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On p-adic Siegel modular forms of non-real Nebentypus

by

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1. Introduction. In [9], Serre defined the notion of p-adic modular forms and applied it to the construction of a p-adic L-function. Recently, several people attempted to generalize this notion to the case of several variables. In particular, Böcherer–Nagaoka [3] defined p-adic Siegel modular forms and showed that all Siegel modular forms with level p and real Nebentypus are p-adic Siegel modular forms. The aim of this paper is to generalize this result to the case of non-real Nebentypus.

We now state our results more precisely. Let k be a positive integer, p an odd prime and χ a Dirichlet character modulo p with $\chi(-1) = (-1)^k$. For the congruence subgroup $\Gamma_0^{(n)}(p)$ of the symplectic group $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$, we denote by $M_k(\Gamma_0^{(n)}(p), \chi)$ the space of corresponding Siegel modular forms of weight k and character χ. For a subring R of $\mathbb{C}$, let $M_k(\Gamma_0^{(n)}(p), \chi)_R \subset M_k(\Gamma_0^{(n)}(p), \chi)$ denote the R-module of all modular forms whose Fourier coefficients belong to R. Let μ_{p-1} denote the group of $(p-1)$th roots of unity in $\mathbb{C}^\times$. We fix an embedding σ from $\mathbb{Q}(\mu_{p-1})$ to $\mathbb{Q}_p$. For $f \in M_k(\Gamma_0^{(n)}(p), \chi)_{\mathbb{Q}(\mu_{p-1})}$, let f^σ denote the formal power series defined by taking σ of each Fourier coefficient of the Fourier expansion of f (see Subsection 2.4). The following theorem is our main result:

Theorem 1.1. *For any modular form $f \in M_k(\Gamma_0^{(2)}(p), \chi)_{\mathbb{Q}(\mu_{p-1})}$, f^σ is a p-adic Siegel modular form. In other words, there exists a sequence $\{g_m\}$ of full modular forms such that*

$$\lim_{m\to\infty} g_m = f^\sigma \quad (p\text{-}adically).$$

We prove Theorem 1.1 in Section 3. The key point of the proof is the following existence theorem:

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THEOREM 1.2. *There exists a sequence* $\{G_{k_m} \in M_{k_m}(\Gamma_0^{(2)}(p), \chi)_{\mathbb{Q}(\mu_{p-1})}\}$ *of modular forms such that*

$$\lim_{m\to\infty} G_{k_m}^{\sigma} = 1 \quad (p\text{-}adically).$$

2. Preliminaries

2.1. Siegel modular forms. Let $\mathbb{H}_n$ be the Siegel upper half-space of degree n. The Siegel modular group $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$ acts on $\mathbb{H}_n$ by the generalized fractional transformation

$$MZ := (AZ+B)(CZ+D)^{-1} \quad \text{for } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n.$$

Let N be a positive integer. The congruence subgroup $\Gamma_0^{(n)}(N)$ is defined by

$$\Gamma_0^{(n)}(N) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \,\middle|\, C \equiv O_n \bmod N \right\}.$$

Let χ be a Dirichlet character modulo N. The space $M_k(\Gamma_0^{(n)}(N), \chi)$ of Siegel modular forms of weight k and character χ consists of all holomorphic functions $f : \mathbb{H}_n \to \mathbb{C}$ satisfying

$$f(MZ) = \chi(\det D)\det(CZ+D)^k f(Z) \quad \text{for } M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0^{(n)}(N).$$

If χ is trivial, we simply write $M_k(\Gamma_0^{(n)}(N))$ for $M_k(\Gamma_0^{(n)}(N), \chi)$. If $f \in M_k(\Gamma_0^{(n)}(N), \chi)$ then f has a Fourier expansion of the form

$$f = \sum_{O \le T \in \Lambda_n} a_f(T) e^{2\pi i \,\mathrm{tr}(TZ)},$$

where T runs over all semi-positive definite elements of

$$\Lambda_n := \{T = (t_{ij}) \in \mathrm{Sym}_n(\mathbb{Q}) \mid t_{ii} \in \mathbb{Z},\ 2t_{ij} \in \mathbb{Z}\}.$$

In this paper, we mainly deal with the case where N is a prime.

2.2. p-adic Siegel modular forms. Let v_p be the additive valuation on $\mathbb{Q}_p$ normalized by $v_p(p) = 1$. We consider a formal power series of the form $f = \sum_{O \le T \in \Lambda_n} a(T) e^{2\pi i \,\mathrm{tr}(TZ)}$ with $a(T) \in \mathbb{Q}_p$. For a more accurate interpretation of f, see [1, 3].

DEFINITION 2.1. A formal power series $f = \sum_{O \le T \in \Lambda_n} a(T) e^{2\pi i \,\mathrm{tr}(TZ)}$ with $a(T) \in \mathbb{Q}_p$ is called a *p-adic Siegel modular form* if there exists a sequence $\{g_m \in M_{k_m}(\Gamma_n)_{\mathbb{Q}}\}$ of full modular forms such that $\lim_{m\to\infty} g_m = f$ (p-adically), where the limit means that $\inf_{T\in\Lambda_n}(v_p(a_{g_m}(T) - a(T))) \to \infty$ as $m \to \infty$.

Böcherer and Nagaoka showed

THEOREM 2.2 (Böcherer–Nagaoka [3]). *Let p be an odd prime. If $f \in M_k(\Gamma_0^{(n)}(p))_{\mathbb{Q}}$ then f is a p-adic Siegel modular form.*

2.3. Jacobi forms and their liftings. In this subsection, we recall some known facts on Jacobi forms and their liftings. Since we do not need the general level case, we only consider the prime level case.

Let p be an odd prime and χ a Dirichlet character modulo p with $\chi(-1) = (-1)^k$. Let ϕ be a Jacobi form of weight k, index 1 and character χ with respect to $\Gamma_0^{(1)}(p)$. Then ϕ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n=0}^{\infty} \sum_{\substack{r\in\mathbb{Z} \\ 4n-r^2\geq 0}} c(n,r)q^n\zeta^r \quad \text{for } (\tau, z) \in \mathbb{H}_1 \times \mathbb{C},$$

where $q := e^{2\pi i\tau}$ and $\zeta := e^{2\pi iz}$. The Maass lift $\mathcal{M}\phi \in M_k(\Gamma_0^{(2)}(p), \chi)$ of ϕ is described by

$$\mathcal{M}\phi(Z) = \biggl(\frac{1}{2}L(1-k,\chi) + \sum_{n=1}^{\infty} \sum_{\substack{0<d|n \\ (p,d)=1}} \chi(d)d^{k-1}q^n\biggr)c(0,0)$$
$$+ \sum_{l=1}^{\infty} \sum_{4nl-r^2\geq 0} \sum_{\substack{0<d|(n,r,l) \\ (p,d)=1}} \chi(d)d^{k-1}c\Bigl(\frac{nl}{d^2}, \frac{r}{d}\Bigr)q^n\zeta^r q'^l$$
$$\text{for } Z = \begin{pmatrix} \tau & z \\ z & w \end{pmatrix} \in \mathbb{H}_2,$$

where $q' := e^{2\pi iw}$. This lift was studied by Ibukiyama. For the precise definitions of Jacobi forms with level and their liftings, see [6, 8].

2.4. Embeddings from $\mathbb{Q}(\mu_{p-1})$ to $\mathbb{Q}_p$. In this subsection, we mention how to determine embeddings from $\mathbb{Q}(\mu_{p-1})$ to $\mathbb{Q}_p$.

Let μ_{p-1} denote the group of $(p-1)$th roots of unity in $\mathbb{C}^\times$. Let us take a generator ζ_{p-1} of μ_{p-1} and consider the prime ideal factorization of p in the ring $\mathbb{Z}[\zeta_{p-1}]$ of integers of $\mathbb{Q}(\mu_{p-1})$. Let $\Phi(X) \in \mathbb{Z}[X]$ be the minimal polynomial of ζ_{p-1}, namely $\Phi(X)$ is the cyclotomic polynomial having the root ζ_{p-1}. We can always decompose $\Phi(X)$ in the form $\Phi(X) \equiv q_1(X)\cdots q_r(X) \bmod p$, where $r = \varphi(p-1)$ and each $q_i(X)$ is a polynomial of degree one with $q_i(X) \not\equiv q_j(X) \bmod p$. Then p is decomposed as a product of r prime ideals $\mathfrak{p}_i := (q_i(\zeta_{p-1}), p)$, namely we have the perfect decomposition

$$(p) = \mathfrak{p}_1 \cdots \mathfrak{p}_r = (q_1(\zeta_{p-1}), p) \cdots (q_r(\zeta_{p-1}), p).$$

Let ω be the Teichmüller character on $\mathbb{Z}_p$. If we write $q_i(X) = X - d_i$ for some $d_i \in \mathbb{Z}$, then an embedding σ_i from $\mathbb{Q}(\zeta_{p-1})$ to $\mathbb{Q}_p$ corresponding to $\mathfrak{p}_i$ is determined by $\sigma_i(\zeta_{p-1}) = \omega(d_i)$.

EXAMPLE 2.3. (1) *Case* $p = 5$ ($\zeta_4 = i$). We see easily that $\Phi(X) = X^2 + 1 \equiv (X-2)(X-3) \bmod 5$. Putting $\mathfrak{p}_1 := (i-2, 5)$ and $\mathfrak{p}_2 := (i-3, 5)$, we have $(5) = \mathfrak{p}_1\mathfrak{p}_2$. In fact, $(i-2,5) = (i-2)$ and $(i-3,5) = (i+2)$. Hence, the embeddings σ_i corresponding to $\mathfrak{p}_i$ are determined by $\sigma_1(i) = \omega(2)$ and $\sigma_2(i) = \omega(3)$.

(2) *Case* $p = 7$ ($\zeta_6 = (1+\sqrt{3}i)/2$). One has $\Phi(X) = X^2 - X + 1 \equiv (X-3)(X-5) \bmod 7$. If we set $\mathfrak{p}_1 := (\zeta_6 - 3, 5)$ and $\mathfrak{p}_2 := (\zeta_6 - 5, 5)$, then $(7) = \mathfrak{p}_1\mathfrak{p}_2$. Hence, the embeddings σ_i are determined by $\sigma_1(\zeta_6) = \omega(3)$ and $\sigma_2(\zeta_6) = \omega(5)$.

For a formal power series of the form $f = \sum_{O \le T \in \Lambda_n} a(T)e^{2\pi i \operatorname{tr}(TZ)}$ with $a(T) \in \mathbb{Q}(\mu_{p-1})$, we define

$$f^\sigma := \sum_{O \le T \in \Lambda_n} a(T)^\sigma e^{2\pi i \operatorname{tr}(TZ)}.$$

3. Proofs

3.1. Proof of Theorem 1.2. Let $\boldsymbol{X} := \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$ denote the group of weights of p-adic Siegel modular forms. Let ω be the Teichmüller character on $\mathbb{Z}_p$. Following Serre's notation in [9], let us write $\zeta^*(s,u) := L_p(s, \omega^{1-u})$ for $(s,u) \in \boldsymbol{X}$, where $L_p(s,\chi)$ is Kubota–Leopoldt's p-adic L-function (e.g. [5]).

As in [4], let $E^J_{k,1}$ be the normalized Jacobi Eisenstein series of weight k and index 1 (i.e. the constant term is 1). It is known that its Fourier coefficients are in $\mathbb{Q}$. Moreover we denote by

$$E^{(1)}_k = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sum_{0<d\mid n} d^{k-1}q^n \in M_k(\Gamma_1)_{\mathbb{Q}},$$

$$E^{(1)}_{k,\chi} = 1 + \frac{2}{L(1-k,\chi)}\sum_{n=1}^{\infty}\sum_{\substack{0<d\mid n\\(p,d)=1}} \chi(d)d^{k-1}q^n \in M_k(\Gamma^{(1)}_0(p),\chi)_{\mathbb{Q}(\mu_{p-1})}$$

the normalized Eisenstein series of weight k for Γ_1 and normalized Hecke's Eisenstein series of weight k and character χ for $\Gamma^{(1)}_0(p)$, respectively.

We choose $\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$ such that $\chi^\sigma = \omega^\alpha$. Moreover, we take a sequence $\{k_m = ap^{m+1}\}$ for $0 < a \in \mathbb{Z}$ with $a \equiv -\alpha \bmod p-1$. Note that a is even or odd according as χ is even or odd. If we put

$$\phi_{k_m} := E^{(1)}_{a(p-2),\chi}E^{(1)}_{ap(p^m-1)}E^J_{2a,1}$$

then ϕ_{k_m} is a Jacobi form of weight k_m and index 1 with character χ for $\Gamma_0^{(1)}(p)$. Here note that $E^{(1)}_{ap(p^m-1)}E^J_{2a,1}$ has rational Fourier coefficients. Moreover if we write its Fourier expansion as $\phi_{k_m} = \sum_{n,r} c_{k_m}(n,r)q^n\zeta^r$, then $c_{k_m}(n,r) \in \mathbb{Q}(\mu_{p-1})$ and $c_{k_m}(0,0) = 1$. Now we can prove

LEMMA 3.1. $\{\phi^\sigma_{k_m}\}$ *converges uniformly in the formal power series ring* $\mathbb{Q}_p[\zeta,\zeta^{-1}][\![q]\!]$.

Proof. Recall that

$$\begin{aligned}\phi^\sigma_{k_m} &= (E^{(1)}_{a(p-2),\chi}E^{(1)}_{ap(p^m-1)}E^J_{2a,1})^\sigma \\ &= (E^{(1)}_{a(p-2),\chi})^\sigma E^{(1)}_{ap(p^m-1)}E^J_{2a,1} \in \mathbb{Q}_p[\zeta,\zeta^{-1}][\![q]\!].\end{aligned}$$

Hence we may only show that $\lim_{m\to\infty} E^{(1)}_{ap(p^m-1)} \in \mathbb{Q}_p[\![q]\!]$. To prove this, we consider the Eisenstein series

$$G^{(1)}_{l_m} := -\frac{B_{l_m}}{2l_m}E^{(1)}_{l_m} = -\frac{B_{l_m}}{2l_m} + \sum_{n=1}^{\infty}\sum_{0<d\mid n} d^{l_m-1}q^n,$$

where we put $l_m := ap(p^m-1)$. It is clear that there exists a limiting value $\lim_{m\to\infty}\sum_{0<d\mid n} d^{l_m-1} \in \mathbb{Q}_p$ for each $n \ge 1$. Obviously, this convergence is uniform with respect to $n \ge 1$. Since l_m tends to $(-ap,0) \ne (0,0)$ in $\boldsymbol{X}$, we can apply Corollaire 2 of [9] to $G^{(1)}_{l_m}$. Therefore we see that the constant term also converges in $\mathbb{Q}_p$,

$$-\lim_{m\to\infty}\frac{B_{l_m}}{2l_m} \in \mathbb{Q}_p.$$

Now we shall show that this value is not zero. If $m \ge 1$ then $p-1 \mid l_m$. Hence the denominator of B_{l_m} is divisible by p according to the von Staudt–Clausen theorem. This means that the numerator of $B_{l_m}/2l_m$ is divisible by p for no $m \ge 1$. It follows immediately that

$$-\lim_{m\to\infty}\frac{B_{l_m}}{2l_m} \ne 0.$$

Therefore

$$\lim_{m\to\infty} E^{(1)}_{l_m} = \lim_{m\to\infty}\left(1 - \frac{2l_m}{B_{l_m}}\sum_{n=1}^{\infty}\sum_{0<d\mid n} d^{l_m-1}q^n\right) \in \mathbb{Q}_p[\![q]\!].$$

This completes the proof of Lemma 3.1. ∎

Let us return to the proof of Theorem 1.2. Taking the Maass lift $\mathcal{M}\phi_{k_m} =: F_{k_m} \in M_{k_m}(\Gamma_0^{(2)}(p),\chi)_{\mathbb{Q}(\mu_{p-1})}$, we have the following Fourier expansion:

$$F_{k_m} = \frac{1}{2}L(1-k_m,\chi) + \sum_{n=1}^{\infty} \sum_{\substack{0<d|n \\ (p,d)=1}} \chi(d)d^{k_m-1}q^n$$
$$+ \sum_{l=1}^{\infty} \sum_{4nl-r^2\geq 0} \sum_{\substack{0<d|(n,r,l) \\ (p,d)=1}} \chi(d)d^{k_m-1}c_{k_m}\left(\frac{nl}{d^2},\frac{r}{d}\right)q^n\zeta^r q'^l.$$

For $l>0$, the lth Fourier Jacobi coefficient is

$$\sum_{4nl-r^2\geq 0} \sum_{\substack{0<d|(n,r,l) \\ (p,d)=1}} \chi(d)d^{k_m-1}c_{k_m}\left(\frac{nl}{d^2},\frac{r}{d}\right)q^n\zeta^r.$$

Since $\chi(d)^\sigma = \omega(d)^\alpha = d^\alpha$, if we take σ then

$$\sum_{4nl-r^2\geq 0} \sum_{\substack{0<d|(n,r,l) \\ (p,d)=1}} d^{k_m+\alpha-1}c_{k_m}\left(\frac{nl}{d^2},\frac{r}{d}\right)^\sigma q^n\zeta^r.$$

The constatnt term of the Fourier Jacobi expansion is Hecke's Eisenstein series of weight k_m and character χ. By a similar argument of Serre, we obtain

$$\left(\frac{1}{2}L(1-k_m,\chi) + \sum_{n=1}^{\infty} \sum_{\substack{0<d|n \\ (p,d)=1}} \chi(d)d^{k_m-1}q^n\right)^\sigma$$
$$= \frac{1}{2}\zeta^*(1-k_m, 1-k_m-\alpha) + \sum_{n=1}^{\infty} \sum_{\substack{0<d|n \\ (p,d)=1}} d^{k_m+\alpha-1}q^n.$$

Finally, we set $G_{k_m} := 2L(1-k_m,\chi)^{-1}F_{k_m}$. Since k_m tends to $(0,-\alpha)$ in $\boldsymbol{X}$, $(1-k_m, 1-k_m-\alpha)$ tends to $(1,1)$ in $\boldsymbol{X}$. Note that $\zeta^*(s,u)$ has a simple pole at $(1,1)$. Combining this fact with Lemma 3.1, we see that $G_{k_m}^\sigma$ tends to 1. In fact, the q-expansion of $G_{k_m}^\sigma$ is given by

$$G_{k_m}^\sigma = 1 + \frac{2}{\zeta^*(1-k_m, 1-k_m-\alpha)} \sum_{n=1}^{\infty} \sum_{\substack{0<d|n \\ (p,d)=1}} d^{k_m+\alpha-1}q^n$$
$$+ \frac{2}{\zeta^*(1-k_m, 1-k_m-\alpha)}$$
$$\times \left(\sum_{l=1}^{\infty} \sum_{4nl-r^2\geq 0} \sum_{\substack{0<d|(n,r,l) \\ (p,d)=1}} d^{k_m+\alpha-1}c_{k_m}\left(\frac{nl}{d^2},\frac{r}{d}\right)^\sigma q^n\zeta^r q'^l \right).$$

This completes the proof of Theorem 1.2. ∎

3.2. Proof of Theorem 1.1. In order to apply Serre's argument, we start by proving

LEMMA 3.2. *Each* $f \in M_k(\Gamma_0^{(n)}(p))_{\mathbb{Q}(\mu_{p-1})}$ *is a* $\mathbb{Q}(\mu_{p-1})$*-linear combination of elements of* $M_k(\Gamma_0^{(n)}(p))_{\mathbb{Q}}$.

Proof. We have $M_k(\Gamma_0^{(n)}(p))_{\mathbb{C}} = M_k(\Gamma_0^{(n)}(p))_{\mathbb{Q}} \otimes \mathbb{C}$ by Shimura's result [10]. Hence $f \in M_k(\Gamma_0^{(n)}(p))_{\mathbb{Q}(\mu_{p-1})}$ can be uniquely written in the form $f = \sum_{i=1}^{N} c_i f_i$ for some $c_i \in \mathbb{C}$ and $f_i \in M_k(\Gamma_0^{(n)}(p))_{\mathbb{Q}}$. For each $\tau \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q}(\mu_{p-1}))$, $f^\tau = \sum_{i=1}^{N} c_i^\tau f_i$ because each f_i has rational Fourier coefficients. On the other hand, since the Fourier coefficients of f are in $\mathbb{Q}(\mu_{p-1})$, we have $f^\tau = f = \sum_{i=1}^{N} c_i f_i$. It follows from uniqueness of description of f that $c_i^\tau = c_i$. The assertion follows. ∎

We are now in a position to prove our main theorem.

Proof of Theorem 1.1. For any $f \in M_k(\Gamma_0^{(2)}(p),\chi)_{\mathbb{Q}(\mu_{p-1})}$, take a sequence $\{G_{k_m} \in M_{k_m}(\Gamma_0^{(2)}(p),\chi^{-1})\}$ of modular forms constructed in Theorem 1.2. We consider $fG_{k_m} \in M_{k+k_m}(\Gamma_0^{(2)}(p))_{\mathbb{Q}(\mu_{p-1})}$. Note that each $k+k_m$ is even. Applying Lemma 3.2 to each fG_{k_m}, we see that fG_{k_m} is a $\mathbb{Q}(\mu_{p-1})$-linear combination of elements of $M_{k+k_m}(\Gamma_0^{(2)}(p))_{\mathbb{Q}}$. Hence, $(fG_{k_m})^\sigma = f^\sigma G_{k_m}^\sigma$ is a p-adic Siegel modular form according to Theorem 2.2. Since $G_{k_m}^\sigma$ tends to 1, $f^\sigma G_{k_m}^\sigma$ tends to f^σ. Thus f^σ is a p-adic Siegel modular form. ∎

4. Towards a generalization. In this section, we make some remarks on possible generalizations.

If we can solve the following problem affirmatively, then we can generalize Theorem 1.1 to the case of any degree.

PROBLEM 4.1. *Let p be an odd prime and χ a Dirichlet character modulo p. Does there exist a sequence* $\{G_{k_m} \in M_{k_m}(\Gamma_0^{(n)}(p),\chi)_{\mathbb{Q}(\mu_{p-1})}\}$ *of Siegel modular forms such that*

$$\lim_{m\to\infty} G_{k_m}^\sigma = 1 \quad (p\text{-}adically)?$$

Now we raise one more question which is equivalent to this problem.

PROBLEM 4.2. *Let p and χ be as above. Does there exist a modular form* $G_a \in M_a(\Gamma_0^{(n)}(p),\chi)_{\mathbb{Q}(\mu_{p-1})}$ *such that*

$$G_a^\sigma \equiv 1 \bmod p?$$

REMARK. (1) If we can solve Problem 4.2 affirmatively, then we can solve Problem 4.1 affirmatively by putting $G_{k_m} := G_a^{p^m}$.

(2) Take $\alpha \in \mathbb{Z}/(p-1)\mathbb{Z}$ such that $\chi^\sigma = \omega^\alpha$. As we have seen in the proof of Theorem 1.1, for $f \in M_k(\Gamma_0^{(2)}(p), \chi^{-1})_{\mathbb{Q}(\mu_{p-1})}$, the p-adic weight of f^σ is $(k, k-\alpha)$. Assume that there is another sequence $\{G'_{k'_m} \in M_{k'_m}(\Gamma_0^{(2)}(p), \chi)_{\mathbb{Q}(\mu_{p-1})}\}$ in Theorem 1.2. The well-definedness of p-adic weights indicates that $\{k+k'_m\}$ also tends to $(k, k-\alpha)$ in $\boldsymbol{X}$ and hence $\{k'_m\}$ converges automatically to $(0, -\alpha)$ in $\boldsymbol{X}$. Similarly, $\{k_m\}$ of Problem 4.1 converges automatically to $(0, -\alpha)$ in $\boldsymbol{X}$. In fact, for the Siegel Φ-operator, $\Phi^{n-2}(G_{k_m}) \in M_{k_m}(\Gamma_0^{(2)}(p), \chi)_{\mathbb{Q}(\mu_{p-1})}$ still satisfies that $\lim_{m\to\infty} \Phi^{n-2}(G_{k_m})^\sigma = 1$ (p-adically). For the p-adic weights and their well-definedness, see [1, 7, 10].

(3) Combining (1) with (2), we see that there is a relation between a and χ in Problem 4.2 such that $a \equiv -\alpha \bmod p-1$ for α satisfying $\chi^\sigma = \omega^\alpha$.

(4) If $p-1 \mid a$ (i.e. χ is trivial), then Problem 4.2 was solved by Böcherer–Nagaoka [2]. They used theta series to construct G_{p-1}, but their arguments do not work for the case of non-real Nebentypus.

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