \times \left(1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{2n^2+n}(q^{2n} + 1)}{(1-zq^{2n})(1-z^{-1}q^{2n})}\right) \\
&\quad - \frac{(-q, q^2; q^2)_\infty}{(1-z)(1-z^{-1})(zq^2, z^{-1}q^2; q^2)_\infty} \\
&= \frac{1}{(1-z)(1-z^{-1})} \left(\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N2(m, n) z^m q^n - M2(m, n) z^m q^n\right). \quad \blacksquare
\end{aligned}$$

Proof of Theorem 2.3. We have

$$\begin{aligned}
\overline{\mathrm{S}}_2(z,q) &= \sum_{n=1}^{\infty} \frac{q^{2n}(-q^{2n+1},q^{2n+1};q)_\infty}{(zq^{2n},z^{-1}q^{2n};q)_\infty} \\
&= \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty} \sum_{n=1}^{\infty} \frac{q^{2n}(z,z^{-1};q)_{2n}}{(-q,q;q)_{2n}} \\
&= \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty} \sum_{n=0}^{\infty} \frac{q^{2n}(z,z^{-1};q)_{2n}}{(-q,q;q)_{2n}} - \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty}.
\end{aligned}$$

Using the Bailey pair in proof of Theorem 2.1 along with the Bailey pair E(2) for $(1,q)$ given by

$$\alpha_n = \begin{cases} 1, & n=0, \\ 2(-1)^n, & n \ge 1, \end{cases} \qquad \beta_n = \frac{(-1)^n}{(-q,q;q)_n}$$

(see [30, p. 469]), we have the Bailey pair

$$\begin{aligned}
\alpha_n &= \begin{cases} 1, & n=0, \\ (-1)^n(1+q^{n^2}), & n \ge 1, \end{cases} \\
\beta_n &= \frac{1}{2(q^2;q^2)_n} + \frac{(-1)^n}{2(q^2;q^2)_n} = \begin{cases} 1/(q^2;q^2)_n, & n \equiv 0 \pmod 2, \\ 0, & n \equiv 1 \pmod 2. \end{cases}
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{2n}(z,z^{-1};q)_{2n}}{(-q,q;q)_{2n}} \\
&\quad = \sum_{n=0}^{\infty} (z,z^{-1};q)_n q^n \beta_n = \frac{(zq,z^{-1}q;q)_\infty}{(q,q;q)_\infty} \sum_{n=0}^{\infty} \frac{(z,z^{-1};q)_n q^n \alpha_n}{(zq,z^{-1}q;q)_n} \\
&\quad = \frac{(zq,z^{-1}q;q)_\infty}{(q,q;q)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^n (1+q^{n^2})}{(1-zq^n)(1-z^{-1}q^n)} \right).
\end{aligned}$$

And so

$$\begin{aligned}
&\overline{\mathrm{S}}_2(z,q) \\
&= \frac{(-q;q)_\infty}{(1-z)(1-z^{-1})(q;q)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^n (1+q^{n^2})}{(1-zq^n)(1-z^{-1}q^n)} \right) \\
&\quad - \frac{(q^2;q^2)_\infty}{(1-z)(1-z^{-1})(zq,z^{-1}q;q)_\infty}. \blacksquare
\end{aligned}$$

Proof of Theorem 2.4. With $\overline{\mathrm{S}}(z,q)$ and $\overline{\mathrm{S}}_2(z,q)$ known, we also know $\overline{\mathrm{S}}_1(z,q)$. However, we can also derive the result from a Bailey pair as we have for the other series.

We start from

$$\overline{\mathrm{S}}_1(z,q)=\sum_{n=0}^{\infty}\frac{q^{2n+1}(-q^{2n+2},q^{2n+2};q)_\infty}{(zq^{2n+1},z^{-1}q^{2n+1};q)_\infty}$$
$$=\frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty}\sum_{n=0}^{\infty}\frac{(z,z^{-1};q)_{2n+1}q^{2n+1}}{(-q,q;q)_{2n+1}}.$$

By combining Bailey pairs as we did for $\overline{\mathrm{S}}_2(z,q)$, we have a Bailey pair for $(1,q)$ given by

$$\alpha_n=\begin{cases}0, & n=0,\\ (-1)^n(q^{n^2}-1), & n\geq 1,\end{cases}$$
$$\beta_n=\frac{1}{2(q^2;q^2)_n}-\frac{(-1)^n}{2(q^2;q^2)_n}=\begin{cases}0, & n\equiv 0 \pmod 2,\\ 1/(q^2;q^2)_n, & n\equiv 1 \pmod 2.\end{cases}$$

By Bailey's Lemma we then see that

$$\sum_{n=0}^{\infty}\frac{(z,z^{-1};q)_{2n+1}q^{2n+1}}{(-q,q;q)_{2n+1}}=\frac{(zq,z^{-1};q)_\infty}{(q,q;q)_\infty}\sum_{n=0}^{\infty}\frac{(z,z^{-1};q)_n q^n\alpha_n}{(zq,z^{-1}q;q)_n}$$
$$=\frac{(zq,z^{-1};q)_\infty}{(q,q;q)_\infty}\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})q^n(-1)^n(q^{n^2}-1)}{(1-zq^n)(1-z^{-1}q^n)}.$$

This gives

$$\overline{\mathrm{S}}_1(z,q)=\frac{(-q;q)_\infty}{(1-z)(1-z^{-1})(q;q)_\infty}\times\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})q^n(-1)^n(q^{n^2}-1)}{(1-zq^n)(1-z^{-1}q^n)}. \tag{4.1}$$ ∎

As pointed out by the referee, it is also possible to deduce these identities from Watson's transformation, rather than from Bailey pairs and Bailey's Lemma.

5. Dissections

Proofs of Theorems 2.5 and 2.7. We are to show

$$\sum_{n=0}^{\infty}\sum_{r=0}^{2}\overline{N}(r,3,3n)\zeta_3^r q^n=\frac{(q^3;q^3)_\infty^4(q^2;q^2)_\infty}{(q;q)_\infty^2(q^6;q^6)_\infty^2}, \tag{5.1}$$

$$\sum_{n=0}^{\infty}\sum_{r=0}^{2}\overline{N}(r,3,3n+1)\zeta_3^r q^n=2\frac{(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q;q)_\infty}, \tag{5.2}$$

$$(5.3)\qquad \sum_{n=0}^{\infty}\sum_{r=0}^{2} N2(r,3,3n+1)\zeta_3^r q^n = \frac{(q^6;q^6)_\infty^4}{(q^2;q^2)_\infty(q^3;q^3)_\infty(q^{12};q^{12})_\infty}.$$

For (5.1) we have

$$(5.4)\qquad \sum_{n=0}^{\infty}\big(\overline{N}(0,3,3n)+\overline{N}(1,3,3n)\zeta_3+\overline{N}(2,3,3n)\zeta_3^2\big)q^n = \sum_{n=0}^{\infty}(\overline{N}(0,3,3n)-\overline{N}(1,3,3n))q^n = \frac{(q^3;q^3)_\infty^2(-q;q)_\infty}{(q;q)_\infty(-q^3;q^3)_\infty^2} = \frac{(q^3;q^3)_\infty^4(q^2;q^2)_\infty}{(q;q)_\infty^2(q^6;q^6)_\infty^2}.$$

The penultimate equality in (5.4) is the first part of Theorem 1.1 of [23], although we have omitted their -1 term. The -1 is due to how one interprets the empty overpartition and its rank. We use the convention that the empty overpartition has rank 0 and do not adjust the q^0 term of the generating function.

Equations (5.2) and (5.3) are also just restatements of results in [23] and [24], respectively. ∎

Proofs of Theorems 2.6 and 2.8. We see we have to prove

$$(5.5)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{4}(\overline{N}(k,5,5n)\zeta_5^k)q^n = \frac{(q^4,q^6;q^{10})_\infty(q^5;q^5)_\infty^2}{(q^2,q^3;q^5)_\infty^2(q^{10};q^{10})_\infty} + 2(\zeta_5+\zeta_5^{-1})q\frac{(q^{10};q^{10})_\infty}{(q^3,q^4,q^6,q^7;q^{10})_\infty},$$

$$(5.6)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{4}(\overline{N}(k,5,5n+3)\zeta_5^k)q^n = \frac{2(1-\zeta_5-\zeta_5^{-1})(q^{10};q^{10})_\infty}{(q^2,q^3;q^5)_\infty},$$

$$(5.7)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{4}(N2(k,5,5n+1)\zeta_5^k)q^n = \frac{(-q^5,q^{10};q^{10})_\infty}{(q^2,q^8;q^{10})_\infty},$$

$$(5.8)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{4}(N2(k,5,5n+3)\zeta_5^k)q^n = (\zeta_5+\zeta_5^4)\frac{(-q^5,q^{10};q^{10})_\infty}{(q^4,q^6;q^{10})_\infty}.$$

But we observe that

$$\begin{aligned}\overline{N}(0,5,5n)&+\overline{N}(1,5,5n)\zeta_5+\overline{N}(2,5,5n)\zeta_5^2+\overline{N}(3,5,5n)\zeta_5^3+\overline{N}(4,5,5n)\zeta_5^4\\ &=\overline{N}(0,5,5n)+\overline{N}(1,5,5n)(\zeta_5+\zeta_5^4)+\overline{N}(2,5,5n)(\zeta_5^2+\zeta_5^3)\\ &=\overline{N}(0,5,5n)-\overline{N}(2,5,5n)+(\zeta_5+\zeta_5^4)(\overline{N}(1,5,5n)-\overline{N}(2,5,5n)).\end{aligned}$$

By the difference formulas in [23] we then have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{4}&(\overline{N}(k,5,5n)\zeta_5^k)q^n\\&=\frac{(-q^2,-q^3;q^5)_\infty(q^5;q^5)_\infty}{(q^2,q^3;q^5)_\infty(-q^5;q^5)_\infty}+\frac{2(\zeta_5+\zeta_5^4)q(q^{10};q^{10})_\infty}{(q^3,q^4,q^6,q^7;q^{10})_\infty}\\&=\frac{(q^4,q^6;q^{10})_\infty(q^5;q^5)_\infty^2}{(q^2,q^3;q^5)_\infty^2(q^{10};q^{10})_\infty}+2(\zeta_5+\zeta_5^{-1})q\frac{(q^{10};q^{10})_\infty}{(q^3,q^4,q^6,q^7;q^{10})_\infty}.\end{aligned}$$

Equations (5.6)–(5.8) are also just restatements of results in [23] and [24]. ∎

Proof of Theorem 2.9. By definition we have

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{M}(m,n)\zeta_3^m q^n=\frac{(q^2;q^2)_\infty}{(\zeta_3 q;q)_\infty(\zeta_3^{-1}q;q)_\infty}=\frac{(q^2;q^2)_\infty(q;q)_\infty}{(q^3;q^3)_\infty}.$$

We see we have to show

$$\begin{aligned}\frac{(q^2;q^2)_\infty(q;q)_\infty}{(q^3;q^3)_\infty}&=\frac{(q^9;q^9)_\infty^4(q^6;q^6)_\infty}{(q^3;q^3)_\infty^2(q^{18};q^{18})_\infty^2}-q\frac{(q^{18};q^{18})_\infty(q^9;q^9)_\infty}{(q^3;q^3)_\infty}\\&\quad-2q^2\frac{(q^{18};q^{18})_\infty^4}{(q^9;q^9)_\infty^2(q^6;q^6)_\infty}.\end{aligned}\tag{5.9}$$

After replacing q by $q^{1/3}$ and multiplying by $\frac{(q;q)_\infty}{(q^3;q^3)_\infty(q^6;q^6)_\infty}$, the proposition is equivalent to

$$\begin{aligned}\frac{(q^{1/3};q^{1/3})_\infty(q^{2/3};q^{2/3})_\infty}{(q^3;q^3)_\infty(q^6;q^6)_\infty}&=\frac{(q^3;q^3)_\infty^3(q^2;q^2)_\infty}{(q;q)_\infty(q^6;q^6)_\infty^3}-q^{1/3}\\&\quad-2q^{2/3}\frac{(q;q)_\infty(q^6;q^6)_\infty^3}{(q^3;q^3)_\infty^3(q^2;q^2)_\infty}.\end{aligned}\tag{5.10}$$

If we let v be the infinite continued fraction

$$v=\cfrac{q^{1/3}}{1+\cfrac{q+q^2}{1+\cfrac{q^2+q^4}{1+\cfrac{q^3+q^6}{1+\ddots}}}}$$

then by Entry 3.3.1(a) of Ramanujan's Lost Notebook Part I (see [3]) we have

$$v=q^{1/3}\frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3}.$$

Thus with $x(q)=q^{-1/3}v$ we get

$$x(q)=\frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3}=\frac{(q;q)_\infty(q^6;q^6)_\infty}{(q^3;q^3)_\infty^3(q^2;q^2)_\infty}.$$

But now (5.10) is exactly Theorem 2 of [16]. ∎

Proof of Theorem 2.10. We have

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{M}(m,n)\zeta_5^m q^n = \frac{(q^2;q^2)_\infty}{(\zeta_5 q,\zeta_5^{-1}q;q)_\infty}$$

and so we need to find a dissection for this product.

By Lemma 3.9 of [19] we have

$$\frac{1}{(\zeta_5 q,\zeta_5^{-1}q;q)_\infty} = \frac{1}{(q^5,q^{20};q^{25})_\infty} + \frac{(\zeta_5+\zeta_5^{-1})q}{(q^{10},q^{15};q^{25})_\infty}.$$

Replacing q by q^2 in [19, Lemma 3.18] we have

$$(q^2;q^2)_\infty = (q^{50};q^{50})_\infty\left(\frac{(q^{20},q^{30};q^{50})_\infty}{(q^{10},q^{40};q^{50})_\infty} - q^2 - q^4\frac{(q^{10},q^{40};q^{50})_\infty}{(q^{20},q^{30};q^{50})_\infty}\right).$$

Expanding the product of these two expressions then gives the result. ∎

Proof of Theorem 2.11. We see

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M2(m,n)\zeta_3^m q^n = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty^2}{(q^6;q^6)_\infty}.$$

We then need to show

$$\begin{aligned}\frac{(-q;q^2)_\infty(q^2;q^2)_\infty^2}{(q^6;q^6)_\infty} &= \frac{(q^{18};q^{18})_\infty^{10}(q^{12};q^{12})_\infty(q^3;q^3)_\infty}{(q^{36};q^{36})_\infty^4(q^9;q^9)_\infty^4(q^6;q^6)_\infty^3} + q\frac{(q^{18};q^{18})_\infty^4}{(q^{36};q^{36})_\infty(q^9;q^9)_\infty(q^6;q^6)_\infty}\\ &\quad - 2q^2\frac{(q^{36};q^{36})_\infty^2(q^9;q^9)_\infty^2(q^6;q^6)_\infty}{(q^{18};q^{18})_\infty^2(q^{12};q^{12})_\infty(q^3;q^3)_\infty}.\end{aligned}$$

Noting $(-q;q^2)_\infty(q^2;q^2)_\infty^2 = (-q;-q)_\infty(q^2;q^2)_\infty$, in equation (5.9) we replace q by $-q$ and multiply by $(-q^3;-q^3)_\infty/(q^6;q^6)_\infty$ to get

$$\begin{aligned}\frac{(-q;q^2)_\infty(q^2;q^2)_\infty^2}{(q^6;q^6)_\infty} &= \frac{(-q^9;-q^9)_\infty^4}{(-q^3;-q^3)_\infty(q^{18};q^{18})_\infty^2} + q\frac{(q^{18};q^{18})_\infty(-q^9;-q^9)_\infty}{(q^6;q^6)_\infty}\\ &\quad - 2q^2\frac{(q^{18};q^{18})_\infty^4(-q^3;-q^3)_\infty}{(-q^9;-q^9)_\infty^2(q^6;q^6)_\infty^2}.\end{aligned} \tag{5.11}$$

But we have

$$(-q^3;-q^3)_\infty = \frac{(q^6;q^6)_\infty^3}{(q^{12};q^{12})_\infty(q^3;q^3)_\infty}, \tag{5.12}$$

$$(-q^9;-q^9)_\infty = \frac{(q^{18};q^{18})_\infty^3}{(q^{36};q^{36})_\infty(q^9;q^9)_\infty}. \tag{5.13}$$

Equations (5.12) and (5.13) with (5.11) then give the theorem. ∎

Proof of Theorem 2.12. We have

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M2(m,n)\zeta_5^m q^n = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(\zeta_5 q^2, \zeta_5^{-1}q^2;q^2)_\infty}$$

and so we need a dissection for this product.

Replacing q by q^2 in Lemma 3.9 of [19] we have

$$\frac{1}{(\zeta_5 q^2,\zeta_5^{-1}q^2;q^2)_\infty} = \frac{1}{(q^{10},q^{40};q^{50})_\infty} + \frac{(\zeta_5+\zeta_5^{-1})q^2}{(q^{20},q^{30};q^{50})_\infty}.$$

Next we note that $(-q;q^2)_\infty(q^2;q^2)_\infty = (-q;-q)_\infty$ and so replacing q by $-q$ in Lemma 3.18 in [19] we get

$$\begin{aligned}
&(-q;q^2)_\infty (q^2;q^2)_\infty \\
&\quad = (-q^{25};-q^{25})_\infty \left(\frac{(q^{10},-q^{15};-q^{25})_\infty}{(-q^5,q^{20};-q^{25})_\infty} + q - q^2\frac{(-q^5,q^{20};-q^{25})_\infty}{(q^{10},-q^{15};-q^{25})_\infty}\right) \\
&\quad = (-q^{25},q^{50};q^{50})_\infty \left(\frac{(q^{10},-q^{15},-q^{35},q^{40};q^{50})_\infty}{(-q^5,q^{20},q^{30},-q^{45};q^{50})_\infty} + q \right. \\
&\qquad\qquad \left. - q^2\frac{(-q^5,q^{20},q^{30},-q^{45};q^{50})_\infty}{(q^{10},-q^{15},-q^{35},q^{40};q^{50})_\infty}\right).
\end{aligned}$$

Multiplying out these two 5-dissections then gives

$$\begin{aligned}
&\frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(\zeta_5 q,\zeta_5^{-1}q;q)_\infty} \\
&= \frac{(-q^{15},-q^{25},-q^{35},q^{50};q^{50})_\infty}{(-q^5,q^{20},q^{30},-q^{45};q^{50})_\infty} + q\frac{(-q^{25},q^{50};q^{50})_\infty}{(q^{10},q^{40};q^{50})_\infty} \\
&\quad + q^2(\zeta_5+\zeta_5^4)\frac{(q^{10},-q^{15},-q^{25},-q^{35},q^{40},q^{50};q^{50})_\infty}{(-q^5,q^{20},q^{20},q^{30},q^{30},-q^{45};q^{50})_\infty} \\
&\quad - q^2\frac{(-q^5,q^{20},-q^{25},q^{30},-q^{45},q^{50};q^{50})_\infty}{(q^{10},q^{10},-q^{15},-q^{35},q^{40},q^{40};q^{50})_\infty} + q^3(\zeta_5+\zeta_5^4)\frac{(-q^{25},q^{50};q^{50})_\infty}{(q^{20},q^{30};q^{50})_\infty} \\
&\quad - q^4(\zeta_5+\zeta_5^4)\frac{(-q^5,-q^{25},-q^{45},q^{50};q^{50})_\infty}{(q^{10},-q^{15},-q^{35},q^{40};q^{50})_\infty}. \ \blacksquare
\end{aligned}$$

Proof of Theorem 2.13. We will use Ramanujan's functions

$$f(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \qquad \phi(q) = f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2}.$$

By Entry 19 in Chapter 16 of [12] we have

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Also,

$$\phi(-q)=\frac{(q;q)_\infty}{(-q;q)_\infty}=\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}. \tag{5.14}$$

PROPOSITION 1.

$$\sum_{n=1}^{\infty}\frac{(-1)^n q^n(1-q^n)}{1-q^{3n}}=\frac{-1}{6}\left(1-\frac{(q;q)_\infty^6(q^6;q^6)_\infty}{(q^2;q^2)_\infty^3(q^3;q^3)_\infty^2}\right) \tag{5.15}$$

$$=\frac{-1}{6}\left(1-\frac{\phi(-q)^3}{\phi(-q^3)}\right). \tag{5.16}$$

Proof. As in [18] we let

$$E_r(N;m)=\sum_{\substack{d\mid N\\ d\equiv r \pmod m}}1-\sum_{\substack{d\mid N\\ d\equiv -r \pmod m}}1.$$

Thus

$$\sum_{N=1}^{\infty}q^N E_r(N;m)=\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}q^{kmn+rn}-q^{kmn+(m-r)n}=\sum_{n=1}^{\infty}\frac{q^{rn}-q^{(m-r)n}}{1-q^{mn}}.$$

Similarly,

$$\sum_{N=1}^{\infty}q^N(E_r(N;m)-2E_r(N/2;m))=\sum_{n=1}^{\infty}\frac{(-1)^n(q^{rn}-q^{(m-r)n})}{1-q^{mn}}.$$

Then equality (5.15) is given by equation (32.64) of [18], and (5.16) follows from (5.14). ∎

With this we then have

$$\begin{aligned}
&\frac{(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(1-\zeta_3)(1-\zeta_3^{-1})(-1)^n q^n}{(1-\zeta_3 q^n)(1-\zeta_3^{-1}q^n)}\right) \qquad (5.17)\\
&=\frac{(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}+3\sum_{n=1}^{\infty}\frac{(-1)^n q^n(1-q^n)}{(1-q^{3n})}\right)\\
&=\frac{(-q;q)_\infty\phi(-q)^3}{2(q;q)_\infty\phi(-q^3)}=\frac{(q;q)_\infty^2(-q^3;q^3)_\infty}{2(-q;q)_\infty^2(q^3;q^3)_\infty}=\frac{(q;q)_\infty^2(q^6;q^6)_\infty}{2(-q;q)_\infty^2(q^3;q^3)_\infty^2}.
\end{aligned}$$

PROPOSITION 2.

$$\begin{aligned}
\frac{(q;q)_\infty^2}{(-q;q)_\infty^2}&=\frac{(q^9;q^9)_\infty^4}{(q^{18};q^{18})_\infty^2}-4q\frac{(q^{18};q^{18})_\infty(q^9;q^9)_\infty(q^3;q^3)_\infty}{(q^6;q^6)_\infty}\\
&\quad+4q^2\frac{(q^{18};q^{18})_\infty^4(q^3;q^3)_\infty^2}{(q^9;q^9)_\infty^2(q^6;q^6)_\infty^2}.
\end{aligned}$$

Proof. By the Corollary (p. 49) to Entry 31 of [12], we have

$$\phi(q)=\phi(q^9)+2qf(q^3,q^{15}).$$

Replacing q by $-q$ we find that

$$\begin{aligned}\phi(-q) &= \frac{(q^9;q^9)_\infty^2}{(q^{18};q^{18})_\infty} - 2q(q^3,q^{15},q^{18};q^{18})_\infty \\ &= \frac{(q^9;q^9)_\infty^2}{(q^{18};q^{18})_\infty} - 2q\frac{(q^{18};q^{18})_\infty^2(q^3;q^3)_\infty}{(q^9;q^9)_\infty(q^6;q^6)_\infty}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{(q;q)_\infty^2}{(-q;q)_\infty^2} = \phi(-q)^2 &= \frac{(q^9;q^9)_\infty^4}{(q^{18};q^{18})_\infty^2} - 4q\frac{(q^{18};q^{18})_\infty(q^9;q^9)_\infty(q^3;q^3)_\infty}{(q^6;q^6)_\infty} \\ &\quad + 4q^2\frac{(q^{18};q^{18})_\infty^4(q^3;q^3)_\infty^2}{(q^9;q^9)_\infty^2(q^6;q^6)_\infty^2}. \ \blacksquare\end{aligned}$$

With (5.17) and Proposition 2, we have finished the proof of Theorem 2.13. ∎

Proof of Theorem 2.15. We can determine $\overline{\mathrm{S}}(i,q)$, $\overline{\mathrm{S}}_1(i,q)$, and $\overline{\mathrm{S}}_2(i,q)$ from formulas involving ϕ:

$$\phi(q) - \phi(-q) = 4\sum_{n=1}^{\infty} q^{(2n-1)^2}, \tag{5.18}$$

$$\phi(q)\phi(-q) = \phi(-q^2)^2, \tag{5.19}$$

$$\phi(-q^2)^2 = 1 + 4\sum_{n=1}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{2n}}, \tag{5.20}$$

$$\phi(-q)^2 = 1 + 4\sum_{n=1}^{\infty}\frac{(-1)^n q^n}{1+q^{2n}}. \tag{5.21}$$

These equalities can all be found in Ramanujan's Notebooks Part III by Berndt [12]. Equation (5.18) is by Entry 22(i) on p. 36, (5.19) is Entry 25(iii) on p. 40, (5.20) is Entry 8(v) on p. 114 with q replaced by q^2, and (5.21) is on p. 116 as part of the proof of Entry 8(v).

As in the proof of Theorem 2.1 we have

$$\begin{aligned}\overline{\mathrm{S}}(z,q) = &\frac{(-q;q)_\infty}{(1-z)(1-z^{-1})(q;q)_\infty}\left(1 + 2\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}\right) \\ &- \frac{(q^2;q^2)_\infty}{(z,z^{-1};q)_\infty},\end{aligned}$$

thus

$$\begin{aligned}\overline{\mathrm{S}}(i,q) &= \frac{(-q;q)_\infty}{2(q;q)_\infty}\left(1 + 4\sum_{n=1}^{\infty}\frac{(-1)^n q^{n^2+n}}{1+q^{2n}}\right) - \frac{(q^2;q^2)_\infty}{2(-q^2;q^2)_\infty} \\ &= \frac{\phi(-q^2)^2}{2\phi(-q)} - \frac{\phi(-q^2)}{2} = \frac{\phi(q)}{2} - \frac{\phi(-q^2)}{2} = \sum_{n=1}^{\infty} q^{n^2} - \sum_{n=1}^{\infty}(-1)^n q^{2n^2}.\end{aligned}$$

By (4.1) we have

$$\begin{aligned}\overline{\mathrm{S}}_1(i,q) &= \frac{1}{\phi(-q)}\sum_{n=1}^{\infty}\frac{(-1)^n q^n(q^{n^2}-1)}{1+q^{2n}} = \frac{1}{\phi(-q)}\left(\frac{\phi(-q^2)^2-\phi(-q)^2}{4}\right)\\ &= \frac{\phi(-q)}{\phi(-q)}\left(\frac{\phi(q)-\phi(-q)}{4}\right) = \sum_{n=1}^{\infty} q^{(2n-1)^2}.\end{aligned}$$

Lastly, since $\overline{\mathrm{S}}_2(z,q) = \overline{\mathrm{S}}(z,q) - \overline{\mathrm{S}}_1(z,q)$, we get

$$\overline{\mathrm{S}}_2(i,q) = \sum_{n=1}^{\infty} q^{(2n)^2} - \sum_{n=1}^{\infty}(-1)^n q^{2n^2}. \ \blacksquare$$

Proof of Theorem 2.14. To start we set

$$C(\tau) = 3 + 10\sum_{n=1}^{\infty}\frac{(-1)^n(q^n-q^{4n})}{1-q^{5n}},$$

$$D(\tau) = 1 + 10\sum_{n=1}^{\infty}\frac{(-1)^n(q^{2n}-q^{3n})}{(1-q^{5n})}.$$

We claim $C(\tau)$ and $D(\tau)$ are elements of $M_1(\Gamma_1(10))$. Here $M_k(\Gamma)$ is the vector space of holomorphic modular forms of weight k with respect to a congruence subgroup Γ of $\Gamma_0(1)$.

First we define a primitive Dirichlet character modulo 5 by

$$\chi_5(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{5},\\ i & \text{if } n \equiv 2 \pmod{5},\\ -i & \text{if } n \equiv 3 \pmod{5},\\ -1 & \text{if } n \equiv 4 \pmod{5},\\ 0 & \text{otherwise.}\end{cases}$$

We then also have a primitive Dirichlet character given by the conjugate $\overline{\chi_5}$.

As in [21] and [20] we set

$$\begin{aligned}V_{\chi_5,1}(\tau) &= \frac{3+i}{10} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\chi_5(n)q^{mn}\\ &= \frac{3+i}{10} + \sum_{m=1}^{\infty}\frac{q^m-q^{4m}}{1-q^{5m}} + i\sum_{m=1}^{\infty}\frac{q^{2m}-q^{3m}}{1-q^{5m}},\\ V_{\overline{\chi_5},1}(\tau) &= \frac{3-i}{10} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{\chi_5}(n)q^{mn}\\ &= \frac{3-i}{10} + \sum_{m=1}^{\infty}\frac{q^m-q^{4m}}{1-q^{5m}} - i\sum_{m=1}^{\infty}\frac{q^{2m}-q^{3m}}{1-q^{5m}}.\end{aligned}$$

Then $V_{\chi_5,1}(\tau) \in M_1(\Gamma_0(5), \chi_5)$ and $V_{\overline{\chi_5},1}(\tau) \in M_1(\Gamma_0(5), \overline{\chi_5})$. Thus we have $V_{\chi_5,1}(2\tau) \in M_1(\Gamma_0(10), \chi_5)$ and $V_{\overline{\chi_5},1}(2\tau) \in M_1(\Gamma_0(10), \overline{\chi_5})$. Here $M_k(\Gamma, \chi)$ is the vector space of holomorphic modular forms of weight k with respect to a congruence subgroup Γ of $\Gamma_0(1)$ and with character χ.

We see that

$$
\begin{aligned}
C(\tau) &= 5(2V_{\chi_5,1}(2\tau) - V_{\chi_5,1}(\tau) + 2V_{\overline{\chi_5},1}(2\tau) - V_{\overline{\chi_5},1}(\tau)),\\
D(\tau) &= -i5(2V_{\chi_5,1}(2\tau) - V_{\chi_5,1}(\tau) - 2V_{\overline{\chi_5},1}(2\tau) + V_{\overline{\chi_5},1}(\tau)),
\end{aligned}
$$

but since the characters are different, we must move from Γ_0 to Γ_1. That is, we have $C(\tau), D(\tau) \in M_1(\Gamma_1(10))$. Noting $\eta(2\tau)^2/\eta(\tau)^4$ is a modular form of weight -1 for $\Gamma_1(8)$, we then deduce that $C(\tau)\eta(2\tau)^2/\eta(\tau)^4$ and $D(\tau)\eta(2\tau)^2/\eta(\tau)^4$ are modular functions with respect to $\Gamma_1(40)$. Here by *modular function*, we mean a modular form of weight zero.

We use the following generalized eta notation as in [29]:

$$
\eta_{\delta,g}(\tau) = e^{\pi i P_2(g/\delta)\delta\tau} \prod_{\substack{m>0\\ m\equiv g \,(\mathrm{mod}\,\delta)}} (1-q^m) \prod_{\substack{m>0\\ m\equiv -g \,(\mathrm{mod}\,\delta)}} (1-q^m)
$$

where

$$
P_2(t) = \{t\}^2 - \{t\} + 1/6.
$$

So for $g = 0$ we have

$$
\eta_{\delta,0}(\tau) = q^{\delta/12}(q^\delta; q^\delta)_\infty^2 = \eta(\delta\tau)^2
$$

and for $0 < g < \delta$ we have

$$
\eta_{\delta,g}(\tau) = q^{P_2(g/\delta)\delta/2}(q^g, q^{\delta-g}; q^\delta)_\infty.
$$

PROPOSITION 3.

$$
C(\tau)\frac{\eta(2\tau)}{\eta(\tau)^2} = C_0(q^5) + qC_1(q^5) + q^2C_2(q^5) + q^3C_3(q^5) + q^4C_4(q^5) \tag{5.22}
$$

where

$$
\begin{aligned}
C_0(q) &= \frac{(q^5; q^5)_\infty^2 (q^2; q^2)_\infty^2}{(q^{10}; q^{10})_\infty (q; q)_\infty^4} C(\tau),\\
C_1(q) &= -4\frac{(q^4, q^6, q^{10}; q^{10})_\infty}{(q^2, q^8; q^{10})_\infty^2 (q^3, q^7; q^{10})_\infty},\\
C_2(q) &= 2\frac{(q^{10}; q^{10})_\infty}{(q^1, q^9; q^{10})_\infty (q^4, q^6; q^{10})_\infty},\\
C_3(q) &= -6\frac{(q^{10}; q^{10})_\infty}{(q^2, q^8; q^{10})_\infty (q^3, q^7; q^{10})_\infty},\\
C_4(q) &= 2\frac{(q^2, q^8, q^{10}; q^{10})_\infty}{(q, q^9; q^{10})_\infty (q^4, q^6; q^{10})_\infty^2}.
\end{aligned}
$$

Proof. Multiplying both sides of (5.22) by $\eta(2\tau)/\eta(\tau)^2$ and noting the powers of q from $\eta_{\delta,g}$ really do match, we see that this proposition is equivalent to

$$
\begin{aligned}
(5.23)\qquad C(\tau)\frac{\eta(2\tau)^2}{\eta(\tau)^4} &= \frac{\eta(2\tau)\eta(25\tau)^2\eta(10\tau)^2}{\eta(\tau)^2\eta(50\tau)\eta(5\tau)^4}C(5\tau)\\
&- 4\frac{\eta_{2,0}(\tau)^{1/2}\eta_{50,0}(\tau)^{1/2}\eta_{50,20}(\tau)}{\eta_{1,0}(\tau)\eta_{50,10}(\tau)^2\eta_{50,15}(\tau)} + 2\frac{\eta_{2,0}(\tau)^{1/2}\eta_{50,0}(\tau)^{1/2}}{\eta_{1,0}(\tau)\eta_{50,5}(\tau)\eta_{50,20}(\tau)}\\
&- 6\frac{\eta_{2,0}(\tau)^{1/2}\eta_{50,0}(\tau)^{1/2}}{\eta_{1,0}(\tau)\eta_{50,10}(\tau)\eta_{50,15}(\tau)} + 2\frac{\eta_{2,0}(\tau)^{1/2}\eta_{50,0}(\tau)^{1/2}\eta_{50,10}(\tau)}{\eta_{1,0}(\tau)\eta_{50,5}(\tau)\eta_{50,20}(\tau)^2}.
\end{aligned}
$$

However, $\frac{\eta(2\tau)\eta(25\tau)^2\eta(10\tau)^2}{\eta(\tau)^2\eta(50\tau)\eta(5\tau)^4}$ is a weight -1 modular form for $\Gamma_1(200)$ by Theorem 1.64 of [26] and so

$$\frac{\eta(2\tau)\eta(25\tau)^2\eta(10\tau)^2}{\eta(\tau)^2\eta(50\tau)\eta(5\tau)^4}C(5\tau)$$

is a modular function for $\Gamma_1(200)$. By Theorem 3 of [29], the other four generalized eta quotients on the right hand side of (5.23) are also modular functions on $\Gamma_1(200)$.

We recall some facts about modular functions from [28] and use the notation from Chapter 20 of [12]. Suppose f is a modular function with respect to the congruence subgroup Γ of $\Gamma_0(1)$. For $A \in \Gamma_0(1)$ we have a *cusp* given by $\zeta = A^{-1}\infty$. The *width* $N = N(\Gamma,\zeta)$ of the cusp is

$$N(\Gamma,\zeta) = \min\{k > 0 : \pm A^{-1}T^kA \in \Gamma\},$$

where T is the translation matrix

$$T = \begin{pmatrix} 1 & 1\\ 0 & 1\end{pmatrix}.$$

If

$$f(A^{-1}\tau) = \sum_{m=m_0}^{\infty} b_m q^{m/N}$$

and $b_{m_0} \neq 0$, then we say m_0 is the *order* of f at ζ with respect to Γ and we denote this value by $\mathrm{Ord}_\Gamma(f;\zeta)$. By $\mathrm{ord}(f;\zeta)$ we mean the *invariant order* of f at ζ given by

$$\mathrm{ord}(f;\zeta) = \frac{\mathrm{Ord}_\Gamma(f;\zeta)}{N}.$$

For z in the upper half-plane $\mathcal{H}$, we write $\mathrm{ord}(f;z)$ for the *order* of f at z as an analytic function in z. We define the order of f at z with respect to Γ by

$$\mathrm{Ord}_\Gamma(f;z) = \frac{\mathrm{ord}(f;z)}{m},$$

where m is the order of z as a fixed point of Γ.

We then have the well-known valence formula for modular functions as the weight zero case of the valence formula for modular forms, which is Theorem 4.1.4 of [28]. Suppose a subset $\mathcal{F}$ of $\mathcal{H}\cup\{\infty\}\cup\mathbb{Q}$ is a fundamental region for the action of Γ along with a complete set of inequivalent cusps; if f is not the zero function then

$$\sum_{z\in\mathcal{F}} \mathrm{Ord}_{\Gamma}(f;z) = 0.$$

To prove (5.23), we use the valence formula with f being the difference of the two sides of (5.23). We note the only poles of f can be at the cusps corresponding to $\Gamma_1(200)$ and so

$$\sum_{z\in\mathcal{F}} \mathrm{Ord}_{\Gamma}(f;z) \geq \sum_{\zeta\in C} \mathrm{Ord}_{\Gamma}(f;\zeta)$$

where C is a set of inequivalent cusps.

But if we have a lower bound on the cusps not equivalent to ∞, say

$$\sum_{\substack{\zeta\in C\\ \zeta\not\equiv\infty}} \mathrm{Ord}_{\Gamma}(f;\zeta) \geq -M,$$

and we know $\mathrm{Ord}_{\Gamma}(f;\infty) > M$, then by the valence formula, f must be identically zero. That is, to prove (5.23) we would need only verify that the q-series expansions agree past q^M.

Since $C(\tau)$ is a holomorphic modular form, we may ignore it when establishing a lower bound on the sum of the orders. Using Theorem 4 of [29], we can compute the order of the generalized eta quotients at the cusps. Including ∞, there are 336 inequivalent cusps for $\Gamma_1(200)$. To get a lower bound on the sum of orders at cusps not equivalent to ∞, at each cusp we take the minimum order of the six generalized eta quotients in (5.23). Using Maple for the calculations, we find

$$\sum_{\substack{\zeta\in C\\ \zeta\not\equiv\infty}} \mathrm{Ord}_{\Gamma}(f;\zeta) \geq -1840.$$

However we also verify in Maple that f vanishes past q^{1840}, and so the equality holds. ∎

PROPOSITION 4.

$$D(\tau)\frac{\eta(2\tau)}{\eta(\tau)^2} = D_0(q^5) + qD_1(q^5) + q^2D_2(q^5) + q^3D_3(q^5) + q^4D_4(q^5)$$

where

$$D_0(q) = \frac{(q^5;q^5)_\infty^2(q^2;q^2)_\infty^2}{(q^{10};q^{10})_\infty(q;q)_\infty^4}D(\tau),$$

$$D_1(q) = 2\frac{(q^4,q^6,q^{10};q^{10})_\infty}{(q^2,q^8;q^{10})_\infty^2(q^3,q^7;q^{10})_\infty},$$
$$D_2(q) = -6\frac{(q^{10};q^{10})_\infty}{(q,q^9;q^{10})_\infty(q^4,q^6;q^{10})_\infty},$$
$$D_3(q) = -2\frac{(q^{10};q^{10})_\infty}{(q^2,q^8;q^{10})_\infty(q^3,q^7;q^{10})_\infty},$$
$$D_4(q) = 4\frac{(q^2,q^8,q^{10};q^{10})_\infty}{(q,q^9;q^{10})_\infty(q^4,q^6;q^{10})_\infty^2}.$$

Proof. Since D is also a weight 1 form for $\Gamma_1(10)$ and these are the same products as in the previous proposition, we also need only verify that the corresponding equality between modular functions holds past q^{1840}. This verification is done in Maple. ∎

PROPOSITION 5.

$$(2C(\tau)-D(\tau))\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^4} = 5\frac{(q^4,q^6;q^{10})_\infty}{(q^2,q^3;q^5)_\infty^2}.$$

Proof. We see that this proposition is equivalent to

$$(2C(\tau)-D(\tau))\frac{\eta(2\tau)^2}{\eta(\tau)^4} = 5\frac{\eta_{10,4}(\tau)}{\eta_{5,2}(\tau)^2}. \tag{5.24}$$

However, we know the left hand side of (5.24) is a modular function for $\Gamma_1(40)$. Using Theorem 3 of [29] we find that the right hand side is as well. Comparing the orders at cusps as we did in the proof of Proposition 3, we find that a lower bound for the sum of orders at the cusps other than ∞ is -48. However, we verify in Maple that (5.24) holds past q^{48}, and so the equality must hold. ∎

PROPOSITION 6.

$$(3D(\tau)-C(\tau))\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty^4} = 10\frac{q}{(q^3,q^4,q^6,q^7;q^{10})_\infty(q^5;q^{10})_\infty^2}.$$

Proof. We see that this proposition is equivalent to

$$(3D(\tau)-C(\tau))\frac{\eta(2\tau)^2}{\eta(\tau)^4} = 10\frac{1}{\eta_{10,3}(\tau)\eta_{10,4}(\tau)\eta_{10,5}(\tau)}. \tag{5.25}$$

Again both sides are modular functions for $\Gamma_1(40)$ and taking the minimum of orders shows that we need only verify that the equality in (5.25) holds past q^{48}. ∎

With these propositions we can complete the proof of Theorem 2.14. We have

$$
\begin{aligned}
&\frac{(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(1-\zeta_5)(1-\zeta_5^{-1})(-1)^nq^n}{(1-\zeta_5q^n)(1-\zeta_5^{-1}q^n)}\right)\\
&\quad=\frac{(-q;q)_\infty}{(q;q)_\infty}\\
&\qquad\times\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(1-\zeta_5)(1-\zeta_5^{-1})(-1)^nq^n(1-q^n)(1-\zeta_5^2q^n)(1-\zeta_5^3q^n)}{(1-q^{5n})}\right)\\
&\quad=\frac{(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}+(3+\zeta_5^2+\zeta_5^3)\sum_{n=1}^{\infty}\frac{(-1)^nq^n}{(1-q^{5n})}\right.\\
&\qquad\qquad-(4+3\zeta_5^2+3\zeta_5^3)\sum_{n=1}^{\infty}\frac{(-1)^nq^{2n}}{(1-q^{5n})}+(4+3\zeta_5^2+3\zeta_5^3)\sum_{n=1}^{\infty}\frac{(-1)^nq^{3n}}{(1-q^{5n})}\\
&\qquad\qquad\left.-(3+\zeta_5^2+\zeta_5^3)\sum_{n=1}^{\infty}\frac{(-1)^nq^{4n}}{(1-q^{5n})}\right)\\
&\quad=\frac{(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}+(3+\zeta_5^2+\zeta_5^3)\sum_{n=1}^{\infty}\frac{(-1)^n(q^n-q^{4n})}{(1-q^{5n})}\right.\\
&\qquad\qquad\left.-(4+3\zeta_5^2+3\zeta_5^3)\sum_{n=1}^{\infty}\frac{(-1)^n(q^{2n}-q^{3n})}{(1-q^{5n})}\right)\\
&\quad=\frac{(-q;q)_\infty}{(q;q)_\infty}\left(\frac{1}{2}-\frac{3(3+\zeta_5^2+\zeta_5^3)}{10}+\frac{4+3\zeta_5^2+3\zeta_5^3}{10}+\frac{3+\zeta_5^2+\zeta_5^3}{10}C(\tau)\right.\\
&\qquad\qquad\left.-\frac{4+3\zeta_5^2+3\zeta_5^3}{10}D(\tau)\right)\\
&\quad=\frac{(-q;q)_\infty}{10(q;q)_\infty}((3+\zeta_5^2+\zeta_5^3)C(\tau)-(4+3\zeta_5^2+3\zeta_5^3)D(\tau))\\
&\quad=B_0(q^5)+qB_1(q^5)+q^2B_2(q^5)+q^3B_3(q^5)+q^4B_4(q^5)
\end{aligned}
$$

where

$$
\begin{aligned}
B_0(q)&=\frac{(q^5;q^5)_\infty^2(q^2;q^2)_\infty^2}{10(q^{10};q^{10})_\infty(q;q)_\infty^4}((3+\zeta_5^2+\zeta_5^3)C(\tau)-(4+3\zeta_5^2+3\zeta_5^3)D(\tau))\\
&=\frac{(q^5;q^5)_\infty^2(q^2;q^2)_\infty^2}{10(q^{10};q^{10})_\infty(q;q)_\infty^4}((2-\zeta_5-\zeta_5^{-1})C(\tau)-(1-3\zeta_5-3\zeta_5^{-1})D(\tau))\\
&=\frac{(q^5;q^5)_\infty^2(q^4,q^6;q^{10})_\infty}{2(q^{10};q^{10})_\infty(q^2,q^3;q^5)_\infty^2}\\
&\quad+(\zeta_5+\zeta_5^{-1})\frac{q(q^5;q^5)_\infty^2}{(q^{10};q^{10})_\infty(q^3,q^4,q^6,q^7;q^{10})_\infty(q^5;q^{10})_\infty^2},\\
B_1(q)&=(\zeta_5+\zeta_5^{-1}-1)\frac{(q^4,q^6,q^{10};q^{10})_\infty}{(q^2,q^8;q^{10})_\infty^2(q^3,q^7;q^{10})_\infty},
\end{aligned}
$$

$$B_2(q) = (1 - 2\zeta_5 - 2\zeta_5^{-1})\frac{(q^{10};q^{10})_\infty}{(q,q^9;q^{10})_\infty(q^4,q^6;q^{10})_\infty},$$
$$B_3(q) = -\frac{(q^{10};q^{10})_\infty}{(q^2,q^8;q^{10})_\infty(q^3,q^7;q^{10})_\infty},$$
$$B_4(q) = (\zeta_5 + \zeta_5^{-1})\frac{(q^2,q^8,q^{10};q^{10})_\infty}{(q,q^9;q^{10})_\infty(q^4,q^6;q^{10})_\infty^2}.$$

This finishes the proof of Theorem 2.14. ∎

6. Remarks. In Section 3 we proved the coefficients of $\overline{\mathrm{S}}(z,q)$, $\overline{\mathrm{S}}_1(z,q)$, and $\overline{\mathrm{S}}_2(z,q)$ are nonnegative by showing each summand $\frac{q^n(-q^{n+1},q^{n+1};q)_\infty}{(zq^n,z^{-1}q^n;q)_\infty}$ has nonnegative coefficients. Numerical evidence suggests $\mathrm{S2}(z,q)$ also has nonnegative coefficients. However, the corresponding individual summands for $\mathrm{S2}(z,q)$ do not have nonnegative coefficients themselves. In particular we find the coefficient of q^{10} in $q^4(-q^5,q^6;q^2)_\infty/(zq^4,z^{-1}q^4;q^2)_\infty$ to be $z^{-1}+z-1$. Thus for $\mathrm{S2}(z,q)$ a more complicated argument is required.

CONJECTURE 1. *For all m and n the coefficient $N_{\mathrm{S2}}(m,n)$ is nonnegative.*

Related to the nonnegativity of these coefficients is the difference between the first rank and the first crank moments. If we let $N(m,n)$ denote the number of partitions of n with rank m and $M(m,n)$ denote the number of partitions of n with crank m, then for $k \ge 1$ the kth *rank moment* $N_k(n)$ and kth *crank moment* $M_k(n)$ are given by

$$N_k(n) = \sum_{n\in\mathbb{Z}} m^k N(m,n), \qquad M_k(n) = \sum_{n\in\mathbb{Z}} m^k M(m,n).$$

These rank and crank moments were introduced by Atkin and the first author [8]. To allow for nontrivial odd moments, Andrews, Chan, and Kim [4] defined the *modified rank* and *crank moments* by

$$N_k^+(n) = \sum_{n=1}^{\infty} m^k N(m,n), \qquad M_k^+(n) = \sum_{n=1}^{\infty} m^k M(m,n).$$

In the same paper they proved that $M_1^+(n) > N_1^+(n)$ for all positive integers n. This was done by manipulating the generating function for $M_1^+(n) - N_1^+(n)$ and carefully grouping the terms in such a way that it is clear the coefficients are positive. However, it turns out that $M_1^+(n) - N_1^+(n) = N_{\mathrm{S}}(0,n)$; the latter was proved to be nonnegative in [7] and so $M_1^+(n) \ge N_1^+(n)$.

Recently Andrews, Chan, Kim, and Osburn [5] considered the *moments for* the rank and crank of *overpartitions*,

$$\overline{N}_k^+(n) = \sum_{n=1}^{\infty} m^k \overline{N}(m,n), \qquad \overline{M}_k^+(n) = \sum_{n=1}^{\infty} m^k \overline{M}(m,n).$$

The generating functions of these moments are

$$\overline{N}_k(q) = \sum_{n=1}^{\infty} \overline{N}_k^+(n) q^n, \qquad \overline{M}_k(q) = \sum_{n=1}^{\infty} \overline{M}_k^+(n) q^n.$$

In that paper they show $\overline{M}_1^+(n) > \overline{N}_1^+(n)$. As we will prove shortly, it also turns out that $\overline{M}_1^+(n) - \overline{N}_1^+(n) = N_{\overline{\mathrm{S}}}(0,n)$. Thus the nonnegativity of the coefficients of $\overline{\mathrm{S}}(z,q)$ gives $\overline{M}_1^+(n) \geq \overline{N}_1^+(n)$.

To begin, we use [19, equation (7.15)]:

$$\frac{(q;q)_\infty}{(zq, z^{-1}q;q)_\infty} = \frac{1}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}\right),$$

so we have

$$\frac{(-q,q;q)_\infty}{(zq, z^{-1};q)_\infty} = \frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}\right).$$

With this we can express $\overline{\mathrm{S}}(z,q)$ as follows:

$$\begin{aligned}
\overline{\mathrm{S}}(z,q) &= \frac{(-q;q)_\infty}{(1-z)(1-z^{-1})(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}\right) \\
&\quad - \frac{(-q,q;q)_\infty}{(z,z^{-1};q)_\infty} \\
&= \frac{(-q;q)_\infty}{(1-z)(1-z^{-1})(q;q)_\infty}\left(1+2\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)}\right) \\
&\quad - \frac{(-q;q)_\infty}{(1-z)(1-z^{-1})(q;q)_\infty} \\
&\qquad\qquad \times\left(1+\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}\right) \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)} \\
&\quad - 2\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n+1)/2}}{(1-q^n)}\left(\sum_{m=0}^{\infty} z^m q^{nm} + \sum_{m=1}^{\infty} z^{-m} q^{nm}\right) \\
&\quad - 2\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n^2+n}}{(1-q^{2n})}\left(\sum_{m=0}^{\infty} z^m q^{nm} + \sum_{m=1}^{\infty} z^{-m} q^{nm}\right).
\end{aligned}$$

In the last equality we have used the identity

$$\frac{1-q^{2n}}{(1-zq^n)(1-z^{-1}q^n)} = \frac{1}{1-zq^n} + \frac{1}{1-z^{-1}q^n} - 1.$$

But $\sum_{n=0}^{\infty} N_{\overline{\mathrm{S}}}(0,n)q^n$ is the coefficient of z^0 in $\overline{\mathrm{S}}(z,q)$. From the above we see that

$$\begin{aligned}(6.1)\qquad \sum_{n=0}^{\infty} N_{\overline{\mathrm{S}}}(0,n)q^n &= \frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n(n+1)/2}}{(1-q^n)} \\ &\quad - 2\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}q^{n^2+n}}{(1-q^{2n})}.\end{aligned}$$

By (6.1) and Proposition 2.1 of [5] we have

$$\overline{M}_1(q) - \overline{N}_1(q) = \sum_{n=1}^{\infty} N_{\overline{\mathrm{S}}}(0,n)q^n.$$

As explained earlier, we know that each $N_{\overline{\mathrm{S}}}(0,n)$ is nonnegative, and so this is another proof that $\overline{M}_1^+(n) \geq \overline{N}_1^+(n)$.

There is also the $d = e = 1$ case for the general spt function, which as noted in [15] reduces to $\overline{pp}(n)/4$, where $\overline{pp}(n)$ is the number of overpartition pairs of n. The methods in the present paper do not give a new proof of the congruences for $\overline{pp}(n)$. Using Bailey's Lemma on a two-variable generating function and applying Watson's transformation to the generating function for the rank of overpartition pairs does at first appear to give a difference between the rank of overpartition pairs and some residual crank. However, the resulting crank is

$$\frac{(-q;q)_\infty^2}{(zq,z^{-1}q;q)_\infty},$$

which can be written in terms of the rank for overpartition pairs as in equation (2.1) of [13]. In particular, the generating function for the rank of overpartition pairs is

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty}\overline{NN}(m,n)z^m q^n = \sum_{n=0}^{\infty}\frac{(-1,-1;q)_n q^n}{(zq,z^{-1}q;q)_n}$$

and

$$\frac{4}{(1+z)(1+z^{-1})} + \sum_{n=1}^{\infty}\frac{(-1,-1;q)_n q^n}{(zq,z^{-1}q;q)_n} = \frac{4(-q;q)_\infty^2}{(zq,z^{-1}q;q)_\infty}.$$

Thus the method of proving congruences in this paper only gives the proofs already given by Bringmann and Lovejoy [13].

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Received on 22.12.2013
and in revised form on 8.8.2014 (7685)