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Points on $X_0^+(N)$ over quadratic fields

by

KEISUKE ARAI and $\boxed{\text{FUMIYUKI MOMOSE}}$ (Tokyo)

1. Introduction. In this article, we study points on the modular curve $X_0^+(N)$ over quadratic fields, and show that such points consist of cusps and CM points under certain conditions.

Let $N \geq 1$ be an integer. Let $X_0(N)$ be the modular curve over $\mathbb{Q}$ associated to the subgroup $\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} \subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ (cf. [5]). A non-cuspidal point on $X_0(N)$ corresponds to a pair (E, A) where E is an elliptic curve and A is a cyclic subgroup of E of order N. For rational points on $X_0(N)$, we know the following.

THEOREM 1.1 ([8, p. 129, Theorem 1]). *If $N > 163$, then $X_0(N)(\mathbb{Q}) = \{cusps\}$.*

The second author studied points on $X_0(N)$ over quadratic fields when N is a prime number.

THEOREM 1.2 ([12, p. 330, Theorem B]). *Let K be a quadratic field which is not an imaginary quadratic field of class number one. Then for every sufficiently large prime number p, we have $X_0(p)(K) = \{cusps\}$.*

For any number field K, it seems likely that

$$X_0(N)(K) = \{\text{cusps, CM points}\}$$

for every sufficiently large integer N (cf. [16, p. 187]). But this still remains unsolved. Here a point x on a modular curve (e.g. $X_0(N)$, $X_0^+(N)$ defined below) is called a *CM point* if x is represented by an elliptic curve with complex multiplication.

Define an involution w_N on $X_0(N)$ by

$$(E, A) \mapsto (E/A, E[N]/A),$$

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The first author deeply regrets the death of his co-author, Professor Fumiyuki Momose during this work, and dedicates this article to his memory.

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where $E[N]$ is the kernel of multiplication by N in E. Put

$$X_0^+(N) := X_0(N)/w_N.$$

We have the following open question: For a number field K, does

$$X_0^+(N)(K) = \{\text{cusps, CM points}\}$$

hold for every sufficiently large integer N? Notice that there are arbitrarily large N such that $X_0^+(N)(\mathbb{Q}) = \{\text{cusps}\}$ does not hold. We know the following partial answers (Theorem 1.3, Theorem 1.5) to the above question.

THEOREM 1.3 ([2]). *For every sufficiently large prime number p, we have $X_0^+(p^2)(\mathbb{Q}) = \{cusps,\ CM\ points\}$.*

REMARK 1.4. We have a natural isomorphism $X_0^+(p^2) \cong X_{\mathrm{split}}(p)$, where $X_{\mathrm{split}}(p)$ is the modular curve (over $\mathbb{Q}$) associated to the subgroup $\{\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & * \\ * & 0 \end{smallmatrix}\right)\} \subseteq \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$.

Let p be a prime number. We have an involution w_p on $X_0(p)$ as above. By abuse of notation, we also write w_p for the induced map $J_0(p) \to J_0(p)$. Put

$$J_0^-(p) := J_0(p)/(1+w_p)J_0(p).$$

Let

$$C := \langle cl((\mathbf{0}) - (\boldsymbol{\infty}))\rangle \subseteq J_0(p)(\mathbb{Q})$$

be the subgroup generated by the divisor class $cl((\mathbf{0}) - (\boldsymbol{\infty}))$ (for the precise definition of the cusps $\mathbf{0}$ and $\boldsymbol{\infty}$, see the next section). Then $C = J_0(p)(\mathbb{Q})_{\mathrm{tor}}$ (the torsion subgroup of $J_0(p)(\mathbb{Q})$) and C maps isomorphically to $J_0^-(p)(\mathbb{Q})_{\mathrm{tor}}$ by the natural map ([6, p. 143, Corollary (1.4)], cf. [14, p. 229]). By abuse of notation we identify $C = J_0^-(p)(\mathbb{Q})_{\mathrm{tor}}$. The order of C is equal to the numerator of $\frac{p-1}{12}$ ([14, p. 228, Theorem] or [6, p. 98, Proposition (11.1)]).

THEOREM 1.5 ([11, p. 269, Theorem (0.1)], cf. [9], [10]). *Let N be a composite number. If N has a prime divisor p which satisfies the following conditions* (1) *and* (2), *then $X_0^+(N)(\mathbb{Q}) = \{cusps,\ CM\ points\}$.*

(1) *$p \geq 17$ or $p = 11$.*
(2) *$p \neq 37$ and $\sharp J_0^-(p)(\mathbb{Q}) < \infty$.*

We generalize Theorem 1.5 to quadratic fields. The following is the main theorem of this article.

THEOREM 1.6. *Let N be a composite number. Let p be a prime divisor of N such that* ($p = 11$ *or* $p \geq 17$) *and* $p \neq 37$. *Suppose* $\mathrm{ord}_p N = 1$ *if* $p = 11$. *Let K be a quadratic field where p is unramified. Assume $X_0(N)(K) = \{cusps\}$ and $J_0^-(p)(K) = C$. Then $X_0^+(N)(K) = \{cusps,\ CM\ points\}$.*

REMARK 1.7. Since the modular curve $X_0(37)$ is peculiar ([15]), we exclude $p = 37$ in the above theorems. But we have recently shown that Theorem 1.5 holds even if $p = 37$, and have generalized the result to certain imaginary quadratic fields ([1]).

REMARK 1.8. (1) For N as in Theorem 1.5, we have $X_0(N)(\mathbb{Q}) = \{\text{cusps}\}$ ([8, pp. 129–131]).

(2) The assumption $X_0(N)(K) = \{\text{cusps}\}$ in Theorem 1.6 is usually satisfied by Theorem 1.2.

We have the following examples of the condition $J_0^-(p)(K) = C$ in Theorem 1.6. For a number field K, let h_K be the class number of K.

PROPOSITION 1.9. *Let K be an imaginary quadratic field.*

(1) *Suppose* 11 *does not split in K and* 5 *does not divide h_K. Then $J_0^-(11)(K) = C$.*
(2) *Suppose* 17 *does not split in K and* 2 *does not divide h_K. Then $J_0^-(17)(K) = C$.*
(3) *Suppose* 19 *does not split in K and* 3 *does not divide h_K. Then $J_0^-(19)(K) = C$.*

In Section 2, we prepare the necessary material on modular curves. In Section 3, we introduce a key proposition (Proposition 3.1) and from it we deduce Theorem 1.6. In Section 4, we prove Proposition 3.1. In Section 5, we prove Proposition 1.9.

2. Modular curves. For a prime number p, let $g : X_0(p) \to X_0^+(p)$ be the quotient map. We know that the Jacobian variety $J_0^+(p)$ of $X_0^+(p)$ is isomorphic to $(1 + w_p)J_0(p)$ and there is an exact sequence of abelian varieties

$$0 \to J_0^+(p) \xrightarrow{g^*} J_0(p) \xrightarrow{u} J_0^-(p) \to 0,$$

where g^* is the pull back and u is the quotient map ([11, p. 278]).

For an integer $N \geq 1$, let $\mathcal{X}_0(N)$ be the normalization of the composite

$$X_0(N) \xrightarrow{j} X_0(1) = \mathbb{P}^1_{\mathbb{Q}} \subseteq \mathbb{P}^1_{\mathbb{Z}},$$

where $j : (E, A) \mapsto E$. If p is a prime divisor of N with $r = \operatorname{ord}_p N$, then the special fiber $\mathcal{X}_0(N) \otimes_{\mathbb{Z}} \mathbb{F}_p$ has $r + 1$ irreducible components $E_0, E_1, \ldots, E_r$. They are defined over $\mathbb{F}_p$ and intersect at the supersingular points. Let $\zeta = \zeta_N$ be a primitive Nth root of unity. For each positive divisor d of N and an integer i, $0 \leq i < d$, prime to d, let $A_{d,i}$ be the subgroup of $\mathbb{G}_{\mathrm{m}} \times \mathbb{Z}/(N/d)\mathbb{Z}$ generated by $(\zeta^i, 1 \bmod N/d)$. Let $\binom{i}{d}$ be the cuspidal section of $\mathcal{X}_0(N)$ which is represented by the pair $(\mathbb{G}_{\mathrm{m}} \times \mathbb{Z}/(N/d)\mathbb{Z}, A_{d,i})$ for the integers d, i as above. For $d = 1, N$, we write $\mathbf{0} = \binom{0}{1}$ and $\boldsymbol{\infty} = \binom{1}{N}$. We choose the irreducible components E_t so that $\binom{i}{d} \otimes \mathbb{F}_p$ are sections of E_t for a positive divisor d of N with $t = \mathrm{ord}_p\, d$. For $0 \leq t \leq r$, let E_t^h be the open subscheme of E_t obtained by excluding the supersingular points.

The special fiber $\mathcal{X}_0(p) \otimes_{\mathbb{Z}} \mathbb{F}_p$ has $g_0(p)+1$ supersingular points. They can be described as follows. Let α_i, $\alpha_i' := w_p(\alpha_i)$ be the non-$\mathbb{F}_p$-rational supersingular points on $\mathcal{X}_0(p) \otimes_{\mathbb{Z}} \mathbb{F}_p$ for $1 \leq i \leq g_0^+(p)$, and let β_i be the $\mathbb{F}_p$-rational supersingular points on $\mathcal{X}_0(p) \otimes_{\mathbb{Z}} \mathbb{F}_p$ for $1 \leq i \leq g_0(p) - 2g_0^+(p) + 1$. The involution w_p exchanges α_i and α_i' and fixes β_i ([11, p. 279]).

For a finite abelian group G and an integer $n \geq 1$, let $G^{(n)}$ be the prime-to-n subgroup of G. For an abelian group (or a commutative group scheme) G and an integer n, let $G[n]$ be the kernel of multiplication by n in G. For a group scheme G, let G^0 be the connected component of the identity in G. For a morphism of schemes $X \to S$, let X^{sm} be the smooth locus of X. For a prime number p, let $\mathbb{Q}_p^{\mathrm{unr}}$ be the maximal unramified extension of $\mathbb{Q}_p$, and let $\mathbb{Z}_p^{\mathrm{unr}}$ be the ring of integers of $\mathbb{Q}_p^{\mathrm{unr}}$. For a number field or a discrete valuation field L, let $\mathcal{O}_L$ be the ring of integers. For an abelian variety J over a number field or a discrete valuation field L, let $J_{/\mathcal{O}_L}$ be the Néron model of J over $\mathcal{O}_L$ (later we take $J_0(p)$ or $J_0^-(p)$ as J).

Let p be a prime number and $M \geq 1$ be an integer. Let

$$\pi : X_0(pM) \to X_0(p), \qquad (E, A) \mapsto (E, A[p]).$$

Define

$$h : X_0(pM) \to J_0(p), \qquad h(x) := cl((w_p\pi(x)) - (\pi w_{pM}(x))).$$

Put

$$\widetilde{h}^- : X_0(pM) \xrightarrow{h} J_0(p) \to J_0^-(p),$$

where $J_0(p) \to J_0^-(p)$ is the quotient map. The map $\widetilde{h}^-$ factors as $X_0(pM) \to X_0^+(pM) \to J_0^-(p)$, where $X_0(pM) \to X_0^+(pM)$ is the quotient map. We call the induced map $h^- : X_0^+(pM) \to J_0^-(p)$. Thus we have the following commutative diagram:

$$\begin{array}{ccc} X_0(pM) & \xrightarrow{h} & J_0(p) \\ \downarrow & & \downarrow \\ X_0^+(pM) & \xrightarrow{h^-} & J_0^-(p) \end{array}$$

See [1, p. 2276].

3. Key proposition

PROPOSITION 3.1. *Let K be a quadratic field. Let p be a prime number such that $p = 11$ or $p \geq 17$. Let $M \geq 2$ be an integer and suppose $X_0(pM)(K) = \{cusps\}$. Let $y \in X_0^+(pM)(K)$ be a non-cuspidal point, and x,*

$w_{pM}(x)$ be sections of the fiber $X_0(pM)_y$. Let L be the quadratic extension of K over which x and $w_{pM}(x)$ are defined. Take a prime $\mathfrak{p}$ of L above p, and let $\kappa(\mathfrak{p})$ be the residue field of $\mathfrak{p}$. Assume $p \nmid M$ if $p = 11$.

(1) *If $p \mid M$ or $x \otimes \kappa(\mathfrak{p})$ is not a supersingular point, then $h(x) \otimes \kappa(\mathfrak{p})$ is a section of the connected component $(J_0(p)_{/\mathcal{O}_L} \otimes \kappa(\mathfrak{p}))^0$ of the identity.*

(2) *Suppose otherwise (i.e. $p \nmid M$ and $x \otimes \kappa(\mathfrak{p})$ is a supersingular point).*

(2-a) *If one of the following three conditions is satisfied, then $h(x) \otimes \kappa(\mathfrak{p})$ is a section of $(J_0(p)_{/\mathcal{O}_L} \otimes \kappa(\mathfrak{p}))^0$.*

- *$\mathfrak{p}$ is unramified in $L/\mathbb{Q}$.*
- *$\mathfrak{p}$ is ramified in L/K and p is split in K.*
- *$\mathfrak{p}$ is inert in L/K and p is ramified in K.*
- *$\mathfrak{p}$ is ramified in L/K and p is ramified in K.*

(2-b) *If $\mathfrak{p}$ is ramified in L/K and p is inert in K, then $h^-(y) \otimes \kappa(\mathfrak{p})$ is a section of $(J_0^-(p)_{/\mathcal{O}_L} \otimes \kappa(\mathfrak{p}))^0$.*

Remark 3.2. (1) In Proposition 3.1, $h^-(y)\otimes\kappa(\mathfrak{p})$ is a section of $(J_0^-(p)_{/\mathcal{O}_L} \otimes \kappa(\mathfrak{p}))^0$ in any case.

(2) We do not treat the case where $\mathfrak{p}$ is split in L/K and p is ramified in K in Proposition 3.1. In that case the proof does not work.

(3) We do not use the last two cases of (2-a) in Proposition 3.1 for proving Theorem 1.6.

Lemma 3.3 ([11, p. 278 Proposition (2.8)]). *Let L' be an extension of $\mathbb{Q}_p^{\mathrm{unr}}$ of degree ≤ 2. Let $\mathcal{C} \subseteq J_0^-(p)_{/\mathcal{O}_{L'}}$ be the finite flat subgroup scheme generated by C. Then $(\mathcal{C} \otimes \overline{\mathbb{F}}_p) \cap (J_0^-(p)_{/\mathcal{O}_{L'}} \otimes \overline{\mathbb{F}}_p)^0 = \{0\}$.*

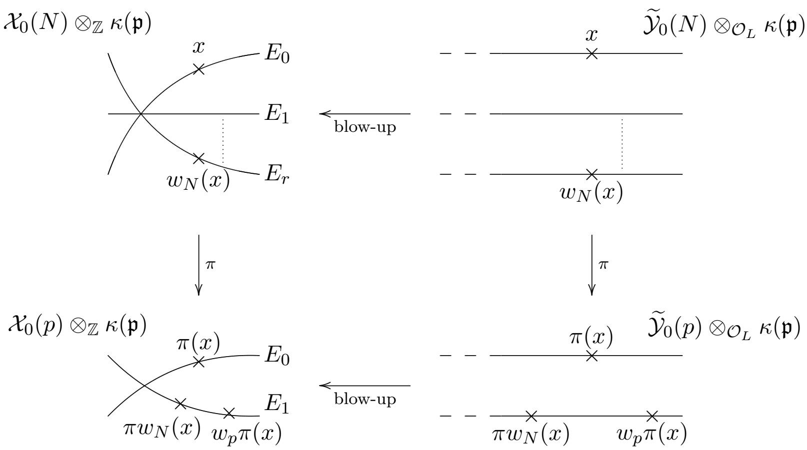
Proposition 3.4. *Under the hypothesis in Proposition 3.1, further assume that p is unramified in K and $J_0^-(p)(K) = C$. Then $h^-(y) = 0$.*

Proof. By assumption we have $h^-(y) \in J_0^-(p)(K) = C$. Let L' be the maximal unramified extension of the completion $L_{\mathfrak{p}}$. Then $[L' : \mathbb{Q}_p^{\mathrm{unr}}] \le 2$ because p is unramified in K. Since $h^-(y) \in C \subseteq J_0^-(p)(L')$, we have $h^-(y) \in \mathcal{C}(\mathcal{O}_{L'}) \subseteq J_0^-(p)_{/\mathcal{O}_{L'}}(\mathcal{O}_{L'})$. Hence $h^-(y) \otimes \overline{\mathbb{F}}_p \in \mathcal{C}(\overline{\mathbb{F}}_p) \subseteq J_0^-(p)_{/\mathcal{O}_{L'}}(\overline{\mathbb{F}}_p)$. On the other hand $h^-(y)\otimes\overline{\mathbb{F}}_p \in (J_0^-(p)_{/\mathcal{O}_L}\otimes\kappa(\mathfrak{p}))^0(\overline{\mathbb{F}}_p) = (J_0^-(p)_{/\mathcal{O}_{L'}}\otimes\overline{\mathbb{F}}_p)^0(\overline{\mathbb{F}}_p)$ by Proposition 3.1. Notice that taking the connected component is compatible with base change since $J_0^-(p)$ is semi-stable ([4, p. 183, Corollary 4]). Then $h^-(y) \otimes \overline{\mathbb{F}}_p = 0$ by Lemma 3.3. Since the order of C is prime to p, the group scheme $\mathcal{C}$ over $\mathcal{O}_{L'}$ is étale. Therefore $h^-(y) = 0$. ∎

The condition $h^-(y) = 0$ implies that y is a CM point since $p \neq 37$ ([11, p. 274, Proposition (2.2)]). Thus Theorem 1.6 follows from Proposition 3.1.

4. Calculation of connected components. Now we prove Proposition 3.1.

For simplicity write $N = pM$. Let $\widetilde{\mathcal{Y}}_0(p) \to \operatorname{Spec}\mathcal{O}_L$ be the minimal proper regular model of $X_0(p) \otimes_{\mathbb{Q}} L$. We may canonically identify $\mathcal{X}_0(N)(\mathcal{O}_L) = X_0(N)(L)$ and $\mathcal{X}_0(p)(\mathcal{O}_L) = X_0(p)(L) = \widetilde{\mathcal{Y}}_0(p)(\mathcal{O}_L)$. If $w_p\pi(x)$ and $\pi w_N(x)$ define sections of the same irreducible component of $\widetilde{\mathcal{Y}}_0(p)^{\mathrm{sm}} \otimes \kappa(\mathfrak{p})$, then $h(x) \otimes \kappa(\mathfrak{p})$ is a section of $(J_0(p)_{/\mathcal{O}_L} \otimes \kappa(\mathfrak{p}))^0$ ([6, p. 179, Proposition (1.4)]). Put $r = \operatorname{ord}_p N$. If $x \otimes \kappa(\mathfrak{p})$ is a section of $E_0^h \cup E_r^h$, then $w_p\pi(x)$ and $\pi w_N(x)$ define sections of the same irreducible component of $\widetilde{\mathcal{Y}}_0(p)^{\mathrm{sm}} \otimes \kappa(\mathfrak{p})$. To see this, we use the following: π maps E_0 to E_0 and E_r to E_1; w_N exchanges E_0 and E_r; w_p exchanges E_0 and E_1 ([10, p. 446]). Notice that here we use the symbol E_i in two ways.



If $p \mid M$, then $x \otimes \kappa(\mathfrak{p})$ is a section of $E_0^h \cup E_r^h$ since $e_{L/\mathbb{Q}}(\mathfrak{p}) \le 4$ and $3e_{L/\mathbb{Q}}(\mathfrak{p}) < p - 1$ ([10, p. 452, Corollary (2.3)], cf. [13, p. 159, Main Theorem]). Here we used $p \ge 17$. If $p \nmid M$ and $x \otimes \kappa(\mathfrak{p})$ is not a supersingular point, then $x \otimes \kappa(\mathfrak{p})$ is a section of $E_0^h \cup E_r^h$ for $r = 1$.

From now on we consider the case when $p \nmid M$ and $x \otimes \kappa(\mathfrak{p})$ is a supersingular point.

Case (i): $\mathfrak{p}$ *is unramified* in $L/\mathbb{Q}$. In this case $j(x \otimes \kappa(\mathfrak{p})) = 0$ or 1728, and

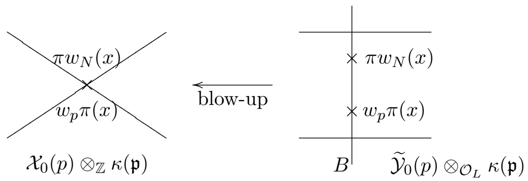
$$\hat{\mathcal{O}}_{\mathcal{X}_0(N)\otimes\mathbb{Z}_p^{\mathrm{unr}},x} \cong \mathbb{Z}_p^{\mathrm{unr}}[[u,v]]/(uv - p^i)$$

where $i = 3$ (resp. 2) if $j(x \otimes \kappa(\mathfrak{p})) = 0$ (resp. 1728) ([6, p. 63]). Here $\hat{\mathcal{O}}_{\mathcal{X}_0(N)\otimes\mathbb{Z}_p^{\mathrm{unr}},x}$ is the completion of the local ring $\mathcal{O}_{\mathcal{X}_0(N)\otimes\mathbb{Z}_p^{\mathrm{unr}},x}$ at the maximal ideal. Since w_N is an automorphism, we have

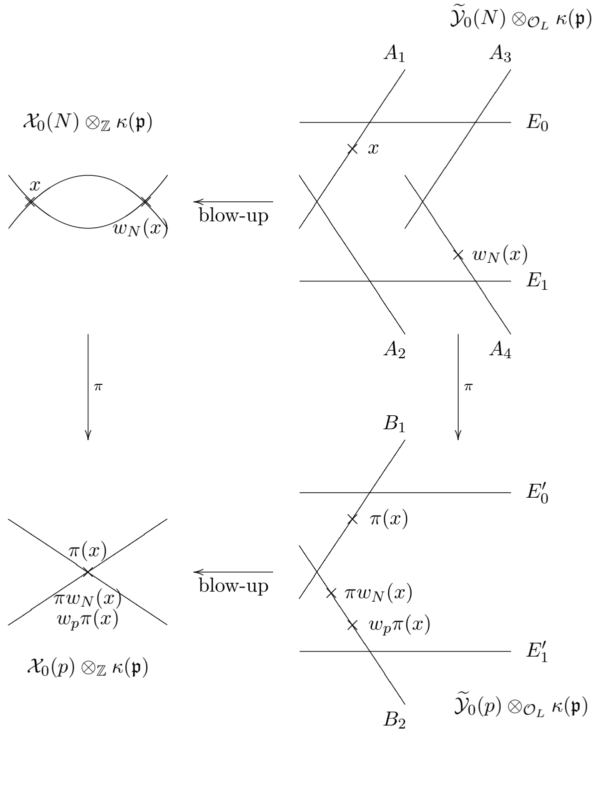
$$\hat{\mathcal{O}}_{\mathcal{X}_0(N)\otimes\mathbb{Z}_p^{\mathrm{unr}},w_N(x)} \cong \mathbb{Z}_p^{\mathrm{unr}}[[u,v]]/(uv - p^i).$$

Then $j(w_N(x)\otimes\kappa(\mathfrak{p})) = j(x\otimes\kappa(\mathfrak{p})) = 0$ (resp. 1728). Hence $j(\pi w_N(x)\otimes\kappa(\mathfrak{p})) = j(\pi(x)\otimes\kappa(\mathfrak{p}))$. Since w_p fixes all the $\mathbb{F}_p$-rational supersingular points on $\mathcal{X}_0(p)\otimes\mathbb{F}_p$, we have $\pi w_N(x)\otimes\kappa(\mathfrak{p}) = \pi(x)\otimes\kappa(\mathfrak{p}) = w_p\pi(x)\otimes\kappa(\mathfrak{p})$.

If $j(x\otimes\kappa(\mathfrak{p})) = 1728$, then $w_p\pi(x)\otimes\kappa(\mathfrak{p})$ and $\pi w_N(x)\otimes\kappa(\mathfrak{p})$ define sections of the unique exceptional irreducible component B of $\widetilde{\mathcal{Y}}_0(p)^{\mathrm{sm}}\otimes_{\mathcal{O}_L}\kappa(\mathfrak{p})$. Therefore $h(x)\otimes\kappa(\mathfrak{p})$ is a section of $(J_0(p)_{/\mathcal{O}_L}\otimes\kappa(\mathfrak{p}))^0$.

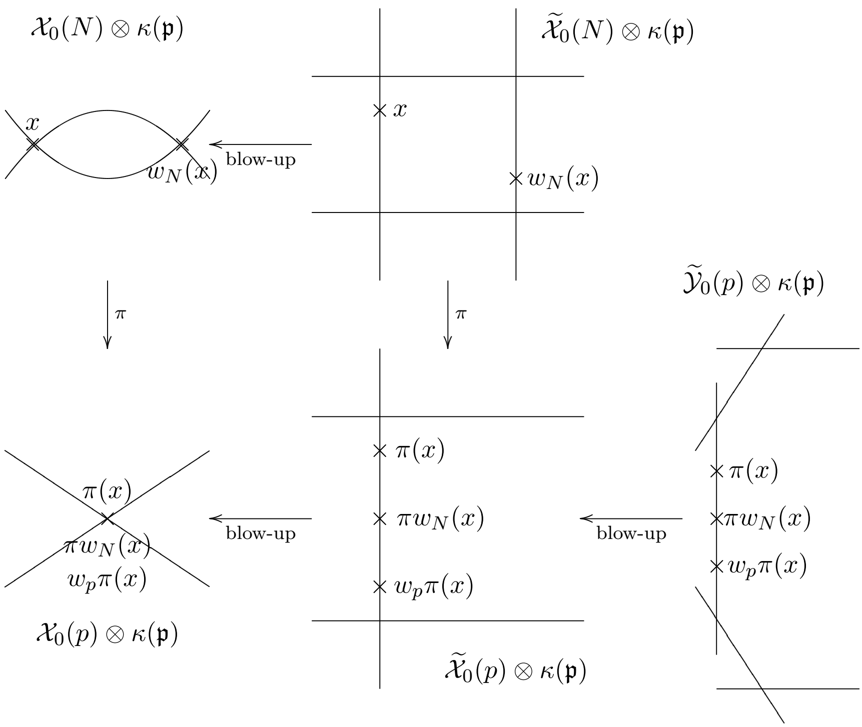


Assume $j(x\otimes\kappa(\mathfrak{p})) = 0$. Then $\widetilde{\mathcal{Y}}_0(p)^{\mathrm{sm}}\otimes_{\mathcal{O}_L}\kappa(\mathfrak{p})$ has two exceptional irreducible components, say B_1, B_2. Also $\widetilde{\mathcal{Y}}_0(N)^{\mathrm{sm}}\otimes_{\mathcal{O}_L}\kappa(\mathfrak{p})$ has two exceptional irreducible components over $x\otimes\kappa(\mathfrak{p})$ (resp. $w_N(x)\otimes\kappa(\mathfrak{p})$), say A_1, A_2 (resp. A_3, A_4). See the figure below. We may assume $x\otimes\kappa(\mathfrak{p})$ is a section of A_1^{sm}. Then $w_N(x)\otimes\kappa(\mathfrak{p})$ is a section of A_4^{sm}. Hence $\pi(x)\otimes\kappa(\mathfrak{p})$ (resp. $\pi w_N(x)\otimes\kappa(\mathfrak{p})$) is a section of B_1^{sm} (resp. B_2^{sm}). Therefore $w_p\pi(x)\otimes\kappa(\mathfrak{p})$ and $\pi w_N(x)\otimes\kappa(\mathfrak{p})$ are sections of the same irreducible component B_2^{sm}, and so $h(x)\otimes\kappa(\mathfrak{p})$ is a section of $(J_0(p)_{/\mathcal{O}_L}\otimes\kappa(\mathfrak{p}))^0$. Note that $x\otimes\kappa(\mathfrak{p})$ and $w_N(x)\otimes\kappa(\mathfrak{p})$ may be equal in $\mathcal{X}_0(N)\otimes_{\mathbb{Z}}\kappa(\mathfrak{p})$. Then $A_1 = A_3$, $A_2 = A_4$.

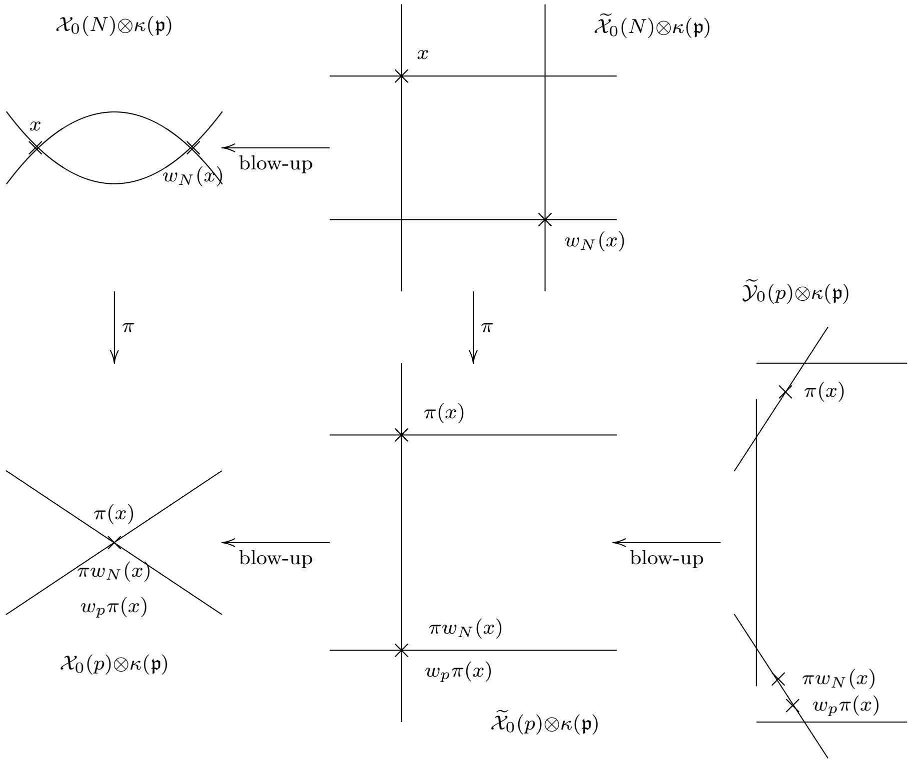


Case (ii): $\mathfrak{p}$ *is ramified in* L/K *and* p *is split in* K. Let $\sigma \in \mathrm{Gal}(L/K)$ be the non-trivial element. Since $\mathfrak{p}$ is ramified in L/K, we have $x^\sigma \otimes \kappa(\mathfrak{p}) = x \otimes \kappa(\mathfrak{p})$. Since $\kappa(\mathfrak{p}) = \mathbb{F}_p$, the sections $x \otimes \kappa(\mathfrak{p})$ and $w_N(x) \otimes \kappa(\mathfrak{p}) = x^\sigma \otimes \kappa(\mathfrak{p})$ are $\mathbb{F}_p$-rational. Thus $\pi(x) \otimes \kappa(\mathfrak{p})$ and $\pi w_N(x) \otimes \kappa(\mathfrak{p})$ are also $\mathbb{F}_p$-rational. Since w_p fixes all the $\mathbb{F}_p$-rational supersingular points on $\mathcal{X}_0(p) \otimes \mathbb{F}_p$, we have $\pi w_N(x) \otimes \kappa(\mathfrak{p}) = \pi(x) \otimes \kappa(\mathfrak{p}) = w_p\pi(x) \otimes \kappa(\mathfrak{p}) \in \mathcal{X}_0(p)(\kappa(\mathfrak{p}))$. If $j(x \otimes \kappa(\mathfrak{p})) \neq 0, 1728$, then $w_p\pi(x) \otimes \kappa(\mathfrak{p})$ and $\pi w_N(x) \otimes \kappa(\mathfrak{p})$ correspond to sections in the unique exceptional irreducible component of $\widetilde{\mathcal{Y}}_0(p)^{\mathrm{sm}} \otimes_{\mathcal{O}_L} \kappa(\mathfrak{p})$.

Suppose $j(x \otimes \kappa(\mathfrak{p})) = 0, 1728$. Let $\widetilde{\mathcal{X}}_0(N)$ (resp. $\widetilde{\mathcal{X}}_0(p)$) be the minimal regular model of $X_0(N)$ (resp. $X_0(p)$) over $\mathbb{Z}_p$. Then $\widetilde{\mathcal{Y}}_0(p) \otimes \mathcal{O}_{L_\mathfrak{p}}$ is obtained from $\widetilde{\mathcal{X}}_0(p) \otimes \mathcal{O}_{L_\mathfrak{p}}$ by blowing-up at the singular points of the special fiber. Assume $j(x \otimes \kappa(\mathfrak{p})) = 1728$. If $x \otimes \kappa(\mathfrak{p})$ define a section of $\widetilde{\mathcal{X}}_0(N)^{\mathrm{sm}} \otimes \kappa(\mathfrak{p})$, then $\pi(x) \otimes \kappa(\mathfrak{p})$, $\pi w_N(x) \otimes \kappa(\mathfrak{p})$ and $w_p\pi(x) \otimes \kappa(\mathfrak{p})$ define sections of the unique exceptional irreducible component of $\widetilde{\mathcal{X}}_0(p)^{\mathrm{sm}} \otimes \kappa(\mathfrak{p})$. Hence $\pi(x) \otimes \kappa(\mathfrak{p})$, $\pi w_N(x) \otimes \kappa(\mathfrak{p})$ and $w_p\pi(x) \otimes \kappa(\mathfrak{p})$ define sections of the same irreducible component of $\widetilde{\mathcal{Y}}_0(p)^{\mathrm{sm}} \otimes \kappa(\mathfrak{p})$.



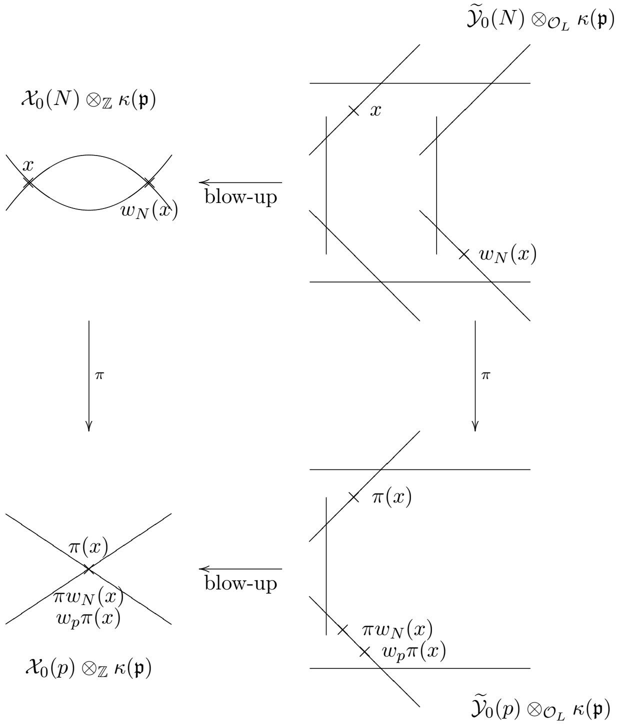
If $x \otimes \kappa(\mathfrak{p})$ corresponds to a singular point of $\widetilde{\mathcal{X}}_0(N) \otimes \kappa(\mathfrak{p})$, then by an easy calculation, $w_p\pi(x) \otimes \kappa(\mathfrak{p})$ and $\pi w_N(x) \otimes \kappa(\mathfrak{p})$ define sections of the same irreducible component of $\widetilde{\mathcal{Y}}_0(p)^{\mathrm{sm}} \otimes \kappa(\mathfrak{p})$ (see the figure below).

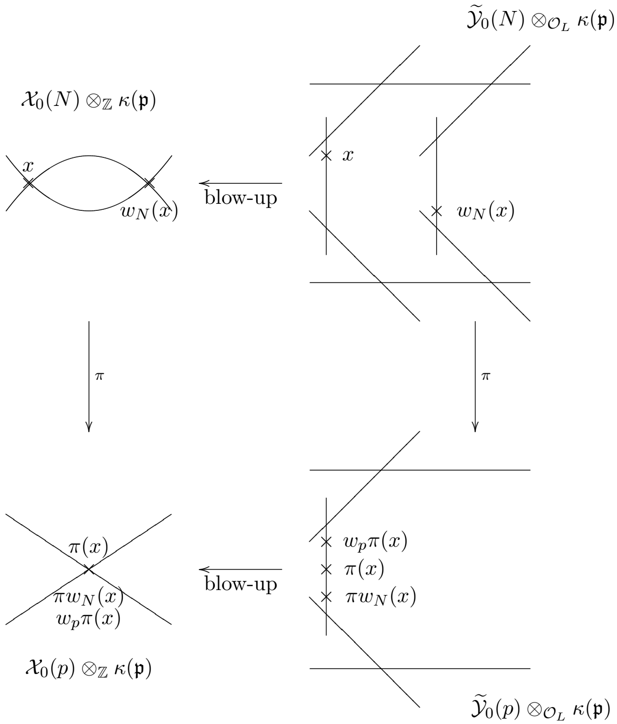


Assume $j(x \otimes \kappa(\mathfrak{p})) = 0$. Looking at a similar figure, we can show $w_p\pi(x) \otimes \kappa(\mathfrak{p})$ and $\pi w_N(x) \otimes \kappa(\mathfrak{p})$ define sections of the same irreducible component of $\widetilde{\mathcal{Y}}_0(p)^{\mathrm{sm}} \otimes \kappa(\mathfrak{p})$.

Case (iii): $\mathfrak{p}$ *is inert in* L/K *and* p *is ramified in* K. We have $\kappa(\mathfrak{p}) = \mathbb{F}_{p^2}$. The sections x and $w_N(x) = x^\sigma$ correspond to $\mathrm{Gal}(L/K)$-conjugate L-rational points. Hence $\pi(x) \otimes \kappa(\mathfrak{p})$ and $\pi w_N(x) \otimes \kappa(\mathfrak{p})$ correspond to $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$-conjugate $\mathbb{F}_{p^2}$-rational supersingular points. If one of them is $\mathbb{F}_p$-rational, they coincide. Then $w_p\pi(x) \otimes \kappa(\mathfrak{p}) = \pi(x) \otimes \kappa(\mathfrak{p}) = \pi w_N(x) \otimes \kappa(\mathfrak{p}) \in \mathcal{X}_0(p)(\mathbb{F}_p)$. (When $j(x \otimes \kappa(\mathfrak{p})) = 0, 1728$, look at some figures.) Otherwise they correspond to distinct but $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$-conjugate $\mathbb{F}_{p^2}$-rational supersingular points. Then $w_p\pi(x) \otimes \kappa(\mathfrak{p}) = \pi w_N(x) \otimes \kappa(\mathfrak{p}) \in \mathcal{X}_0(p)(\kappa(\mathfrak{p}))$. In any case $w_p\pi(x) \otimes \kappa(\mathfrak{p})$ and $\pi w_N(x) \otimes \kappa(\mathfrak{p})$ define sections of the same irreducible component of $\widetilde{\mathcal{Y}}_0(p)^{\mathrm{sm}} \otimes \kappa(\mathfrak{p})$.

Case (iv): $\mathfrak{p}$ *is ramified in* L/K *and* p *is ramified in* K. We have $\kappa(\mathfrak{p}) = \mathbb{F}_p$ and $x \otimes \kappa(\mathfrak{p}) = x^\sigma \otimes \kappa(\mathfrak{p}) = w_N(x) \otimes \kappa(\mathfrak{p}) \in \mathcal{X}_0(N)(\mathbb{F}_p)$. Then $\pi(x) \otimes \kappa(\mathfrak{p}) = \pi w_N(x) \otimes \kappa(\mathfrak{p})$, which is $\mathbb{F}_p$-rational. Hence $w_p\pi(x) \otimes \kappa(\mathfrak{p}) = \pi(x) \otimes \kappa(\mathfrak{p}) = \pi w_N(x) \otimes \kappa(\mathfrak{p}) \in \mathcal{X}_0(p)(\mathbb{F}_p)$. For $j(x \otimes \kappa(\mathfrak{p})) \neq 0, 1728$, see the figures below (there are two cases).


$\widetilde{\mathcal{Y}}_0(N) \otimes_{\mathcal{O}_L} \kappa(\mathfrak{p})$
$\mathcal{X}_0(N) \otimes_{\mathbb{Z}} \kappa(\mathfrak{p})$
x
$w_N(x)$
blow-up
π
$\pi(x)$
$\pi w_N(x)$
$w_p\pi(x)$
$\mathcal{X}_0(p) \otimes_{\mathbb{Z}} \kappa(\mathfrak{p})$
$\widetilde{\mathcal{Y}}_0(p) \otimes_{\mathcal{O}_L} \kappa(\mathfrak{p})$


$\widetilde{\mathcal{Y}}_0(N) \otimes_{\mathcal{O}_L} \kappa(\mathfrak{p})$
$\mathcal{X}_0(N) \otimes_{\mathbb{Z}} \kappa(\mathfrak{p})$
x
$w_N(x)$
blow-up
π
$\pi(x)$
$\pi w_N(x)$
$w_p\pi(x)$
$\mathcal{X}_0(p) \otimes_{\mathbb{Z}} \kappa(\mathfrak{p})$
$\widetilde{\mathcal{Y}}_0(p) \otimes_{\mathcal{O}_L} \kappa(\mathfrak{p})$

For $j(x\otimes\kappa(\mathfrak{p}))=0,1728$ we need more complicated figures, but we omit them.

Case (v): $\mathfrak{p}$ *is ramified in* L/K *and* p *is inert in* K. We have $\kappa(\mathfrak{p})=\mathbb{F}_{p^2}$. Since L/K is ramified at $\mathfrak{p}$, we have $x\otimes\kappa(\mathfrak{p})=x^\sigma\otimes\kappa(\mathfrak{p})=w_N(x)\otimes\kappa(\mathfrak{p})$. Hence $\pi(x)\otimes\kappa(\mathfrak{p})=\pi w_N(x)\otimes\kappa(\mathfrak{p})$.

If $\pi(x)\otimes\kappa(\mathfrak{p})$ is $\mathbb{F}_p$-rational, we have $w_p\pi(x)\otimes\kappa(\mathfrak{p})=\pi(x)\otimes\kappa(\mathfrak{p})=\pi w_N(x)\otimes\kappa(\mathfrak{p})\in\mathcal{X}_0(p)(\mathbb{F}_p)$. (When $j(x\otimes\kappa(\mathfrak{p}))=0,1728$, look at some figures.) Then $w_p\pi(x)\otimes\kappa(\mathfrak{p})$ and $\pi w_N(x)\otimes\kappa(\mathfrak{p})$ define sections of the same irreducible component of $\widetilde{\mathcal{Y}}_0(p)^{\mathrm{sm}}\otimes_{\mathcal{O}_L}\kappa(\mathfrak{p})$.

Suppose $\pi(x)\otimes\kappa(\mathfrak{p})$ is not $\mathbb{F}_p$-rational. Note that $j(\pi(x)\otimes\kappa(\mathfrak{p}))\neq 0,1728$ in this case. Then $w_p\pi(x)\otimes\kappa(\mathfrak{p})$ and $\pi w_N(x)\otimes\kappa(\mathfrak{p})$ $(=\pi(x)\otimes\kappa(\mathfrak{p}))$ correspond to distinct $\mathbb{F}_{p^2}$-rational supersingular points. Hence $w_p\pi(x)\otimes\kappa(\mathfrak{p})$ and $\pi w_N(x)\otimes\kappa(\mathfrak{p})$ define sections of two distinct exceptional irreducible components of $\widetilde{\mathcal{Y}}_0(p)^{\mathrm{sm}}\otimes_{\mathcal{O}_L}\kappa(\mathfrak{p})$. Let $\mathcal{J}$ (resp. $\mathcal{J}^+$, $\mathcal{J}^-$) be the Néron model of $J_0(p)\otimes L_{\mathfrak{p}}$ (resp. $J_0^+(p)\otimes L_{\mathfrak{p}}$, $J_0^-(p)\otimes L_{\mathfrak{p}}$) over $\mathcal{O}_{L_{\mathfrak{p}}}$. Considering the ramification index $e(L_{\mathfrak{p}}/\mathbb{Q}_p)=2<p-1$, we have an induced exact sequence

$$0\to\mathcal{J}^+\to\mathcal{J}\to\mathcal{J}^-$$

([4, p. 187, Theorem 4]). To simplify the notation let $\mathcal{J}_s$ (resp. $\mathcal{J}_s^+$, $\mathcal{J}_s^-$) be the geometric special fiber $\mathcal{J}\otimes_{\mathcal{O}_{L_{\mathfrak{p}}}}\overline{\mathbb{F}}_p$ (resp. $\mathcal{J}^+\otimes_{\mathcal{O}_{L_{\mathfrak{p}}}}\overline{\mathbb{F}}_p$, $\mathcal{J}^-\otimes_{\mathcal{O}_{L_{\mathfrak{p}}}}\overline{\mathbb{F}}_p$). Then the natural composite map

$$\mathcal{J}_s^+/(\mathcal{J}_s^+)^0\to\mathcal{J}_s/(\mathcal{J}_s)^0\to\mathcal{J}_s^-/(\mathcal{J}_s^-)^0$$

is the zero map. Let $\widetilde{\mathcal{Y}}^+\to\operatorname{Spec}\mathcal{O}_{L_{\mathfrak{p}}}$ be the minimal proper regular model of $X_0^+(p)\otimes_{\mathbb{Q}}L_{\mathfrak{p}}$. Let $\{C_i\}$ (resp. $\{C'_j\}$) be the set of irreducible components of $\widetilde{\mathcal{Y}}_0(p)\otimes\overline{\mathbb{F}}_p$ (resp. $\widetilde{\mathcal{Y}}^+\otimes\overline{\mathbb{F}}_p$). Let $\mathcal{D}$ (resp. $\mathcal{D}_+$) be the free abelian group generated by the divisors C_i (resp. C'_j). Let $\mathcal{D}^0\subseteq\mathcal{D}$ (resp. $\mathcal{D}_+^0\subseteq\mathcal{D}_+$) be the subgroup of divisors of degree 0. Let $\alpha:\mathcal{D}\to\mathcal{D}$ (resp. $\alpha_+:\mathcal{D}_+\to\mathcal{D}_+$) be the $\mathbb{Z}$-linear map defined by

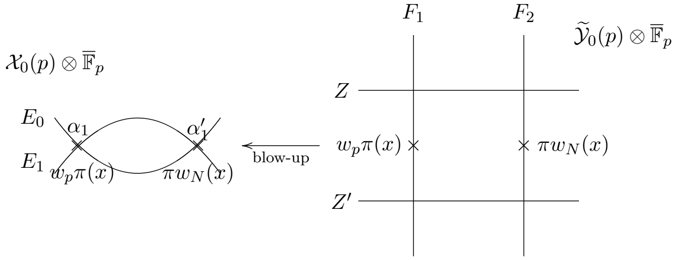
$$\alpha(B)=\sum_i(B,C_i)C_i\quad\text{(resp. }\alpha_+(B')=\sum_j(B',C'_j)C'_j)$$

where (B,C_i) (resp. (B',C'_j)) is the intersection number. Then we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{J}_s^+/(\mathcal{J}_s^+)^0 \arrow[r] \arrow[d, "\cong"'] & \mathcal{J}_s/(\mathcal{J}_s)^0 \arrow[r] \arrow[d, "\cong"'] & \mathcal{J}_s^-/(\mathcal{J}_s^-)^0 \\
\mathcal{D}_+^0/\alpha_+(\mathcal{D}_+) \arrow[r, "g^*"] & \mathcal{D}^0/\alpha(\mathcal{D}) &
\end{tikzcd}$$

where g^* is the natural map induced by the quotient map $g:X_0(p)\to X_0^+(p)$ and the vertical maps are the natural isomorphisms ([6, p. 179, Proposition (1.4)]). Let Z (resp. Z') be the irreducible component of $\widetilde{\mathcal{Y}}_0(p)\otimes\overline{\mathbb{F}}_p$ over E_0

(resp. E_1), and let F_{2i-1} (resp. F_{2i}) be the exceptional divisor of $\widetilde{\mathcal{Y}}_0(p)\otimes\overline{\mathbb{F}}_p$ over α_i (resp. α'_i) for $1\le i\le g_0^+(p)$. Let $\overline{F}_i := F_i - Z'$ and $\overline{Z} := Z - Z'$ be the elements of $\mathcal{D}^0$ (cf. [11, p. 281]).



We may assume $w_p\pi(x)\otimes\overline{\mathbb{F}}_p = \alpha_1$, $\pi w_N(x)\otimes\overline{\mathbb{F}}_p = \alpha'_1$ in $\mathcal{X}_0(p)\otimes\overline{\mathbb{F}}_p$. Then $w_p\pi(x)\otimes\overline{\mathbb{F}}_p$ (resp. $\pi w_N(x)\otimes\overline{\mathbb{F}}_p$) defines a section of F_1^{sm} (resp. F_2^{sm}) in $\widetilde{\mathcal{Y}}_0(p)\otimes\overline{\mathbb{F}}_p$. In the isomorphism $\mathcal{J}_s/(\mathcal{J}_s)^0 \cong \mathcal{D}^0/\alpha(\mathcal{D})$, the section $h(x)\otimes\overline{\mathbb{F}}_p$ corresponds to $F_1 - F_2$. We have $F_1 - F_2 = \overline{F}_1 - \overline{F}_2 \in g^*(\mathcal{D}_+^0/\alpha_+(\mathcal{D}_+)) \subseteq \mathcal{D}^0/\alpha(\mathcal{D})$ by the discussion in [11, pp. 279–281] (especially by the line "$g^*(\overline{K}_i) \equiv \overline{F}_{2i-1} + \overline{F}_{2i} - \overline{Z} \equiv \overline{F}_{2i-1} - \overline{F}_{2i} \bmod \alpha(\mathcal{D})$" on p. 281). Therefore we get $h^-(y)\otimes\overline{\mathbb{F}}_p = 0$ in $\mathcal{J}_s^-/(\mathcal{J}_s^-)^0$.

Now we have completed the proof of Proposition 3.1 and hence that of Theorem 1.6. □

5. Mordell–Weil groups over quadratic fields. In this section we prove Proposition 1.9. Notice that $g_0(p) = 1$ if and only if $p\in\{11,17,19\}$. In this case we have $J_0^-(p) = J_0(p)\cong X_0(p)$ and $J_0(p)(\mathbb{Q}) = C$ ([6, p. 151, Theorem (4.1)]). Let F (resp. G, H) be the Néron models of $J_0(11)$ (resp. $J_0(17)$, $J_0(19)$) over $\mathbb{Z}$.

Proposition 5.1.

(1) *We have $F(\mathbb{F}_2) = F(\mathbb{F}_4)\cong\mathbb{Z}/5\mathbb{Z}$. For any quadratic field K, we have $F(K)_{\mathrm{tor}} = C$.*
(2) *We have $G(\mathbb{Q}(\sqrt{-1}))_{\mathrm{tor}}\cong G(\mathbb{F}_5)\cong\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$. For any quadratic field K other than $\mathbb{Q}(\sqrt{-1})$, we have $G(K)_{\mathrm{tor}} = C$.*
(3) *We have $H(\mathbb{F}_2)\cong\mathbb{Z}/3\mathbb{Z}$ and $H(\mathbb{Q}(\sqrt{-3}))_{\mathrm{tor}}\cong H(\mathbb{F}_4)\cong(\mathbb{Z}/3\mathbb{Z})^2$. For any quadratic field K other than $\mathbb{Q}(\sqrt{-3})$, we have $H(K)_{\mathrm{tor}} = C$.*

Proof. (1) Let f_{11} be the cusp form of weight 2 and level 11 corresponding to $J_0(11)$. Then $a_2(f_{11}) = -2$ and $a_3(f_{11}) = -1$, where $a_i(f_{11})$ is the ith Fourier coefficient of f_{11} for $i = 2,3$ ([3, p. 117]). We then have $\sharp F(\mathbb{F}_2) = \sharp F(\mathbb{F}_3) = \sharp F(\mathbb{F}_4) = 5$, $\sharp F(\mathbb{F}_9) = 15$. Now $F(\mathbb{F}_2) = F(\mathbb{F}_4)\cong\mathbb{Z}/5\mathbb{Z}$ has been shown.

For any quadratic field K, we have inclusions $C = F(\mathbb{Q})[5]\subseteq F(K)[5]\subseteq F(K)_{\mathrm{tor}}^{(2)}\hookrightarrow F(\mathbb{F}_4)\cong\mathbb{Z}/5\mathbb{Z}$, where $F(K)_{\mathrm{tor}}^{(2)}$ is the prime-to-2 subgroup of

$F(K)_{\mathrm{tor}}$ (the notation introduced in Section 2). Since $\sharp C = 5$, the above inclusions are all isomorphisms. Finally we show $F(K)_{\mathrm{tor}}^{(2)} = F(K)_{\mathrm{tor}}$. Since $F(K)[2] \hookrightarrow F(\mathbb{F}_9)$ and $\sharp F(\mathbb{F}_9) = 15$, we have $F(K)[2] = \{0\}$. Thus indeed $F(K)_{\mathrm{tor}}^{(2)} = F(K)_{\mathrm{tor}}$.

(2) Let f_{17} be the cusp form of weight 2 and level 17 corresponding to $J_0(17)$. Then we know the Fourier coefficients $a_2(f_{17}) = -1$, $a_3(f_{17}) = 0$ and $a_5(f_{17}) = -2$ (loc. cit.). We then have $\sharp G(\mathbb{F}_4) = 8$, $\sharp G(\mathbb{F}_3) = 4$, $\sharp G(\mathbb{F}_9) = 16$, $\sharp G(\mathbb{F}_5) = 8$.

For any quadratic field K, we have an inclusion $\mathbb{Z}/4\mathbb{Z} \cong C = G(\mathbb{Q}) \subseteq G(K)_{\mathrm{tor}}$. Since $G(K)_{\mathrm{tor}}^{(2)} \hookrightarrow G(\mathbb{F}_4)$ and $\sharp G(\mathbb{F}_4) = 8$, we have $G(K)_{\mathrm{tor}}^{(2)} = \{0\}$.

We know that $G(\mathbb{Q}(\sqrt{-1}))$ has a subgroup which is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ ([6, p. 103]). Since $G(\mathbb{Q}(\sqrt{-1}))[5] = \{0\}$, we have $G(\mathbb{Q}(\sqrt{-1}))_{\mathrm{tor}} = G(\mathbb{Q}(\sqrt{-1}))_{\mathrm{tor}}^{(5)} \hookrightarrow G(\mathbb{F}_5)$. By using $\sharp G(\mathbb{F}_5) = 8$, we conclude $G(\mathbb{Q}(\sqrt{-1}))_{\mathrm{tor}} \cong G(\mathbb{F}_5) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Let $\mathrm{G}_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$. Let $r : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_2)$ be the Galois representation determined by the $G_{\mathbb{Q}}$-action on $G(\overline{\mathbb{Q}})[2]$. Since $G(\mathbb{Q}) = C \cong \mathbb{Z}/4\mathbb{Z}$, we have $G(\mathbb{Q})[2] \cong \mathbb{Z}/2\mathbb{Z}$. Then the image $r(\mathrm{G}_{\mathbb{Q}})$ is conjugate to the subgroup $\{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\}$. Since $G(\mathbb{Q}(\sqrt{-1}))[2] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the restriction $r|_{\mathrm{G}_{\mathbb{Q}(\sqrt{-1})}}$ is trivial, where $\mathrm{G}_{\mathbb{Q}(\sqrt{-1})} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{-1}))$ is the absolute Galois group of $\mathbb{Q}(\sqrt{-1})$ considered as a subgroup of $\mathrm{G}_{\mathbb{Q}}$. Then $\mathrm{Ker}\, r$ corresponds to the quadratic field $\mathbb{Q}(\sqrt{-1})$. So, for any quadratic field K other than $\mathbb{Q}(\sqrt{-1})$, the restriction $r|_{\mathrm{G}_K}$ is not trivial. Then $G(K)[2] \cong \mathbb{Z}/2\mathbb{Z}$. Since $G(K)_{\mathrm{tor}}^{(2)} = \{0\}$ and $G(\mathbb{Q}) = C \cong \mathbb{Z}/4\mathbb{Z}$, we have $G(K)_{\mathrm{tor}} \cong \mathbb{Z}/2^n\mathbb{Z}$ for $n \geq 2$.

Since $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong G(\mathbb{Q}(\sqrt{-1}))_{\mathrm{tor}} = G(\mathbb{Q}(\sqrt{-1}))_{\mathrm{tor}}^{(3)} \hookrightarrow G(\mathbb{F}_9)$ and $\sharp G(\mathbb{F}_9) = 16$, we have $G(\mathbb{F}_9) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Let $\mathrm{G}_{\mathbb{F}_3} = \mathrm{Gal}(\overline{\mathbb{F}}_3/\mathbb{F}_3)$ be the absolute Galois group of $\mathbb{F}_3$. Let $\rho : \mathrm{G}_{\mathbb{F}_3} \to \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z})$ be the Galois representation determined by the $G_{\mathbb{F}_3}$-action on $G(\overline{\mathbb{F}}_3)[4]$. Since $\mathbb{Z}/4\mathbb{Z} \cong C = G(\mathbb{Q}) = G(\mathbb{Q})_{\mathrm{tor}}^{(3)} \hookrightarrow G(\mathbb{F}_3)$ and $\sharp G(\mathbb{F}_3) = 4$, we have $G(\mathbb{F}_3) \cong \mathbb{Z}/4\mathbb{Z}$. Then $G(\mathbb{F}_3)[4] \cong \mathbb{Z}/4\mathbb{Z}$, and so we may assume that ρ is of the form $\left(\begin{smallmatrix}1&*\\0&\chi\end{smallmatrix}\right)$, where χ is the mod 4 cyclotomic character. Let $\overline{\rho} : \mathrm{G}_{\mathbb{F}_3} \to \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ be the reduction of ρ modulo 2. Since $G(\mathbb{F}_3)[2] \cong \mathbb{Z}/2\mathbb{Z}$, we have $\overline{\rho}(\mathrm{G}_{\mathbb{F}_3}) = \{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\}$. Since $\chi(\mathrm{G}_{\mathbb{F}_3}) = \{1, -1\}$ and the Galois group $\mathrm{G}_{\mathbb{F}_3}$ is topologically generated by one element, we have $\rho(\mathrm{G}_{\mathbb{F}_3}) = \{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1\\0&-1\end{smallmatrix}\right)\}$ or $\{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&-1\\0&-1\end{smallmatrix}\right)\}$.

Let $\mathrm{G}_{\mathbb{F}_9} = \mathrm{Gal}(\overline{\mathbb{F}}_3/\mathbb{F}_9)$ be the absolute Galois group of $\mathbb{F}_9$ considered as a subgroup of $\mathrm{G}_{\mathbb{F}_3}$. Since $G(\mathbb{F}_9)[2] \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the restriction $\overline{\rho}|_{\mathrm{G}_{\mathbb{F}_9}}$ is trivial. Then $\rho(\mathrm{G}_{\mathbb{F}_9}) \subseteq \{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2\\0&1\end{smallmatrix}\right)\}$, because $\chi|_{\mathrm{G}_{\mathbb{F}_9}}$ is trivial. This combined with the above consideration of $\rho(\mathrm{G}_{\mathbb{F}_3})$ implies that the restriction $\rho|_{\mathrm{G}_{\mathbb{F}_9}}$ is trivial. Therefore $G(\mathbb{F}_9) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Hence, for any quadratic field K other than $\mathbb{Q}(\sqrt{-1})$, we have $\mathbb{Z}/2^n\mathbb{Z} \cong G(K)_{\mathrm{tor}} = G(K)_{\mathrm{tor}}^{(3)} \hookrightarrow G(\mathbb{F}_9) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Since $n \geq 2$, we have $n = 2$. Therefore we conclude $G(K)_{\mathrm{tor}} = C$.

(3) Let f_{19} be the cusp form of weight 2 and level 19 corresponding to $J_0(19)$. Then $a_2(f_{19}) = 2$ and $a_5(f_{19}) = 3$ (loc. cit.). We then have $\sharp H(\mathbb{F}_2) = \sharp H(\mathbb{F}_5) = 3$, $\sharp H(\mathbb{F}_4) = 9$ and $\sharp H(\mathbb{F}_{25}) = 27$. Thus $H(\mathbb{F}_2) \cong \mathbb{Z}/3\mathbb{Z}$.

By [6, p. 125, Corollary (16.3)], we have $H[3] \cong \mathbb{Z}/3\mathbb{Z} \oplus \mu_3$ as group schemes over $\mathbb{Z}$, where $\mu_3 = \mathrm{Spec}(\mathbb{Z}[X]/(X^3 - 1))$. Then we have $H[3](\mathbb{Q}(\sqrt{-3})) \cong (\mathbb{Z}/3\mathbb{Z})^2$ and $H[3](K) \cong \mathbb{Z}/3\mathbb{Z}$ for any quadratic field K other than $\mathbb{Q}(\sqrt{-3})$. Since $H(\mathbb{F}_{25})$ has an odd order, so do $H(\mathbb{Q}(\sqrt{-3}))_{\mathrm{tor}}$ and $H(K)_{\mathrm{tor}}$. Then we have inclusions $H[3](\mathbb{Q}(\sqrt{-3})) \subseteq H(\mathbb{Q}(\sqrt{-3}))_{\mathrm{tor}} \hookrightarrow H(\mathbb{F}_4)$. Comparing the orders, we get $H(\mathbb{Q}(\sqrt{-3}))_{\mathrm{tor}} \cong H(\mathbb{F}_4) \cong (\mathbb{Z}/3\mathbb{Z})^2$. So, for any quadratic field K other than $\mathbb{Q}(\sqrt{-3})$, we have $C = H[3](K) \subseteq H(K)_{\mathrm{tor}} \hookrightarrow H(\mathbb{F}_4) \cong (\mathbb{Z}/3\mathbb{Z})^2$. Therefore $H(K)_{\mathrm{tor}} = H[3](K) = C$. ∎

Proof of Proposition 1.9. It suffices to show $\sharp J_0(p)(K) < \infty$ for $p = 11, 17, 19$. But this is done in [7, p. 143, Corollary 1]. For $p = 11, 19$, the same method as in [1, p. 2278, Proposition 4.3] also works. ∎

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Keisuke Arai
Department of Mathematics
School of Engineering
Tokyo Denki University
2-2 Kanda-Nishiki-cho, Chiyoda-ku
Tokyo, Japan 101-8457
E-mail: araik@mail.dendai.ac.jp

Fumiyuki Momose
Department of Mathematics
Faculty of Science and Engineering
Chuo University
1-13-27 Kasuga, Bunkyo-ku
Tokyo, Japan 112-8551
E-mail: momose@math.chuo-u.ac.jp

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