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Elementary Deuring–Heilbronn phenomenon

by

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Introduction. In a long series of papers in *Acta Arithmetica*, János Pintz gave remarkable elementary proofs of theorems concerning $L(s,\chi)$, with χ the Kronecker symbol attached to a fundamental discriminant $-D$. These include theorems of Hecke, Landau, Siegel, Page, Deuring, and Heilbronn [8]–[13]. In [11], for example, he gives his version of the Deuring phenomenon [2]: under the very strong assumption that the class number satisfies $h(-D)\le\log^{3/4} D$, he obtains a zero free region for $\zeta(s)L(s,\chi)$. As the reviewer in *Math. Reviews* noted, by Siegel's theorem this can hold for only finitely many D (with an ineffective constant). Subsequently the Goldfeld–Gross–Zagier theorem shows this can happen for only finitely many D with an effective constant (1). This is unfortunate, as the proof Pintz gave actually depends on the fact that the exponent of the class group $\mathcal{C}(-D)$ (v. the order) is small.

In [12] he gives an elementary version of (the contrapositive of) the Heilbronn phenomenon [4]: a zero off the critical line of an L-function $L(s,\chi_k)$ attached to any primitive real character can be used to give lower bounds on $L(1,\chi)$. The same *Math. Reviews* reviewer called the proof "ingenious and quite brief" (2).

Pintz's idea is very roughly as follows: With λ denoting the Liouville function, the convolution $1*\lambda$ is the characteristic function of squares. Thus for ρ a hypothetical zero of $L(s,\chi_k)$ with $\operatorname{Re}(\rho)>1/2$, one can consider finite sums of the form

$$\sum_{n<X}\frac{\chi_k(n)}{n^\rho}1*\lambda(n).$$

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(1) In fact there are 61 such fundamental discriminants, all with $-1555\le -D$.

(2) See also [6], [7, §4.2] for an elementary proof by Motohashi which is based on the Selberg sieve.

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Since $\chi_k(m^2) = 1$ or 0, one can compare this sum to a partial sum of $\zeta(2\rho)$, and obtain a lower bound. Pintz decomposes the sum into two pieces, carefully chosen so that $L(\rho, \chi_k) = 0$ shows one piece is not too big, and therefore the other piece is not too small. But if $L(1, \chi)$ were small due to the existence of a Landau–Siegel zero, χ would be a good approximation to λ, and (he can show) this second term would necessarily be small.

In this paper we adapt the method of [12] to apply to $\zeta(s)$, and thus give an elementary demonstration of the Deuring phenomenon. Because $\zeta(s)$ does not converge even conditionally in the critical strip, we assume first that D is even, and consider instead

$$\phi(s) = (2^{1-s} - 1)\zeta(s) = \sum_n \frac{(-1)^n}{n^s}.$$

Suppose $\rho = \beta + i\gamma$ is a zero of $\zeta(s)$ off the critical line. Let $\delta/2\pi$ be the fractional part of $\log 2 \cdot \gamma/2\pi$ so that for integer n,

$$\log 2 \cdot \gamma = 2\pi n + \delta, \qquad -\pi < \delta \le \pi, \qquad 2^{-i\gamma} = \exp(-i\delta).$$

Theorem 1. *If $\beta > 7/8$ and $|\delta| > \pi/100$, then for any real primitive character χ modulo $D \equiv 0 \bmod 4$, $D > 10^9$, we have the lower bound*

$$L(1, \chi) > \frac{1}{5400 \cdot U^{12(1-\beta)} \log^3 U},$$

where $U = |\rho| D^{1/4} \log D$.

The proof actually gives some kind of nontrivial bound as long as $\beta > 5/6$. We assume $\beta > 7/8$ simply to get a precise constant in the theorem.

In the last section we discuss general D, adapting the proof with Ramanujan sums $c_q(n)$ for a fixed prime $q \mid D$.

Arithmetic function preliminaries. Generalizing Liouville's λ function, we begin by defining $\lambda_{\mathrm{odd}}(n)$ via

$$\lambda_{\mathrm{odd}}(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \lambda(n) & \text{if } n \text{is odd.} \end{cases}$$

So

$$\sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{odd}}(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}(1 + 2^{-s}),$$

and the convolution $1 * \lambda_{\mathrm{odd}}(n)$ satisfies

$$1 * \lambda_{\mathrm{odd}}(n) = \begin{cases} 1 & \text{if } n = m^2 \text{ or } n = 2m^2, \\ 0 & \text{otherwise.} \end{cases}$$

With $\tau(n)$ the divisor function and $\nu(n)$ the number of distinct primes dividing n, we have

$$1 * \lambda(n) = \sum_{d \mid n} 2^{\nu(d)} \lambda(d) \tau(n/d).$$

(One needs to verify this only for $n = p^k$ as both sides are multiplicative.) We generalize this by defining $\tau_{\mathrm{odd}}(n)$ to be the number of odd divisors of n, so that

$$1 * \lambda_{\mathrm{odd}}(n) = \sum_{d|n} 2^{\nu(d)}\lambda_{\mathrm{odd}}(d)\tau_{\mathrm{odd}}(n/d).$$

(For n odd this follows from $\lambda_{\mathrm{odd}}(d) = \lambda(d)$ and $\tau_{\mathrm{odd}}(n/d) = \tau(n/d)$, while for $n = 2^k$ both sides are equal to 1.)

Following Pintz we define, relative to the quadratic character χ modulo D, sets

$$\begin{aligned} A_j &= \{u : p \,|\, u \Rightarrow \chi(p) = j\} \quad \text{for } j = -1, 0, 1, \\ C &= \{c = ab : a \in A_1,\ b \in A_0\}. \end{aligned}$$

We are assuming that $2 \in A_0$, so integers in A_{-1} and A_1 are odd. We factor an arbitrary n as

$$n = abm = cm, \qquad \text{where } a \in A_1,\ b \in A_0,\ m \in A_{-1},\ c \in C.$$

We then see that

- for $a \in A_1,\ 1 * \chi(a) = \tau(a) = \tau_{\mathrm{odd}}(a)$,
- for $b \in A_0,\ 1 * \chi(b) = 1$,
- for $m \in A_{-1},\ 1 * \chi(m) = 1 * \lambda(m) = 1 * \lambda_{\mathrm{odd}}(m)$.

Using this and multiplicativity, for $n = abm = cm$ as above we see that

$$\begin{aligned} (1) \qquad 1 * \lambda_{\mathrm{odd}}(n) &= 1 * \lambda_{\mathrm{odd}}(a) \cdot 1 * \lambda_{\mathrm{odd}}(b) \cdot 1 * \lambda_{\mathrm{odd}}(m) \\ &= \Big(\sum_{a'|a} 2^{\nu(a')}\lambda_{\mathrm{odd}}(a') \cdot 1 * \chi(a/a')\Big)\Big(\sum_{b'|b} \lambda_{\mathrm{odd}}(b') \cdot 1 * \chi(b/b')\Big) \cdot 1 * \chi(m) \\ &= \sum_{\substack{c'|c \\ c'=a'b'}} 2^{\nu(a')}\lambda_{\mathrm{odd}}(c') \cdot 1 * \chi(n/c'). \end{aligned}$$

Lower bounds

LEMMA 2.

$$\frac{1}{25} \cdot \frac{\zeta(4\beta)}{\zeta(2\beta)} - U^{6-12\beta} \le \Big|\sum_{n \le U^{12}} \frac{(-1)^n \cdot 1 * \lambda_{\mathrm{odd}}(n)}{n^\rho}\Big|.$$

Proof. We have

$$\begin{aligned} &\Big|\sum_{n \le U^{12}} \frac{(-1)^n \cdot 1 * \lambda_{\mathrm{odd}}(n)}{n^\rho}\Big| \\ &\qquad \ge \Big|\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 * \lambda_{\mathrm{odd}}(n)}{n^\rho}\Big| - \Big|\sum_{U^{12}<n} \frac{(-1)^n \cdot 1 * \lambda_{\mathrm{odd}}(n)}{n^\rho}\Big|. \end{aligned}$$

Now

$$\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1*\lambda_{\text{odd}}(n)}{n^\rho}=\sum_{m=1}^{\infty}\frac{(-1)^{m^2}}{m^{2\rho}}+\sum_{m=1}^{\infty}\frac{(-1)^{2m^2}}{2^\rho m^{2\rho}}.$$

Observe that $(-1)^{m^2}=(-1)^m$, and of course $(-1)^{2m^2}=1$. This gives

$$(2^{1-2\rho}-1)\zeta(2\rho)+2^{-\rho}\zeta(2\rho)=(1+2^{-\rho})(2^{1-\rho}-1)\zeta(2\rho).$$

We compare Euler products to see

$$\frac{1}{|\zeta(2\rho)|}<\frac{\zeta(2\beta)}{\zeta(4\beta)},\quad\text{or}\quad|\zeta(2\rho)|>\frac{\zeta(4\beta)}{\zeta(2\beta)}.$$

Finally a calculation in *Mathematica* shows that

$$|(1+2^{-\rho})(2^{1-\rho}-1)|>\frac{1}{25}$$

as long as $|\delta|>\pi/100$. This gives the main term of the lemma.

Meanwhile

$$\Bigg|\sum_{U^{12}<n}\frac{(-1)^n\cdot 1*\lambda_{\text{odd}}(n)}{n^\rho}\Bigg|\le\Bigg|\sum_{U^6<m}\frac{(-1)^m}{m^{2\rho}}\Bigg|+\Bigg|\frac{1}{2^\rho}\sum_{U^6/\sqrt{2}<m}\frac{1}{m^{2\rho}}\Bigg|.$$

The first sum on the right is bounded by $U^{-12\beta}$, by Abel's inequality. And the second sum, via Euler summation formula [1, Theorem 3.2(c)], is $O(U^{6-12\beta})$. In fact, the proof given there shows the implied constant can be taken as $1/(\sqrt{2}\,(2\beta-1))<1$ for $\beta>7/8$. ∎

Upper bounds. We now follow Pintz in writing

$$\Bigg|\sum_{n\le U^{12}}\frac{(-1)^n}{n^\rho}\cdot 1*\lambda_{\text{odd}}(n)\Bigg|=\Bigg|\sum_{n\le U^{12}}\frac{(-1)^n}{n^\rho}\sum_{c\in C,\,c|n}2^{\nu(a)}\lambda_{\text{odd}}(c)\cdot 1*\chi(n/c)\Bigg|$$
$$=:S,$$

via (1). We change variables $n=rc$, and use the fact that for odd c we have

$$(-1)^{rc}=(-1)^r,\ \text{and }\lambda_{\text{odd}}(c)=0\text{ unless }c\text{ is odd.}$$

(The fact that $(-1)^n$ is not a multiplicative function is the reason we have introduced $\lambda_{\text{odd}}(n)$.) Now

$$S=\Bigg|\sum_{c\le U^{12},c\in C}\frac{2^{\nu(a)}\lambda_{\text{odd}}(c)}{c^\rho}\sum_{r\le U^{12}/c}\frac{(-1)^r}{r^\rho}\cdot 1*\chi(r)\Bigg|\le\Sigma_1'+\Sigma_2',$$

where

$$\Sigma_1' = \sum_{\substack{c \le U^6 \\ c \in C}} \frac{2^{\nu(a)}}{c^\beta} \Big| \sum_{r \le U^{12}/c} \frac{(-1)^r}{r^\rho} \cdot 1 * \chi(r) \Big|,$$

$$\Sigma_2' = \sum_{\substack{U^6 < c \le U^{12} \\ c \in C}} \frac{2^{\nu(a)}}{c^\beta} \sum_{r \le U^{12}/c} \frac{1 * \chi(r)}{r^\beta}.$$

Using the inequalities

$$2^{\nu(a)} \le 1 * \chi(c) \le \tau_{\mathrm{odd}}(c) \le \tau(c), \qquad 1 * \chi(r) \le \tau(r),$$

and dropping the condition $c \in C$ in the outer sums, we see that

$$\Sigma_1' \le \Sigma_1 = \sum_{n \le U^6} \frac{\tau(n)}{n^\beta} \Big| \sum_{r \le U^{12}/n} \frac{(-1)^r}{r^\rho} \cdot 1 * \chi(r) \Big|,$$

$$\Sigma_2' \le \Sigma_2 = \sum_{U^6 < n \le U^{12}} \frac{1 * \chi(n)}{n^\beta} \sum_{r \le U^{12}/n} \frac{\tau(r)}{r^\beta}.$$

Remark. The main idea of the proof is to use the fact that $\zeta(\rho) = 0$ to show that Σ_1 cannot be too big. This then implies that Σ_2 cannot be too small, from which we can bound $L(1, \chi)$ from below.

Lemma 3. *We estimate the inner sum in* Σ_1 *as*

$$\Big| \sum_{r \le y} \frac{(-1)^r}{r^\rho} \sum_{d \mid r} \chi(d) \Big| < \frac{2}{3} \cdot y^{1/2-\beta} |\rho| D^{1/4} \log D \log(y/\sqrt{D}).$$

Proof. We write $(-1)^r = (-1)^{ld}$. Since we are assuming D is even, $\chi(d) = 0$ unless d is odd and so $(-1)^{ld} = (-1)^l$. This gives

$$\Big| \sum_{r \le y} \frac{(-1)^r}{r^\rho} \sum_{d \mid r} \chi(d) \Big| = \Big| \sum_{d \le y} \frac{\chi(d)}{d^\rho} \sum_{l \le y/d} \frac{(-1)^l}{l^\rho} \Big|$$

$$\le \Big| \sum_{d \le z} \frac{\chi(d)}{d^\rho} \sum_{l \le y/d} \frac{(-1)^l}{l^\rho} \Big| + \Big| \sum_{l \le y/z} \frac{(-1)^l}{l^\rho} \sum_{z < d \le y/l} \frac{\chi(d)}{d^\rho} \Big|.$$

The parameter z will be chosen later to make these two terms approximately the same size. Summation by parts [1, Theorem 4.2] gives

$$\phi(s) = \sum_{l=1}^{y/d} \frac{(-1)^l}{l^s} - \frac{S(y/d)}{(y/d)^s} + s \int_{y/d}^{\infty} \frac{S(x) - S(y/d)}{x^{s+1}}\, dx,$$

where $S(x) = \sum_{n \le x} (-1)^n$ is -1 or 0. Set $s = \rho$ and use $\phi(\rho) = 0$; we bound the integral getting

$$\left| s \int_{y/d}^{\infty} \frac{S(x) - S(y/d)}{x^{s+1}}\, dx \right| \le \frac{|\rho|}{\beta (y/d)^\beta}, \qquad \left| \frac{S(y/d)}{(y/d)^s} \right| \le \frac{1}{(y/d)^\beta}.$$

So we claim

$$\left| \sum_{l=1}^{y/d} \frac{(-1)^l}{l^\rho} \right| \le \frac{|\rho|}{\beta (y/d)^\beta},$$

since $1 < 1/\beta$ and [3] shows that $10^{12} < |\rho|$.

Thus we can estimate the first term in the previous sum:

$$\left| \sum_{d \le z} \frac{\chi(d)}{d^\rho} \sum_{l \le y/d} \frac{(-1)^l}{l^\rho} \right| \le \sum_{d \le z} \frac{1}{d^\beta} \cdot \frac{|\rho|}{\beta (y/d)^\beta} = \frac{z|\rho|}{y^\beta \beta}.$$

Another summation by parts gives

$$\sum_{z < d \le y/l} \frac{\chi(d)}{d^s} = \frac{S_D(y/l)}{(y/l)^s} - \frac{S_D(z)}{z^s} + s \int_z^{y/l} \frac{S_D(x) - S_D(\sqrt{y})}{x^{s+1}}\, dx,$$

where $S_D(x) = \sum_{n \le x} \chi(n)$. By the Pólya–Vinogradov inequality [1, Theorem 8.21], $|S_D(x)| < \sqrt{D} \log D$. Neglecting the boundary terms as before, we bound the integral as

$$\left| \sum_{z < d \le y/l} \frac{\chi(d)}{d^\rho} \right| \le \frac{|\rho| \sqrt{D} \log D}{\beta z^\beta},$$

and so bound the second sum above as

$$\left| \sum_{l \le y/z} \frac{(-1)^l}{l^\rho} \sum_{z < d \le y/l} \frac{\chi(d)}{d^\rho} \right| \le \sum_{l \le y/z} \frac{|\rho| \sqrt{D} \log D}{\beta l^\beta z^\beta} = \frac{|\rho| \sqrt{D} \log D}{\beta} \sum_{l \le y/z} \frac{1}{l^\beta z^\beta}.$$

Now

$$\sum_{l \le y/z} \frac{1}{l^\beta z^\beta} = \frac{y^{1-\beta}}{z} \sum_{l \le y/z} \frac{1}{l^\beta (y/z)^{1-\beta}} < \frac{y^{1-\beta}}{z} \sum_{l \le y/z} \frac{1}{l^\beta \cdot l^{1-\beta}} \sim \frac{y^{1-\beta} \log(y/z)}{z},$$

where the inequality follows since $l < y/z$. This gives, for the second sum, the bound

$$\frac{|\rho| \sqrt{D} \log D}{\beta} \cdot \frac{y^{1-\beta} \log(y/z)}{z}.$$

Comparing the two estimates, we see they are approximately the same size when

$$\frac{z}{y^\beta} = \frac{\sqrt{D}\, y^{1-\beta}}{z}, \quad \text{or} \quad z = D^{1/4} y^{1/2}.$$

Combining the two sum estimates, and with

$$\frac{1}{\beta} < \frac{6}{5} \quad \text{and} \quad 1 < \frac{\log(y/\sqrt{D})\log D}{18},$$

we have

$$\frac{y^{1/2-\beta}|\rho|D^{1/4}}{\beta} + \frac{y^{1/2-\beta}|\rho|\log(y/\sqrt{D})D^{1/4}\log D}{2\beta}$$
$$< \frac{6}{5}\left(\frac{1}{18}+\frac{1}{2}\right)y^{1/2-\beta}\log(y/\sqrt{D})|\rho|D^{1/4}\log D$$
$$= \frac{2}{3}y^{1/2-\beta}\log(y/\sqrt{D})|\rho|D^{1/4}\log D. \ \blacksquare$$

Lower bounds again. Applying Lemma 3 with $y = U^{12}/n$, so $U^6 < y < U^{12}$, we get

$$\Sigma_1 < 8U^{6-12\beta}\log U|\rho|D^{1/4}\log D\sum_{n\le U^6}\frac{\tau(n)}{\sqrt{n}} = 8U^{7-12\beta}\log U\sum_{n\le U^6}\frac{\tau(n)}{\sqrt{n}}.$$

With an estimate by the standard 'method of the hyperbola' (e.g. [5, (2.9), p. 37]), we get

$$\sum_{n\le X}\frac{\tau(n)}{\sqrt{n}} = X^{1/2}(2\log X + 4C - 4) + O(1).$$

Thus

$$\Sigma_1 < 96U^{10-12\beta}\log^2 U,$$

and so, for $\beta > 5/6$, Σ_1 is small. In fact, from

$$\frac{1}{25}\cdot\frac{\zeta(4\beta)}{\zeta(2\beta)} - U^{6-12\beta} \le \Sigma_1 + \Sigma_2,$$

Mathematica tells us $1/50 < \Sigma_2$ when $\beta > 7/8$ and $U > 10^{16}$. (We are assuming $D > 10^9$, and Gourdon [3] has verified the Riemann Hypothesis for the first 10^{13} zeros. Therefore our hypothetical ρ satisfies $|\rho| > 2.4\cdot 10^{12}$, so necessarily $U = |\rho|D^{1/4}\log D > 10^{16}$.)

We now convert the lower bound for Σ_2 to a lower bound for $L(1,\chi)$. Recall that

$$\Sigma_2 = \sum_{U^6<n\le U^{12}}\frac{1*\chi(n)}{n^\beta}\sum_{r\le U^{12}/n}\frac{\tau(r)}{r^\beta}.$$

Writing $r^{-\beta} = r^{1-\beta}/r$ and using $r^{1-\beta} < U^{12(1-\beta)}n^{\beta-1}$ we see that

$$\frac{1}{50} < \Sigma_2 < U^{12(1-\beta)}\sum_{U^6<n\le U^{12}}\frac{1*\chi(n)}{n}\sum_{r\le U^{12}/n}\frac{\tau(r)}{r}.$$

The 'method of the hyperbola' argument shows in [5, Ex. 11.2.1 (g)] ([3]) that

$$\sum_{U^6\le n\le U^{12}} \frac{1*\chi(n)}{n} = \log(U^6)L(1,\chi)+O(D^{1/4}U^{-3}\log D\log(U^6))$$
$$= \log(U^6)L(1,\chi)+O(U^{-2}\log(U^6))$$
$$= \log(U^6)(L(1,\chi)+O(U^{-2})).$$

Meanwhile one more application of this same tool (along with Euler summation) gives

$$\sum_{r<X}\frac{\tau(r)}{r} = \frac{1}{2}\log^2 X + 2C\log X + O(1).$$

So

$$\sum_{r\le U^{12}/n}\frac{\tau(r)}{r} \sim \frac{1}{2}\log^2(U^{12}/n) < \frac{1}{2}\log^2(U^6),$$

as $U^6 < n$. Finally

$$\frac{1}{50} < \Sigma_2 < U^{12(1-\beta)}\log(U^6)(L(1,\chi)+O(U^{-2}))\cdot\frac{1}{2}\log^2(U^6)$$
$$= 108U^{12(1-\beta)}\log^3 U\,(L(1,\chi)+O(U^{-2})).$$

The implied constant is no worse than 6, and

$$U^{-2} = \frac{1}{|\rho|^2\sqrt{D}\log^2 D} < \frac{1}{\sqrt{D}},$$

so the theorem follows.

The general case. We fix a prime $q\mid D$ and consider

$$\sum_{n=1}^{\infty}\frac{c_q(n)}{n^s} = (q^{1-s}-1)\zeta(s),$$

where $c_q(n)$ is the Ramanujan sum

$$c_q(n) = \sum_{k=1}^{q-1}\exp(2\pi ikn/q) = \begin{cases} -1 & \text{if } (n,q)=1,\\ q-1 & \text{if } q\mid n.\end{cases}$$

(Observe that $c_2(n) = (-1)^n$.) Since $|\sum_{n<x} c_q(n)| < q$, the Dirichlet series converges conditionally for $\mathrm{Re}(s) > 0$. The Ramanujan sums are not multiplicative in n, but we have $c_q(dm) = c_q(m)$ if $(d,q)=1$. Instead of λ_{odd} we define a function $\lambda_q(n) = 0$ if $q\mid n$. The proof goes through as before. We

([3]) The implied constant in that exercise, combining six big Oh terms with implied constant equal to 1, can be taken to be 6.

find that in Lemma 2 we have

$$\sum_{n=1}^{\infty} \frac{c_q(n) \cdot 1 * \lambda_q(n)}{n^\rho} = (1 + q^{-\rho})(1 - q^{1-\rho})\zeta(2\rho),$$

so the trivial zeros along $\mathrm{Re}(s) = 1$ when $\gamma = 2\pi n/\log q$ still cause a problem. In fact, the constant $1/25$ in Lemma 2 which works for $q = 2$ is a decreasing function of q in the general case.

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