Elementary Deuring–Heilbronn phenomenon

by

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Introduction. In a long series of papers in *Acta Arithmetica*, János Pintz gave remarkable elementary proofs of theorems concerning \( L(s, \chi) \), with \( \chi \) the Kronecker symbol attached to a fundamental discriminant \(-D\). These include theorems of Hecke, Landau, Siegel, Page, Deuring, and Heilbronn \([8, 13]\). In [11], for example, he gives his version of the Deuring phenomenon \([2]\): under the very strong assumption that the class number satisfies \( h(-D) \leq \log^{3/4} D \), he obtains a zero free region for \( \zeta(s)L(s, \chi) \). As the reviewer in *Math. Reviews* noted, by Siegel’s theorem this can hold for only finitely many \( D \) (with an ineffective constant). Subsequently the Goldfeld–Gross–Zagier theorem shows this can happen for only finitely many \( D \) with an effective constant \([3]\). This is unfortunate, as the proof Pintz gave actually depends on the fact that the exponent of the class group \( C(-D) \) (v. the order) is small.

In [12] he gives an elementary version of (the contrapositive of) the Heilbronn phenomenon \([4]\): a zero off the critical line of an \( L \)-function \( L(s, \chi_k) \) attached to any primitive real character can be used to give lower bounds on \( L(1, \chi) \). The same *Math. Reviews* reviewer called the proof “ingenious and quite brief” \([2]\).

Pintz’s idea is very roughly as follows: With \( \lambda \) denoting the Liouville function, the convolution \( 1* \lambda \) is the characteristic function of squares. Thus for \( \rho \) a hypothetical zero of \( L(s, \chi_k) \) with \( \text{Re}(\rho) > 1/2 \), one can consider finite sums of the form

\[
\sum_{n<X} \frac{\chi_k(n)}{n^\rho} 1* \lambda(n).
\]

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\( ^{(1)} \) In fact there are 61 such fundamental discriminants, all with \(-1555 \leq -D \).

\( ^{(2)} \) See also [6, 7, §4.2] for an elementary proof by Motohashi which is based on the Selberg sieve.
Since $\chi_k(m^2) = 1$ or 0, one can compare this sum to a partial sum of $\zeta(2\rho)$, and obtain a lower bound. Pintz decomposes the sum into two pieces, carefully chosen so that $L(\rho, \chi) = 0$ shows one piece is not too big, and therefore the other piece is not too small. But if $L(1, \chi)$ were small due to the existence of a Landau–Siegel zero, $\chi$ would be a good approximation to $\lambda$, and (he can show) this second term would necessarily be small.

In this paper we adapt the method of [12] to apply to $\zeta(s)$, and thus give an elementary demonstration of the Deuring phenomenon. Because $\zeta(s)$ does not converge even conditionally in the critical strip, we assume first that $D$ is even, and consider instead

$$\phi(s) = (2^{1-s} - 1)\zeta(s) = \sum_n \frac{(-1)^n}{n^s}.$$

Suppose $\rho = \beta + i\gamma$ is a zero of $\zeta(s)$ off the critical line. Let $\delta/2\pi$ be the fractional part of $\log 2 \cdot \gamma/2\pi$ so that for integer $n$,

$$\log 2 \cdot \gamma = 2\pi n + \delta, \quad -\pi < \delta \leq \pi, \quad 2^{-i\gamma} = \exp(-i\delta).$$

**Theorem 1.** If $\beta > 7/8$ and $|\delta| > \pi/100$, then for any real primitive character $\chi$ modulo $D \equiv 0 \mod 4$, $D > 10^9$, we have the lower bound

$$L(1, \chi) > \frac{1}{5400 \cdot U^{12(1-\beta)} \log^3 U},$$

where $U = |\rho| D^{1/4} \log D$.

The proof actually gives some kind of nontrivial bound as long as $\beta > 5/6$. We assume $\beta > 7/8$ simply to get a precise constant in the theorem.

In the last section we discuss general $D$, adapting the proof with Ramanujan sums $c_q(n)$ for a fixed prime $q | D$.

**Arithmetic function preliminaries.** Generalizing Liouville’s $\lambda$ function, we begin by defining $\lambda_{\text{odd}}(n)$ via

$$\lambda_{\text{odd}}(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \lambda(n) & \text{if } n \text{ is odd.} \end{cases}$$

So

$$\sum_{n=1}^\infty \frac{\lambda_{\text{odd}}(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)} (1 + 2^{-s}),$$

and the convolution $1 * \lambda_{\text{odd}}(n)$ satisfies

$$1 * \lambda_{\text{odd}}(n) = \begin{cases} 1 & \text{if } n = m^2 \text{ or } n = 2m^2, \\ 0 & \text{otherwise.} \end{cases}$$

With $\tau(n)$ the divisor function and $\nu(n)$ the number of distinct primes dividing $n$, we have

$$1 * \lambda(n) = \sum_{d|n} 2^{\nu(d)} \lambda(d) \tau(n/d).$$
(One needs to verify this only for \( n = p^k \) as both sides are multiplicative.) We generalize this by defining \( \tau_{\text{odd}}(n) \) to be the number of odd divisors of \( n \), so that

\[
1 \ast \lambda_{\text{odd}}(n) = \sum_{d|n} 2^{\nu(d)} \lambda_{\text{odd}}(d) \tau_{\text{odd}}(n/d).
\]

(For \( n \) odd this follows from \( \lambda_{\text{odd}}(d) = \lambda(d) \) and \( \tau_{\text{odd}}(n/d) = \tau(n/d) \), while for \( n = 2^k \) both sides are equal to 1.)

Following Pintz we define, relative to the quadratic character \( \chi \mod D \), sets

\[
A_j = \{ u : p \mid u \Rightarrow \chi(p) = j \} \quad \text{for } j = -1, 0, 1,
\]

\[
C = \{ c = ab : a \in A_1, b \in A_0 \}.
\]

We are assuming that \( 2 \in A_0 \), so integers in \( A_{-1} \) and \( A_1 \) are odd. We factor an arbitrary \( n \) as

\[
n = abm = cm, \quad \text{where } a \in A_1, b \in A_0, m \in A_{-1}, c \in C.
\]

We then see that

- for \( a \in A_1 \), \( 1 \ast \chi(a) = \tau(a) = \tau_{\text{odd}}(a) \),
- for \( b \in A_0 \), \( 1 \ast \chi(b) = 1 \),
- for \( m \in A_{-1} \), \( 1 \ast \chi(m) = 1 \ast \lambda(m) = 1 \ast \lambda_{\text{odd}}(m) \).

Using this and multiplicativity, for \( n = abm = cm \) as above we see that

\[
(1) \quad 1 \ast \lambda_{\text{odd}}(n) = 1 \ast \lambda_{\text{odd}}(a) \cdot 1 \ast \lambda_{\text{odd}}(b) \cdot 1 \ast \lambda_{\text{odd}}(m)
\]

\[
= \left( \sum_{a'|a} 2^{\nu(a')} \lambda_{\text{odd}}(a') \cdot 1 \ast \chi(a/a') \right) \left( \sum_{b'|b} \lambda_{\text{odd}}(b') \cdot 1 \ast \chi(b/b') \right) \cdot 1 \ast \chi(m)
\]

\[
= \sum_{c'|c} 2^{\nu(a')} \lambda_{\text{odd}}(c') \cdot 1 \ast \chi(n/c').
\]

**Lower bounds**

**Lemma 2.**

\[
\frac{1}{25} \cdot \frac{\zeta(4\beta)}{\zeta(2\beta)} U^{6-12\beta} \leq \left| \sum_{n \leq U^{12}} \frac{(-1)^n \cdot 1 \ast \lambda_{\text{odd}}(n)}{n^\rho} \right|.
\]

**Proof.** We have

\[
\left| \sum_{n \leq U^{12}} \frac{(-1)^n \cdot 1 \ast \lambda_{\text{odd}}(n)}{n^\rho} \right|
\]

\[
\geq \left| \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \ast \lambda_{\text{odd}}(n)}{n^\rho} \right| - \left| \sum_{U^{12} < n} \frac{(-1)^n \cdot 1 \ast \lambda_{\text{odd}}(n)}{n^\rho} \right|.
\]
Now
\[ \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \ast \lambda_{\text{odd}}(n)}{n^\rho} = \sum_{m=1}^{\infty} \frac{(-1)^{m^2}}{m^{2\rho}} + \sum_{m=1}^{\infty} \frac{(-1)^{2m^2}}{2^\rho m^{2\rho}}. \]

Observe that \((-1)^{m^2} = (-1)^m\), and of course \((-1)^{2m^2} = 1\). This gives
\[ (2^{1-2\rho} - 1)\zeta(2\rho) + 2^{-\rho}\zeta(2\rho) = (1 + 2^{-\rho})(2^{1-\rho} - 1)\zeta(2\rho). \]

We compare Euler products to see
\[ \frac{1}{|\zeta(2\rho)|} < \frac{\zeta(2\beta)}{\zeta(4\beta)}, \quad \text{or} \quad |\zeta(2\rho)| > \frac{\zeta(4\beta)}{\zeta(2\beta)}. \]

Finally a calculation in Mathematica shows that
\[ |(1 + 2^{-\rho})(2^{1-\rho} - 1)| > \frac{1}{25} \]
as long as \(|\delta| > \pi/100\). This gives the main term of the lemma.

Meanwhile
\[ \left| \sum_{U^{12} < n} \frac{(-1)^n \cdot 1 \ast \lambda_{\text{odd}}(n)}{n^\rho} \right| \leq \left| \sum_{U^6 < m} \frac{(-1)^m}{m^{2\rho}} \right| + \left| \frac{1}{2^\rho} \sum_{U^6 / \sqrt{2} < m} \frac{1}{m^{2\rho}} \right|. \]

The first sum on the right is bounded by \(U^{-12\beta}\), by Abel’s inequality. And the second sum, via Euler summation formula [1, Theorem 3.2(c)], is \(O(U^6 - 12\beta)\). In fact, the proof given there shows the implied constant can be taken as \(1/(\sqrt{2} (2\beta - 1)) < 1\) for \(\beta > 7/8\).

**Upper bounds.** We now follow Pintz in writing
\[ \left| \sum_{n \leq U^{12}} \frac{(-1)^n \cdot 1 \ast \lambda_{\text{odd}}(n)}{n^\rho} \right| = \left| \sum_{n \leq U^{12}} \frac{(-1)^n}{n^\rho} \sum_{c \in C, c \mid n} 2^{\nu(a)} \lambda_{\text{odd}}(c) \cdot 1 \ast \chi(n/c) \right| \]
\[ =: S, \]

via (1). We change variables \(n = rc\), and use the fact that for odd \(c\) we have
\[ (-1)^{rc} = (-1)^r, \quad \text{and} \quad \lambda_{\text{odd}}(c) = 0 \quad \text{unless} \quad c \text{ is odd}. \]
(The fact that \((-1)^n\) is not a multiplicative function is the reason we have introduced \(\lambda_{\text{odd}}(n)\).) Now
\[ S = \left| \sum_{c \leq U^{12}, c \in C} \frac{2^{\nu(a)} \lambda_{\text{odd}}(c)}{c^\rho} \sum_{r \leq U^{12}/c} \frac{(-1)^r}{r^\rho} \cdot 1 \ast \chi(r) \right| \leq \Sigma_1 + \Sigma_2, \]
where

$$
\Sigma_1' = \sum_{c \leq U^6 \atop c \in C} \frac{2^{\nu(a)}}{c^{\beta}} \left| \sum_{r \leq U^{12}/c} \frac{(-1)^r}{r^\rho} \cdot 1 \ast \chi(r) \right|,
$$

$$
\Sigma_2' = \sum_{U^6 < c \leq U^{12}} \frac{2^{\nu(a)}}{c^{\beta}} \sum_{r \leq U^{12}/c} \frac{1 \ast \chi(r)}{r^\beta}.
$$

Using the inequalities

$$2^{\nu(a)} \leq 1 \ast \chi(c) \leq \tau_{\text{odd}}(c) \leq \tau(c), \quad 1 \ast \chi(r) \leq \tau(r),$$

and dropping the condition $c \in C$ in the outer sums, we see that

$$\Sigma_1' \leq \Sigma_1 = \sum_{n \leq U^6} \frac{\tau(n)}{n^{\beta}} \left| \sum_{r \leq U^{12}/n} \frac{(-1)^r}{r^\rho} \cdot 1 \ast \chi(r) \right|,
$$

$$\Sigma_2' \leq \Sigma_2 = \sum_{U^6 < n \leq U^{12}} \frac{1 \ast \chi(n)}{n^{\beta}} \sum_{r \leq U^{12}/n} \frac{\tau(r)}{r^\beta}.$$

**Remark.** The main idea of the proof is to use the fact that $\zeta(\rho) = 0$ to show that $\Sigma_1$ cannot be too big. This then implies that $\Sigma_2$ cannot be too small, from which we can bound $L(1, \chi)$ from below.

**Lemma 3.** We estimate the inner sum in $\Sigma_1$ as

$$\left| \sum_{r \leq y} \frac{(-1)^r}{r^\rho} \sum_{d | r} \chi(d) \right| < \frac{2}{3} \cdot y^{1/2-\beta} |\rho| D^{1/4} \log D \log(y/\sqrt{D}).$$

**Proof.** We write $(-1)^r = (-1)^{ld}$. Since we are assuming $D$ is even, $\chi(d) = 0$ unless $d$ is odd and so $(-1)^{ld} = (-1)^l$. This gives

$$\left| \sum_{r \leq y} \frac{(-1)^r}{r^\rho} \sum_{d | r} \chi(d) \right| = \left| \sum_{d \leq y} \frac{\chi(d)}{d^\rho} \sum_{l \leq y/d} \frac{(-1)^l}{l^\rho} \right| \leq \left| \sum_{d \leq z} \frac{\chi(d)}{d^\rho} \sum_{l \leq y/d} \frac{(-1)^l}{l^\rho} \right| + \left| \sum_{l \leq y/z} \frac{(-1)^l}{l^\rho} \sum_{d < y/z} \frac{\chi(d)}{d^\rho} \right|.$$

The parameter $z$ will be chosen later to make these two terms approximately the same size. Summation by parts \[1, \text{Theorem 4.2}\] gives

$$\phi(s) = \sum_{l=1}^{y/d} \frac{(-1)^l}{l^s} - \frac{S(y/d)}{(y/d)^s} + s \int_{y/d}^{\infty} \frac{S(x) - S(y/d)}{x^{s+1}} \, dx,$$

where $S(x) = \sum_{n \leq x} (-1)^n$ is $-1$ or $0$. Set $s = \rho$ and use $\phi(\rho) = 0$; we bound
the integral getting
\[ \left| s \int_{y/d}^{\infty} \frac{S(x) - S(y/d)}{x^{s+1}} \, dx \right| \leq \frac{|\rho|}{\beta(y/d)^\beta}, \quad \left| S(y/d) \right| \leq \frac{1}{(y/d)^\beta}. \]

So we claim
\[ \left| \sum_{l=1}^{y/d} \frac{(-1)^l}{l^p} \right| \leq \frac{|\rho|}{\beta(y/d)^\beta}, \]
since \(1 < 1/\beta\) and \(3\) shows that \(10^{12} < |\rho|\).

Thus we can estimate the first term in the previous sum:
\[ \left| \sum_{d \leq z} \frac{\chi(d)}{d^p} \sum_{l \leq y/d} \frac{(-1)^l}{l^p} \right| \leq \sum_{d \leq z} \frac{1}{d^\beta} \cdot \frac{|\rho|}{\beta(y/d)^\beta} = \frac{z|\rho|}{y^\beta \beta}. \]

Another summation by parts gives
\[ \sum_{z < d \leq y/l} \frac{\chi(d)}{d^p} = S_D(y/l)(y/l)^s - S_D(z) z^s + s \int_{z}^{y/l} \frac{S_D(x) - S_D(\sqrt{y})}{x^{s+1}} \, dx, \]
where \(S_D(x) = \sum_{n \leq x} \chi(n)\). By the Pólya–Vinogradov inequality \(1\) Theorem 8.21, \(|S_D(x)| < \sqrt{D} \log D\). Neglecting the boundary terms as before, we bound the integral as
\[ \left| \sum_{z < d \leq y/l} \frac{\chi(d)}{d^p} \right| \leq \frac{|\rho|\sqrt{D} \log D}{\beta z^\beta}, \]
and so bound the second sum above as
\[ \left| \sum_{l \leq y/z} \frac{(-1)^l}{l^p} \sum_{z < d \leq y/l} \frac{\chi(d)}{d^p} \right| \leq \sum_{l \leq y/z} \frac{|\rho|\sqrt{D} \log D}{\beta l^\beta z^\beta} = \frac{|\rho|\sqrt{D} \log D}{\beta} \sum_{l \leq y/z} \frac{1}{l^\beta z^\beta}. \]

Now
\[ \sum_{l \leq y/z} \frac{1}{l^\beta z^\beta} = \frac{y^{1-\beta}}{z} \sum_{l \leq y/z} \frac{1}{l^\beta (y/z)^{1-\beta}} < \frac{y^{1-\beta}}{z} \sum_{l \leq y/z} \frac{1}{l^\beta} \cdot \frac{1}{l^{1-\beta}} \sim \frac{y^{1-\beta} \log(y/z)}{z}, \]
where the inequality follows since \(l < y/z\). This gives, for the second sum, the bound
\[ \frac{|\rho|\sqrt{D} \log D}{\beta} \cdot \frac{y^{1-\beta} \log(y/z)}{z}. \]

Comparing the two estimates, we see they are approximately the same size when
\[ \frac{z}{y^\beta} = \frac{\sqrt{D} y^{1-\beta}}{z}, \quad \text{or} \quad z = D^{1/4} y^{1/2}. \]
Combining the two sum estimates, and with
\[ \frac{1}{\beta} < \frac{6}{5} \quad \text{and} \quad 1 < \frac{\log(y/\sqrt{D}) \log D}{18}, \]
we have
\[
\frac{y^{1/2-\beta}|\rho|D^{1/4}}{\beta} + \frac{y^{1/2-\beta}|\rho| \log(y/\sqrt{D})D^{1/4} \log D}{2\beta}
\]
\[< \frac{6}{5} \left( \frac{1}{18} + \frac{1}{2} \right) y^{1/2-\beta} \log(y/\sqrt{D})|\rho|D^{1/4} \log D \]
\[= \frac{2}{3} y^{1/2-\beta} \log(y/\sqrt{D})|\rho|D^{1/4} \log D. \]

**Lower bounds again.** Applying Lemma 3 with \( y = U^{12}/n \), so \( U^6 < y < U^{12} \), we get
\[
\Sigma_1 < 8U^{6-12\beta} \log U|\rho|D^{1/4} \log D \sum_{n \leq U^6} \frac{\tau(n)}{\sqrt{n}} = 8U^{7-12\beta} \log U \sum_{n \leq U^6} \frac{\tau(n)}{\sqrt{n}}.
\]

With an estimate by the standard ‘method of the hyperbola’ (e.g. [5] (2.9), p. 37), we get
\[
\sum_{n \leq X} \frac{\tau(n)}{\sqrt{n}} = X^{1/2}(2 \log X + 4C - 4) + O(1).
\]
Thus
\[
\Sigma_1 < 96U^{10-12\beta} \log^2 U,
\]
and so, for \( \beta > 5/6 \), \( \Sigma_1 \) is small. In fact, from
\[
\frac{1}{25} \cdot \frac{\zeta(4\beta)}{\zeta(2\beta)} - U^{6-12\beta} \leq \Sigma_1 + \Sigma_2,
\]
Mathematica tells us \( 1/50 < \Sigma_2 \) when \( \beta > 7/8 \) and \( U > 10^{16} \). (We are assuming \( D > 10^9 \), and Gourdon [3] has verified the Riemann Hypothesis for the first \( 10^{13} \) zeros. Therefore our hypothetical \( \rho \) satisfies \( |\rho| > 2.4 \cdot 10^{12} \), so necessarily \( U = |\rho|D^{1/4} \log D > 10^{16} \).)

We now convert the lower bound for \( \Sigma_2 \) to a lower bound for \( L(1, \chi) \). Recall that
\[
\Sigma_2 = \sum_{U^6 < n \leq U^{12}} \frac{1 \ast \chi(n)}{n^{\beta}} \sum_{r \leq U^{12}/n} \frac{\tau(r)}{r^\beta}.
\]
Writing \( r^{-\beta} = r^{1-\beta}/r \) and using \( r^{1-\beta} < U^{12(1-\beta)}n^{\beta-1} \) we see that
\[
\frac{1}{50} < \Sigma_2 < U^{12(1-\beta)} \sum_{U^6 < n \leq U^{12}} \frac{1 \ast \chi(n)}{n} \sum_{r \leq U^{12}/n} \frac{\tau(r)}{r}.
\]
The ‘method of the hyperbola’ argument shows in \[5\] Ex. 11.2.1 (g) that
\[
\sum_{U^6 \leq n \leq U^{12}} \frac{1 \cdot \chi(n)}{n} = \log(U^6)L(1, \chi) + O(D^{1/4}U^{-3}\log D \log(U^6))
\]
\[
= \log(U^6)L(1, \chi) + O(U^{-2}\log(U^6))
\]
\[
= \log(U^6)(L(1, \chi) + O(U^{-2})).
\]
Meanwhile one more application of this same tool (along with Euler summation) gives
\[
\sum_{r < X} \frac{\tau(r)}{r} = \frac{1}{2} \log^2 X + 2C \log X + O(1).
\]
So
\[
\sum_{r \leq U^{12}/n} \frac{\tau(r)}{r} \sim \frac{1}{2} \log^2(U^{12}/n) < \frac{1}{2} \log^2(U^6),
\]
as \(U^6 < n\). Finally
\[
\frac{1}{50} < \sum_2 < U^{12(1-\beta)} \log(U^6)(L(1, \chi) + O(U^{-2})) \cdot \frac{1}{2} \log^2(U^6)
\]
\[
= 108U^{12(1-\beta)} \log^3 U \left(L(1, \chi) + O(U^{-2})\right).
\]
The implied constant is no worse than 6, and
\[
U^{-2} = \frac{1}{|\rho|^2 \sqrt{D} \log^2 D} < \frac{1}{\sqrt{D}},
\]
so the theorem follows.

The general case. We fix a prime \(q \mid D\) and consider
\[
\sum_{n=1}^{\infty} \frac{c_q(n)}{n^s} = (q^{1-s} - 1)\zeta(s),
\]
where \(c_q(n)\) is the Ramanujan sum
\[
c_q(n) = \sum_{k=1}^{q-1} \exp(2\pi i kn/q) = \begin{cases} -1 & \text{if } (n, q) = 1, \\ q-1 & \text{if } q \mid n. \end{cases}
\]
(Observe that \(c_2(n) = (-1)^n\).) Since \(|\sum_{n<x} c_q(n)| < q\), the Dirichlet series converges conditionally for \(\Re(s) > 0\). The Ramanujan sums are not multiplicative in \(n\), but we have \(c_q(dm) = c_q(m)\) if \((d, q) = 1\). Instead of \(\lambda_{\text{odd}}\) we define a function \(\lambda_q(n) = 0\) if \(q \mid n\). The proof goes through as before. We

\(^{(3)}\) The implied constant in that exercise, combining six big Oh terms with implied constant equal to 1, can be taken to be 6.
find that in Lemma 2 we have
\[ \sum_{n=1}^{\infty} c_q(n) \cdot 1 \ast \lambda_q(n) = (1 + q^{-\rho})(1 - q^{1-\rho})\zeta(2\rho), \]
so the trivial zeros along Re(\(s\)) = 1 when \(\gamma = 2\pi n/\log q\) still cause a problem. In fact, the constant 1/25 in Lemma 2 which works for \(q = 2\) is a decreasing function of \(q\) in the general case.

References


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