The joint distribution of \(q\)-additive functions

by

Michael Drmota (Wien)

1. Introduction. Let \(q > 1\) be a given integer. A real-valued function \(f\), defined on the non-negative integers, is said to be \(q\)-additive if \(f(0) = 0\) and
\[
f(n) = \sum_{j \geq 0} f(a_{q,j}(n)q^j) \quad \text{for} \quad n = \sum_{j \geq 0} a_{q,j}(n)q^j,
\]
where \(a_{q,j}(n) \in E_q := \{0, 1, \ldots, q - 1\}\). A special \(q\)-additive function is the sum-of-digits function
\[
s_q(n) = \sum_{j \geq 0} a_{q,j}(n).
\]
The statistical behaviour of the sum-of-digits function and, more generally, of \(q\)-additive functions has been very well studied by several authors.

The most general result concerning the mean value of \(q\)-additive functions is due to Manstavičius [20] (extending earlier work of Coquet [3]). Let
\[
m_{k,q} := \frac{1}{q} \sum_{c \in E_q} f(cq^k), \quad m_{2,k,q}^2 := \frac{1}{q} \sum_{c \in E_q} f^2(cq^k)
\]
and
\[
M_q(x) := \sum_{k=0}^{[\log_q x]} m_{k,q}, \quad \quad \quad B_q^2(x) := \sum_{k=0}^{[\log_q x]} m_{2,k,q}^2.
\]
Then
\[
\frac{1}{x} \sum_{n < x} (f(n) - M_q(x))^2 \leq cB_q^2(x),
\]

2000 Mathematics Subject Classification: Primary 11A63; Secondary 11N60.
This research was supported by the Austrian Science Foundation FWF, grant S8302-MAT.
which implies
\[ \frac{1}{x} \sum_{n<x} f(n) = M_q(x) + O(B_q(x)) . \]

For the sum-of-digits function \( s_q(n) \) much more precise results are known, e.g. Delange [5] proved (for integral \( x \)) that
\[ \frac{1}{x} \sum_{n<x} s_q(n) = \frac{q-1}{2} \log_q x + \gamma(\log_q x) , \]
where \( \gamma \) is a continuous, nowhere differentiable and periodic function with period 1. (Higher moments of \( a_q(n) \) were considered by Kirschenhofer [19] and by Kennedy and Cooper [17] (for the variance) and by Grabner, Kirschenhofer, Prodinger and Tichy [12].)

There also exist distributional results for \( q \)-additive functions. In 1972 Delange [4] proved an analogue to the Erdős–Wintner theorem. There exists a distribution function \( F(y) \) such that, as \( x \to \infty \),
\[ \frac{1}{x} \# \{ n < x \mid f(n) < y \} \to F(y) \]
if and only if the two series \( \sum_{k>0} m_{k,q} \), \( \sum_{k>0} m_{2;k,q}^2 \) converge. This theorem was generalized by Kátai [16] who proved that there exists a distribution function \( F(y) \) such that, as \( x \to \infty \),
\[ \frac{1}{x} \# \{ n < x \mid f(n) - M_q(x) < y \} \to F(y) \]
if and only if the series \( \sum_{k>0} m_{2;k,q}^2 \) converges.

The most general theorem known concerning a central limit theorem is again due to Manstavicius [20]. Suppose that, as \( x \to \infty \),
\[ \max_{c q^j < x} |f(c q^j)| = o(B_q(x)) \]
and that \( D_q(x) \to \infty \), where
\[ D_q^2(x) = \sum_{k=0}^{\log_q x} \sigma_{k,q}^2 \quad \text{and} \quad \sigma_{k,q}^2 := \frac{1}{q} \sum_{c \in E_q} f^2(c q^k) - m_{k,q}^2 . \]
Then, as \( x \to \infty \),
\[ \frac{1}{x} \# \left\{ n < x \mid \frac{f(n) - M_q(x)}{D_q(x)} < y \right\} \to \Phi(y) , \]
where \( \Phi \) is the normal distribution function.

Similar distribution results for the sum-of-digits function of number systems related to substitution automata were considered by Dumont and Thomas [8]. For number systems whose bases satisfy linear recurrences we refer to [6].
Furthermore, Bassily and Kátai [1] studied the distribution of \( q \)-additive functions on polynomial sequences.

**Theorem 1.** Let \( f \) be a \( q \)-additive function such that \( f(cq^j) = O(1) \) as \( j \to \infty \) and \( c \in E_q \). Assume that \( D_q(x)/(\log x)^\eta \to \infty \) as \( x \to \infty \) for some \( \eta > 0 \) and let \( P(x) \) be a polynomial with integer coefficients, degree \( r \), and positive leading term. Then, as \( x \to \infty \),

\[
\frac{1}{x} \# \left\{ n < x \mid \frac{f(P(n)) - M_q(x^r)}{D_q(x^r)} < y \right\} \to \Phi(y),
\]

\[
\frac{1}{\pi(x)} \# \left\{ p < x \mid \frac{f(P(p)) - M_q(x^r)}{D_q(x^r)} < y \right\} \to \Phi(y).
\]

This result relies on the fact that suitably modified centralized moments converge (cf. Lemma 4). Note also that this theorem was only stated (and proved) for \( \eta = 1/3 \). However, a short inspection of the proof shows that \( \eta > 0 \) is sufficient.

**2. Joint distributions.** It is a natural question to ask whether there are analogue results for the joint distribution of \( q_l \)-additive functions \( f_l(n) \) (if \( q_1, \ldots, q_d > 1 \) are pairwise coprime integers). For example, Hildebrand [14] announced that one always has

\[
\frac{1}{x} \# \{ n < x \mid f_l(n) < y_l, \ 1 \leq l \leq d \} \to F_1(y_1) \ldots F_d(y_d)
\]

if \( f_l \) satisfies (1.2) for all \( l = 1, \ldots, d \) and that there is a joint central limit theorem of the form

\[
\frac{1}{x} \# \left\{ n < x \mid \frac{f_l(n) - M_{q_l}(x)}{D_{q_l}(x)} < y_l, \ 1 \leq l \leq d \right\} \to \Phi(y_1) \ldots \Phi(y_d)
\]

if \( B_{q_l}(x) \to \infty \) and \( B_{q_l}(x^n) \sim B_{q_l}(x) \) for every \( \eta > 0 \) as \( x \to \infty \). (Note that the sum-of-digits function \( s_q(n) \) is not covered by this result.)

In this paper we will first extend the above result of Bassily and Kátai to the joint distribution of \( q_l \)-additive functions \( f_l \) (1 \( \leq l \leq d \)) on specific polynomial sequences if \( q_1, \ldots, q_d \) are pairwise coprime.

**Theorem 2.** Let \( q_1, \ldots, q_d > 1 \) be pairwise coprime integers and let \( f_l \), 1 \( \leq l \leq d \), be \( q_l \)-additive functions such that \( f_l(cq^j_l) = O(1) \) as \( j \to \infty \) and \( c \in E_{q_l} \). Assume that \( D_{q_l}(x)/(\log x)^\eta \to \infty \) as \( x \to \infty \), 1 \( \leq l \leq d \), for some \( \eta > 0 \) and let \( P_l(x) \) be polynomials with integer coefficients of different degrees \( r_l \) and positive leading terms, 1 \( \leq l \leq d \). Then, as \( x \to \infty \),

\[
\frac{1}{x} \# \left\{ n < x \mid \frac{f_l(P_l(n)) - M_{q_l}(x^{r_l})}{D_{q_l}(x^{r_l})} < y_l, \ 1 \leq l \leq d \right\} \to \Phi(y_1) \ldots \Phi(y_d),
\]

\[
\frac{1}{\pi(x)} \# \left\{ p < x \mid \frac{f_l(P_l(p)) - M_{q_l}(x^{r_l})}{D_{q_l}(x^{r_l})} < y_l, \ 1 \leq l \leq d \right\} \to \Phi(y_1) \ldots \Phi(y_d).
\]
COROLLARY 1. Let \( q_1, \ldots, q_d > 1 \) be pairwise coprime integers and let \( P_i(x) \) be polynomials with integer coefficients of different degrees \( r_i \) and positive leading terms, \( 1 \leq l \leq d \). Then, as \( x \to \infty \),

\[
\frac{1}{x} \# \left\{ n < x \mid \frac{s_{q_l}(P_l(n))}{\sqrt{\frac{q^2-1}{12} \log q_l x^{r_l}}} < y_l, \ 1 \leq l \leq d \right\} \to \Phi(y_1) \ldots \Phi(y_d),
\]

\[
\frac{1}{\pi(x)} \# \left\{ p < x \mid \frac{s_{q_l}(P_l(p))}{\sqrt{\frac{q^2-1}{12} \log q_l x^{r_l}}} < y_l, \ 1 \leq l \leq d \right\} \to \Phi(y_1) \ldots \Phi(y_d).
\]

This theorem contains an unnatural condition, namely that one has to consider polynomials \( P_i(x) \) with different degrees \( r_i \). It would seem that this condition is not necessary. However, this is the crux of the matter. By using a variation of Bassily and Kátai’s proof (combined with Baker’s theorem on linear forms of logarithms) we could handle the case \( d = 2 \) with linear polynomials \( P_i(x) = A_i x + B_i \).

THEOREM 3. Let \( q_1, q_2 > 1 \) be coprime integers and let \( f_i \) be \( q_i \)-additive functions such that \( f_i(cq_1^j) = O(1) \) as \( j \to \infty \) and \( c \in E_{q_1}, \ l = 1, 2 \). Assume that \( D_{q_l}(x)/\log x \eta \to \infty \) as \( x \to \infty \), \( l = 1, 2 \), for some \( \eta > 0 \). Let \( P_i(x) = A_i x + B_i, \ l = 1, 2 \), be arbitrary linear polynomials with integer coefficients and positive leading terms \( A_l \) coprime to \( q_l \). Then, as \( x \to \infty \),

\[
\frac{1}{x} \# \left\{ n < x \mid \frac{f_i(P_l(n))}{D_{q_l}(x)} - M_{q_l}(x) < y_l, \ l = 1, 2 \right\} \to \Phi(y_1) \Phi(y_2).
\]

COROLLARY 2. Let \( q_1, q_2 > 1 \) be coprime integers. Then, as \( x \to \infty \),

\[
\frac{1}{x} \# \left\{ n < x \mid \frac{s_{q_l}(n)}{\sqrt{\frac{q^2-1}{12} \log q_l x}} < y_l, \ l = 1, 2 \right\} \to \Phi(y_1) \Phi(y_2).
\]

Interestingly, there is even a local version of Corollary 2.

THEOREM 4. Let \( q_1, q_2 > 1 \) be coprime integers and set \( d = \gcd(q_1 - 1, q_2 - 1) \). Then, as \( x \to \infty \),

\[
\frac{1}{x} \# \{ n < x \mid s_{q_l}(n) = k_1, s_{q_2}(n) = k_2 \} = d \prod_{l=1}^{2} \left( \frac{1}{\sqrt{2\pi \frac{q^2-1}{12} \log q_l x}} \exp \left( -\frac{(k_l - \frac{q_l-1}{2} \log q_l x)^2}{2 \frac{q^2-1}{12} \log q_l x} \right) \right) + o((\log x)^{-1})
\]

uniformly for all integers \( k_1, k_2 \geq 0 \) with \( k_1 \equiv k_2 \pmod{d} \).
Note that $s_{q_1}(n) \equiv n \mod (q_l - 1)$. Thus we always have $s_{q_1}(n) \equiv s_{q_2}(n) \mod d$ and consequently

$$\#\{n < x \mid s_{q_1}(n) = k_1, s_{q_2}(n) = k_2\} = 0$$

if $k_1 \not\equiv k_2 \mod d$.

There are some other results indicating that the $q_l$-ary digital expansions are \textit{asymptotically independent} for different bases $q_l$; e.g. Kim [18] \(^{(1)}\) showed that for all integers $c_1, \ldots, c_d$,

$$\frac{1}{x}|\{n < x \mid s_{q_j}(n) \equiv c_j \mod m_j \ (1 \leq j \leq d)\}| = \frac{1}{m_1 \ldots m_d} + O(x^{-\delta})$$

with

$$\delta = \frac{1}{120d^2q^2m^2},$$

where $q_1, \ldots, q_d > 1$ are pairwise coprime integers and $m_1, \ldots, m_d$ are positive integers such that

$$\gcd(q_j - 1, m_j) = 1 \quad (1 \leq j \leq d);$$

$q = \max\{q_1, \ldots, q_d\}$, $m = \max\{m_1, \ldots, m_d\}$ and the $O$-constant depends only on $d$ and $q$. (This result sharpens a result by Bésineau [2] and solves a conjecture of Gelfond [11].)

Drmota and Larcher [7] used a variation of Kim’s method to prove that a $d$-dimensional sequence $(\alpha_1s_{q_1}(n), \ldots, \alpha_ds_{q_d}(n))_{n \geq 0}$ is uniformly distributed modulo 1 if and only if $\alpha_1, \ldots, \alpha_d$ are irrational. (Grabner, Llardi and Tichy [13] could prove a similar theorem by ergodic means.)

Another problem has been considered by Senge and Straus [26]. They proved that if $q_1$ and $q_2$ are coprime and $c$ is any given positive constant then there are only finitely many $n \geq 0$ such that

$$s_{q_1}(n) \leq c \quad \text{and} \quad s_{q_2}(n) \leq c.$$  

This result was later generalized and sharpened by Stewart [28], Schlickewei [22, 23] and by Pethó and Tichy [21]. The proofs use Baker’s method for linear forms of logarithms and the $p$-adic version of Schmidt’s subspace theorem by Schlickewei applied to $S$-unit equations.

One would get a much deeper insight into all these results if one could prove a local version of Theorem 2, e.g. asymptotic expansions or general estimates for the numbers

$$\frac{1}{x}\#\{n < x \mid s_q(n^2) = k\}$$

or for

$$\frac{1}{\pi(x)}\#\{p < x \mid s_q(p) = k\}$$

\(^{(1)}\) For brevity we restrict to the sum-of-digits function $s_q(n)$.  

(and of course multivariate versions). It seems that problems of this kind are extremely difficult, e.g. it is an open question whether there are infinitely many primes \( p \) with even sum-of-digits function \( s_2(p) \). The best known results concerning these questions are due to Fouvry and Mauduit [9, 10] who proved that

\[
\frac{1}{x} \sum_{n < x} \left\lvert \sum_{\substack{n = n_1 \cdot n_2 \wedge n_1, n_2 \in \mathbb{P}, s_q(n) \equiv 0 \mod 2}} \right\rvert \geq c > 0
\]

for some constant \( c > 0 \). (\( \mathbb{P} \) denotes the set of primes.)

These questions are also related to two other conjectures of Gelfond [11], namely that \( s_q(P(n)) \) and \( s_q(p) \) are uniformly distributed modulo \( m \).

**Remark.** Schmidt [25] and Schmid [24] discussed the joint distribution of \( s_2(k_1n) \) for different odd integers \( k_1, 1 \leq l \leq d \). (The distribution modulo \( m \) was investigated by Solinas [27].) It is surely possible to extend their result to the joint distribution of \( f_l(P_l(n)), 1 \leq l \leq d \), where \( f_l \) are \( q_l \)-additive functions, \( P_l \) are (certain) integer polynomials, and \( q_l > 1 \) arbitrary integers (e.g. all equal). However, we will not discuss this question here.

### 3. Proof of Theorem 2

As already mentioned, Theorem 2 is a direct generalization of Bassily and Kátai’s result of [1]. Therefore we can proceed as in [1].

The first two lemmata on exponential sums are stated in [1]; a proof can also be found in [15].

**Lemma 1.** Let \( f(y) \) be a polynomial of degree \( k \) of the form

\[
f(y) = \frac{a}{b} y^k + \alpha_1 y^{k-1} + \ldots + \alpha_k
\]

with \( \gcd(a, b) = 1 \). Let \( \tau \) be a positive number satisfying

\[
\tau \geq 2^{3(k-2)} \quad \text{and} \quad (\log x)^\tau < b < x^k (\log x)^{-\tau}.
\]

Then, as \( x \to \infty \),

\[
\frac{1}{x} \sum_{n < x} e(f(n)) = O((\log x)^{-\tau}).
\]

**Lemma 2.** Let \( f(y) \) be as in Lemma 1 and \( \tau_0, \tau \) arbitrary positive numbers satisfying

\[
\tau \geq 2^{6k} \tau_0 \quad \text{and} \quad (\log x)^\tau < b < x^k (\log x)^{-\tau}.
\]

Then, as \( x \to \infty \),

\[
\frac{1}{\pi(x)} \sum_{p < x} e(f(p)) = O((\log x)^{-\tau_0}).
\]

The third lemma is proved in [1] with the help of Lemmata 1 and 2 and the inequality of Erdős–Turán.
Lemma 3. Let $0 < \Delta < 1$ and 

$$U_{b,q,\Delta} := [0, \Delta] \cup \bigcup_{b=1}^{q-1} [b/q - \Delta, b/q + \Delta] \cup [1 - \Delta, 1].$$

Suppose that $P(x)$ is an integer polynomial of degree $r$ with positive leading term. Then for every $\varepsilon > 0$ and arbitrary $\lambda > 0$ we have uniformly for $(\log_q x)^\varepsilon < j < r \log_q x - (\log_q x)^\varepsilon$ and $0 < \Delta < 1/(2q)$, as $x \to \infty$,

$$\frac{1}{x} \# \left\{ n < x \left| \left\{ \frac{P(n)}{q^{j+1}} \right\} \in U_{b,q,\Delta} \right. \right\} \ll \Delta + (\log x)^{-\lambda},$$

$$\frac{1}{\pi(x)} \# \left\{ p < x \left| \left\{ \frac{P(p)}{q^{j+1}} \right\} \in U_{b,q,\Delta} \right. \right\} \ll \Delta + (\log x)^{-\lambda}.$$

We will also make use of the following limiting relations for centralized moments of $q$-additive functions (see [1]).

Lemma 4. Let $f$ be a $q$-additive function such that $f(cq^j) = O(1)$ as $j \to \infty$ and $c \in E_q$ and let $P(x)$ be a polynomial with integer coefficients, degree $r$, and positive leading term. Furthermore, suppose that for some $\eta > 0$ we have $D_q(x^r)/(\log x)^\eta \to \infty$ as $x \to \infty$. Define $f_1$ for $n < x^r$ by

$$f_1(n) = \sum_{(\log_q x)^\eta \leq j \leq r \log_q x - (\log_q x)^\eta} f(a_{q,j}(n)q^j)$$

and set

$$M_{q,1}(x^r) := \sum_{(\log_q x)^\eta \leq k \leq r \log_q x - (\log_q x)^\eta} m_{k,q},$$

$$D^2_{q,1}(x^r) := \sum_{(\log_q x)^\eta \leq k \leq r \log_q x - (\log_q x)^\eta} \sigma^2_{k,q}.$$

Then, as $x \to \infty$,

$$\frac{1}{x} \sum_{n < x} \left( \frac{f_1(P(n)) - M_{q,1}(x^r)}{D_{q,1}(x^r)} \right)^k \to \int_{-\infty}^{\infty} z^k \, d\Phi(z),$$

$$\frac{1}{\pi(x)} \sum_{p < x} \left( \frac{f_1(P(p)) - M_{q,1}(x^r)}{D_{q,1}(x^r)} \right)^k \to \int_{-\infty}^{\infty} z^k \, d\Phi(z).$$

In [1] this property is only proved for $\eta = 1/3$. However, as already mentioned, it is also true for any $\eta > 0$.

Proposition 1. Let $N_l = \lfloor \log_q x \rfloor$, $1 \leq l \leq d$, let $\lambda > 0$ be an arbitrary constant and $h_l$, $1 \leq l \leq d$, be positive integers. Furthermore, let $P_l(x)$, $1 \leq l \leq d$, be integer polynomials with non-negative leading terms and different degrees $r_l \geq 1$. Then for integers
(3.1) \[ N_l^\eta < k_1^{(l)} < k_2^{(l)} < \ldots < k_{h_l}^{(l)} \leq r_l N_l - N_l^{\eta} \quad (1 \leq l \leq d) \]
(with some \( \eta > 0 \)) we have, as \( x \to \infty \),

(3.2) \[
\frac{1}{x} \# \{ n < x \mid a_{q_l, k_j^{(l)}}(P_l(n)) = b_j^{(l)}, \ 0 \leq j \leq h_l, \ 1 \leq l \leq d \} = \frac{1}{q_1^{h_1} \cdots q_d^{h_d}} + O((\log x)^{-\lambda})
\]

and

(3.3) \[
\frac{1}{\pi(x)} \# \{ p < x \mid a_{q_l, k_j^{(l)}}(P_l(p)) = b_j^{(l)}, \ 0 \leq j \leq h_l, \ 1 \leq l \leq d \} = \frac{1}{q_1^{h_1} \cdots q_d^{h_d}} + O((\log x)^{-\lambda})
\]

uniformly for \( b_j^{(l)} \in E_{q_l} \) and \( k_j^{(l)} \) in the given range, where the implicit constant of the error term may depend on \( q_l \), on the polynomials \( P_l \), on \( h_l \) and on \( \lambda \).

Proof. We follow [1]. Let \( f_{b,q,\Delta}(x) \) be defined by

\[
f_{b,q,\Delta}(x) := \frac{1}{\Delta} \int_{-\Delta/2}^{\Delta/2} 1_{[b/q,(b+1)/q]}(\{x + z\}) \, dz,
\]

where \( 1_A \) is the characteristic function of the set \( A \) and \( \{x\} = x - [x] \) the fractional part of \( x \). The Fourier coefficients of the Fourier series \( f_{b,q,\Delta}(x) = \sum_{m \in \mathbb{Z}} d_{m,b,q,\Delta} e(mx) \) are given by

\[ d_{0,b,q,\Delta} = 1/q \]

and for \( m \neq 0 \) by

\[ d_{m,b,q,\Delta} = \frac{e(-mb/q) - e(-m(b+1)/q)}{2\pi im} \cdot \frac{e(m\Delta/2) - e(-m\Delta/2)}{2\pi im\Delta}. \]

Note that \( d_{m,b,q,\Delta} = 0 \) if \( m \neq 0 \) and \( m \equiv 0 \mod q \) and that

\[ |d_{m,b,q,\Delta}| \leq \min \left( \frac{1}{|m|}, \frac{1}{\Delta |m|^2} \right). \]

By definition we have

\[ 0 \leq f_{b,q,\Delta}(x) \leq 1 \]

and

\[ f_{b,q,\Delta}(x) = \begin{cases} 1 & \text{if } x \in [b/q + \Delta, (b+1)/q - \Delta], \\ 0 & \text{if } x \in [0,1] \setminus [b/q - \Delta, (b+1)/q + \Delta]. \end{cases} \]

So if we set

\[ t(y_1, \ldots, y_d) := \prod_{l=1}^{d} \prod_{j=1}^{h_l} f_{b_j^{(l)}, q_l, \Delta} \left( \frac{y_l}{k_j^{(l)} + 1} \right) \]
then for $\Delta < 1/(2q)$ we get
\[
\left| \# \{ n < x \mid a_{q_lk^{(l)}_j}(P_l(n)) = b^{(l)}_j, \ 0 \leq j \leq h_l, \ 1 \leq l \leq d \} \right|
- \sum_{n < x} t(P_1(n), \ldots, P_d(n)) \right| 
\leq \sum_{l=1}^d \sum_{j=1}^{h_l} \# \left\{ n < x \left| \frac{P_l(n)}{k^{(l)}_j+1} \in U_{b^{(l)}_j, q_l, \Delta} \right. \right\} \ll \Delta x + x (\log x)^{-\lambda}
\]
and
\[
\left| \# \{ p < x \mid a_{q_lk^{(l)}_j}(P_l(p)) = b^{(l)}_j, \ 0 \leq j \leq h_l, \ 1 \leq l \leq d \} \right|
- \sum_{p < x} t(P_1(p), \ldots, P_d(p)) \right| 
\leq \sum_{l=1}^d \sum_{j=1}^{h_l} \# \left\{ n < x \left| \frac{P_l(p)}{k^{(l)}_j+1} \in U_{b^{(l)}_j, q_l, \Delta} \right. \right\} \ll \Delta x + x (\log x)^{-\lambda},
\]
where $U_{b^{(l)}_j, q_l, \Delta}$ is given in Lemma 3.

For convenience, let $m_l = (m^{(l)}_1, \ldots, m^{(l)}_{h_l})$ denote $h_l$-dimensional integer vectors and $v_l = (q_l^{-k^{(l)}_1-1}, \ldots, q_l^{-k^{(l)}_{h_l}-1}), 1 \leq l \leq d$. Furthermore set
\[
T_{m_1, \ldots, m_d} := \prod_{l=1}^d \prod_{j=1}^{h_l} d_{m^{(l)}_j, b^{(l)}_j, q_l, \Delta}.
\]
Then $t(P_1(n), \ldots, P_d(n))$ has Fourier series expansion
\[
t(y_1, \ldots, y_d) = \sum_{m_1, \ldots, m_d} T_{m_1, \ldots, m_d} e(m_1 \cdot v_1 y_1 + \ldots + m_d \cdot v_d y_d).
\]
Thus, we are led to consider the exponential sums
\[
(3.4) \quad S_1 = \sum_{m_1, \ldots, m_d} T_{m_1, \ldots, m_d} \sum_{n < x} e(m_1 \cdot v_1 P_1(n) + \ldots + m_d \cdot v_d P_d(n)),
\]
\[
(3.5) \quad S_2 = \sum_{m_1, \ldots, m_d} T_{m_1, \ldots, m_d} \sum_{p < x} e(m_1 \cdot v_1 P_1(p) + \ldots + m_d \cdot v_d P_d(p)).
\]

Let us consider for a moment just the first sum $S_1$. If $m_1, \ldots, m_d$ are all zero then
\[
T_{m_1, \ldots, m_d} \sum_{n < x} e(m_1 \cdot v_1 P_1(n) + \ldots + m_d \cdot v_d P_d(n)) = \frac{x + O(1)}{q_1^{h_1} \ldots q_d^{h_d}},
\]
which provides the leading term. Furthermore, if there exist $l$ and $j$ with $m^{(l)}_j \neq 0$ and $m^{(l)}_j \equiv 0 \mod q_l$ then $T_{m_1, \ldots, m_d} = 0$. So it remains to consider
the case where there exist \( l \) and \( j \) with \( m_j^{(l)} \neq 0 \) mod \( q_l \). Here the exponent is of the form

\[
m_1 \cdot v_1 P_1(n) + \ldots + m_d \cdot v_d P_d(n) = \frac{a_1}{b_1} P_1(n) + \ldots + \frac{a_d}{b_d} P_d(n)
\]

in which we assume that \( \gcd(a_l, b_l) = 1, 1 \leq l \leq d \). The first observation is that for any \( l \) for which there exists \( j \) with \( m_j^{(l)} \neq 0 \) mod \( q_l \) there exists \( \eta_l > 0 \) (only depending on \( q_l \)) such that \( b_l \geq q_l^{\eta_l k_s^{(l)}} \) if \( m_s^{(l)} \neq 0, m_j^{(l)} \neq 0 \) mod \( q_l \) and \( m_s^{(l)} = m_{s+2}^{(l)} = \ldots = m_{h_l}^{(l)} = 0 \) (cf. [1]). For the reader’s convenience we repeat the argument. Suppose that the prime factorization of \( q_l \) is given by \( q_l = p_1^{e_1} \ldots p_k^{e_k} \). If \( m_s^{(l)} \neq 0 \) mod \( q_l \) then there exists \( t \) such that \( m_s^{(l)} \neq 0 \) mod \( p_t^{e_t} \). Now we have

\[
b_l (m_s^{(l)} + q_l^{k_s^{(l)} - k_{s-1}^{(l)}} m_{s-1}^{(l)} + \ldots + q_l^{k_s^{(l)} - k_1^{(l)}} m_1^{(l)}) = a_t q_l^{k_s^{(l)} + 1}
\]

Hence \( b_l \equiv 0 \) mod \( p_t^{k_s^{(l)} e_t} \) and consequently \( b_l \geq p_t^{k_s^{(l)} e_t} \geq q_l^{\eta_l k_s^{(l)}} \). Note that we also have \( b_l \leq q_l^{\eta_l k_s^{(l)}} \).

Now let \( D \) denote the set of \( l \in \{1, \ldots, d\} \) such that there exists \( j \) with \( m_j^{(l)} \neq 0 \) mod \( q_l \). Since all degrees \( r_l \) are different there exists a unique \( l_0 \) with \( r_{l_0} = \max\{r_l \mid l \in D\} \). We now want to apply Lemma 1 with \( k = r_{l_0} \) and \( b = b_{l_0} \). If \( k_s^{(l)} \) are in the range (3.1) then for every \( \tau > 0 \) there exists \( x_0(\tau) \) such that for \( x \geq x_0(\tau) \),

\[
\log(x)^\tau < b_{l_0} < x^{r_{l_0}}(\log x)^{-\tau}.
\]

Consequently, we can apply Lemma 1 to obtain

\[
\frac{1}{x} \#\{n < x \mid a_{q_l, k_j^{(l)}}(P(n)) = b_j^{(l)}, 0 \leq j \leq h_l, 1 \leq l \leq d\} = \frac{1}{q_1^{h_1} \ldots q_d^{h_d}} + O\left((\log x)^{-\lambda} \sum_{\mathbf{m} \neq 0} |T_{\mathbf{m}_1, \ldots, \mathbf{m}_d}| \right) + O(\Delta + (\log x)^{-\lambda}),
\]

where \( \mathbf{m} = (\mathbf{m}_1, \ldots, \mathbf{m}_d) \). Since

\[
\sum_{\mathbf{m} \neq 0} |T_{\mathbf{m}_1, \ldots, \mathbf{m}_d}| \leq (2 + 2 \log(1/\Delta))^{h_1+\ldots+h_d}
\]

it is possible to choose \( \Delta = (\log x)^{-\lambda_1} \) for a sufficiently large constant \( \lambda_1 \) such that (3.2) holds.

The proof of (3.3) runs along the same lines. ■

Corollary 3. Let \( N_l = [\log q_l x], 1 \leq l \leq d, \) and \( \lambda, \eta > 0 \). Then for integers \( k_j^{(l)} \) satisfying

\[
N_l^{\eta} \leq k_j^{(l)} < r_l N_l - N_l^{\eta} \quad (1 \leq j \leq h_l, 1 \leq l \leq d)
\]
and $b^{(l)}_j \in E_{q_l}$, we uniformly have, as $x \to \infty$,

$$\frac{1}{x} \#\{n < x \mid a_{q_l,k^{(l)}_j}(P_l(n)) = b^{(l)}_j, \ 0 \leq j \leq h_l, \ 1 \leq l \leq d\}$$

$$= \prod_{l=1}^{d} \left( \frac{1}{x} \#\{n < x \mid a_{q_l,k^{(l)}_j}(P_l(n)) = b^{(l)}_j, \ 0 \leq j \leq h_l\} \right) + O((\log x)^{-\lambda})$$

and

$$\frac{1}{\pi(x)} \#\{p < x \mid a_{q_l,k^{(l)}_j}(P_l(p)) = b^{(l)}_j, \ 0 \leq j \leq h_l, \ 1 \leq l \leq d\}$$

$$= \prod_{l=1}^{d} \left( \frac{1}{\pi(x)} \#\{p < x \mid a_{q_l,k^{(l)}_j}(P_l(p)) = b^{(l)}_j, \ 0 \leq j \leq h_l\} \right) + O((\log x)^{-\lambda}).$$

**Proof.** If there exist $l$ and $j_1, j_2$ with $k^{(l)}_{j_1} = k^{(l)}_{j_2}$ but $b^{(l)}_{j_1} \neq b^{(l)}_{j_2}$, then both sides are zero.

So it remains to consider the case where for every $l$ the integers $k^{(l)}_j$, $1 \leq j \leq h_l$, are different, and without loss of generality we can assume that they are increasing. Hence we can directly apply Proposition 1. □

**Corollary 4.** For any choice of integers $k_l$, $1 \leq l \leq d$, we have, as $x \to \infty$,

$$\frac{1}{x} \sum_{n < x} \prod_{l=1}^{d} \left( \frac{f_{l,1}(P_l(n)) - M_{q_l,1}(x^{r_l})}{D_{q_l,1}(x^{r_l})} \right)^{k_l}$$

$$- \prod_{l=1}^{d} \left( \frac{1}{x} \sum_{n < x} \left( \frac{f_{l,1}(P_l(n)) - M_{q_l,1}(x^{r_l})}{D_{q_l,1}(x^{r_l})} \right)^{k_l} \right) \to 0$$

and

$$\frac{1}{\pi(x)} \sum_{p < x} \prod_{l=1}^{d} \left( \frac{f_{l,1}(P_l(p)) - M_{q_l,1}(x^{r_l})}{D_{q_l,1}(x^{r_l})} \right)^{k_l}$$

$$- \prod_{l=1}^{d} \left( \frac{1}{\pi(x)} \sum_{p < x} \left( \frac{f_{l,1}(P_l(p)) - M_{q_l,1}(x^{r_l})}{D_{q_l,1}(x^{r_l})} \right)^{k_l} \right) \to 0.$$

**Proof.** In order to demonstrate how this property can be derived, we consider the case $d = 2$ and $k_1 = k_2 = 2$. Set $A_l = [(\log q_l x)^\eta]$ and $B_l = [\log q_l x - (\log q_l x)^\eta]$ and observe that

$$f_{l,1}(P_l(n)) - M_{q_l,1}(x^{r_l}) = \sum_{j=A_l}^{B_l} \sum_{b \in E_{q_l}} \left( f_l(bq_l^j)\delta(a_{q_l,j}(P_l(n)), b) - \frac{m_{j,q_l}}{q_l} \right),$$
where \( \delta(x,y) \) denotes the Kronecker delta. Hence we have

\[
\frac{1}{x} \sum_{n<x} \left( \frac{f_{1,1}(P_1(n)) - M_{q_1,1}(x^{r_1})}{D_{q_1,1}(x^{r_1})} \right)^2 \left( \frac{f_{2,1}(P_2(n)) - M_{q_2,1}(x^{r_2})}{D_{q_2,1}(x^{r_2})} \right)^2 \\
= \sum_{j_1=A_1}^{B_1} \sum_{j_2=A_2}^{B_2} \sum_{j_3=A_3}^{B_3} \sum_{j_4=A_4}^{B_4} \sum_{b_1 \in E_{q_1}} \sum_{b_2 \in E_{q_2}} \sum_{b_3 \in E_{q_2}} \sum_{b_4 \in E_{q_2}} \frac{1}{D_{q_1,1}(x^{r_1})D_{q_2,1}(x^{r_2})} \\
\times \frac{1}{x} \sum_{n<x} \left( f_1(b_1 q_1^{j_1}) \delta(a_{q_1,j_1}(P_1(n)), b_1) - \frac{m_{j_1,q_1}}{q_1} \right) \\
\times \left( f_1(b_2 q_1^{j_2}) \delta(a_{q_1,j_2}(P_1(n)), b_2) - \frac{m_{j_2,q_1}}{q_1} \right) \\
\times \left( f_2(b_3 q_2^{j_3}) \delta(a_{q_2,j_3}(P_2(n)), b_3) - \frac{m_{j_3,q_2}}{q_2} \right) \\
\times \left( f_2(b_4 q_2^{j_4}) \delta(a_{q_2,j_4}(P_2(n)), b_4) - \frac{m_{j_4,q_2}}{q_2} \right).
\]

By Corollary 3 it follows that

\[
\frac{1}{x} \sum_{n<x} \left( f_1(b_1 q_1^{j_1}) \delta(a_{q_1,j_1}(P_1(n)), b_1) - \frac{m_{j_1,q_1}}{q_1} \right) \\
\times \left( f_1(b_2 q_1^{j_2}) \delta(a_{q_1,j_2}(P_1(n)), b_2) - \frac{m_{j_2,q_1}}{q_1} \right) \\
\times \left( f_2(b_3 q_2^{j_3}) \delta(a_{q_2,j_3}(P_2(n)), b_3) - \frac{m_{j_3,q_2}}{q_2} \right) \\
\times \left( f_2(b_4 q_2^{j_4}) \delta(a_{q_2,j_4}(P_2(n)), b_4) - \frac{m_{j_4,q_2}}{q_2} \right) \\
= f_1(b_1 q_1^{j_1}) f_1(b_2 q_1^{j_2}) f_2(b_3 q_2^{j_3}) f_2(b_4 q_2^{j_4}) \\
\times \frac{1}{x} \# \{ n < x \mid a_{q_1,j_1}(P_1(n)) = b_1, \ a_{q_1,j_2}(P_1(n)) = b_2, \ a_{q_2,j_3}(P_2(n)) = b_3, \ a_{q_2,j_4}(P_2(n)) = b_4 \} \\
\quad - f_1(b_1 q_1^{j_1}) f_1(b_2 q_1^{j_2}) f_2(b_3 q_2^{j_3}) \\
\times \frac{1}{x} \# \{ n < x \mid a_{q_1,j_1}(P_1(n)) = b_1, \ a_{q_1,j_2}(P_1(n)) = b_2, \ a_{q_2,j_3}(P_2(n)) = b_3 \} \\
\times \frac{m_{j_4,q_2}}{q_2} + \ldots + \frac{m_{j_1,q_1}}{q_1} \cdot \frac{m_{j_2,q_1}}{q_1} \cdot \frac{m_{j_3,q_2}}{q_2} \cdot \frac{m_{j_4,q_2}}{q_2}.\]
\[
= \left( f_1(b_1 q_{j1}^{j1}) f_1(b_2 q_{j2}^{j2}) \frac{1}{x} \# \{ n < x \mid a_{q_1,j1}(P_1(n)) = b_1, \; a_{q_1,j2}(P_1(n)) = b_2 \} \right) \\
\times \left( f_2(b_3 q_{j3}^{j3}) f_2(b_4 q_{j4}^{j4}) \frac{1}{x} \# \{ n < x \mid a_{q_2,j3}(P_2(n)) = b_3, \; a_{q_2,j4}(P_2(n)) = b_4 \} \right) \\
- \left( f_1(b_1 q_{j1}^{j1}) f_1(b_2 q_{j2}^{j2}) \frac{1}{x} \# \{ n < x \mid a_{q_1,j1}(P_1(n)) = b_1, \; a_{q_1,j2}(P_1(n)) = b_2 \} \right) \\
\times \left( f_2(b_3 q_{j3}^{j3}) \frac{1}{x} \# \{ n < x \mid a_{q_2,j3}(P_2(n)) = b_3 \} \right) \frac{m_{j_4,q_2}}{q_2} \\
\equiv \ldots + \left( \frac{m_{j_1,q_1}}{q_1} \frac{m_{j_2,q_1}}{q_1} \frac{m_{j_3,q_2}}{q_2} \frac{m_{j_4,q_2}}{q_2} \right) + O((\log x)^{-\lambda}) \\
= \left( \frac{1}{x} \sum_{n<x} \left( f_1(b_1 q_{j1}^{j1}) \delta(a_{q_1,j1}(P_1(n)), b_1) - \frac{m_{j_1,q_1}}{q_1} \right) \right) \\
\times \left( f_1(b_2 q_{j2}^{j2}) \delta(a_{q_1,j2}(P_1(n)), b_2) - \frac{m_{j_2,q_1}}{q_1} \right) \\
\times \left( \frac{1}{x} \sum_{n<x} \left( f_2(b_3 q_{j3}^{j3}) \delta(a_{q_2,j3}(P_2(n)), b_3) - \frac{m_{j_3,q_2}}{q_2} \right) \right) \\
\times \left( f_2(b_4 q_{j4}^{j4}) \delta(a_{q_2,j4}(P_2(n)), b_4) - \frac{m_{j_4,q_2}}{q_2} \right) + O((\log x)^{-\lambda}).
\]

So we directly obtain the claimed result with an error term of the form \(O((\log x)^{-\lambda+4-4\eta})\). ■

By combining Lemma 4, Corollary 4, and the Fréchet–Shohat theorem it follows that, as \(x \to \infty\),

\[
\frac{1}{x} \# \left\{ n < x \mid \frac{f_{l_1}(P_l(n)) - M_{q_1,1}(x^{r_l})}{D_{q_1,1}(x^{r_l})} < y_l, \; 1 \leq l \leq d \right\} \to \Phi(y_1) \ldots \Phi(y_d),
\]

\[
\frac{1}{\pi(x)} \# \left\{ p < x \mid \frac{f_{l_1}(P_l(p)) - M_{q_1,1}(x^{r_l})}{D_{q_1,1}(x^{r_l})} < y_l, \; 1 \leq l \leq d \right\} \to \Phi(y_1) \ldots \Phi(y_d).
\]

Since

\[
M_{q_1}(x^{r_l}) - M_{q_1,1}(x^{r_l}) = O((\log x)^{\eta}),
\]

\[
D_{q_1}(x^{r_l}) - D_{q_1,1}(x^{r_l}) = O((\log x)^{\eta}),
\]

Joint distribution of \(q\)-additive functions
it also follows that
\[
\max_{n < x} \left| \frac{f_l(P_l(n)) - M_{q_l}(x^{r_l})}{D_{q_l}(x^{r_l})} - \frac{f_{l,1}(P_l(n)) - M_{q_l,1}(x^{r_l})}{D_{q_l,1}(x^{r_l})} \right| \to 0
\]
as \(x \to \infty\). Consequently we finally obtain the limiting relations stated in Theorem 2.

\section*{4. Proof of Theorem 3.} The proof of Theorem 3 is similar to that of Theorem 2, i.e., we will prove an analogue to Proposition 1. However, the proof requires an additional ingredient, namely a proper version of Baker’s theorem on linear forms. More precisely, we will use the following version due to Waldschmidt \cite{29}.

\begin{lemma}
Let \(a_1, \ldots, a_n \) be non-zero algebraic numbers and \(b_1, \ldots, b_n \) integers such that
\[
a_1^{b_1} \cdots a_n^{b_n} \neq 1
\]
and let \(A_1, \ldots, A_n \geq e \) be real numbers with \(\log A_j \geq h(\alpha_j)\), where \(h(\cdot)\) denotes the absolute logarithmic height. Set \(d = [\mathbb{Q}(\alpha_1, \ldots, \alpha_n) : \mathbb{Q}]\). Then
\[
|a_1^{b_1} \cdots a_n^{b_n} - 1| \geq \exp(-U),
\]
where
\[
U = 2^{6n+32n^3n+6}d^{n+2}(1 + \log d)(\log B + \log d) \log A_1 \cdots \log A_n,
\]
\[
B = \max\{2, |b_1|, \ldots, |b_n|\}.
\]
\end{lemma}

\begin{corollary}
Let \(q_1, q_2 > 1 \) be coprime integers and \(m_1, m_2 \) integers such that \(m_1 \neq 0 \mod q_1 \) and \(m_2 \neq 0 \mod q_2 \). Then there exists a constant \(C > 0 \) such that for all integers \(k_1, k_2 > 1, \)
\[
\left| \frac{m_1}{q_1^{k_1}} + \frac{m_2}{q_2^{k_2}} \right| \geq \max \left( \left| \frac{m_1}{q_1^{k_1}} \right|, \left| \frac{m_2}{q_2^{k_2}} \right| \right) \cdot e^{-C \log q_1 \log q_2 \log(\max(k_1, k_2)) \cdot \log(\max(|m_1|, |m_2|))}.
\]
\end{corollary}

\begin{proof}
Since \(q_1, q_2 > 1 \) are coprime integers and \(m_1 \neq 0 \mod q_1, m_2 \neq 0 \mod q_2 \) we surely have \(m_1 q_1^{-k_1} + m_2 q_2^{-k_2} \neq 0 \). So we can apply Lemma 5 for \(n = 3, \alpha_1 = q_1, \alpha_2 = q_2, \alpha_3 = -m_2/m_1, b_1 = k_1, b_2 = -k_2, b_3 = 1 \) and directly obtain
\[
\left| \frac{m_1}{q_1^{k_1}} + \frac{m_2}{q_2^{k_2}} \right| = |m_1| \cdot q_1^{-k_1} \cdot \left| -q_1^{k_1} q_2^{-k_2} \frac{m_2}{m_1} - 1 \right| \geq |m_1| q_1^{k_1} e^{-C \log q_1 \log q_2 \log(\max(k_1, k_2)) \cdot \log(\max(|m_1|, |m_2|))}.
\]
Since the problem is symmetric it is no loss of generality to assume that \(|m_1| q_1^{-k_1} \geq |m_2| q_2^{-k_2}\). \(\blacksquare\)
Finally we will use the following (trivial) lemma on exponential sums.

**Lemma 6.** Let $\alpha$ be a real number with $0 < |\alpha| \leq 1/2$. Then, as $x \to \infty$,

$$
\sum_{n<x} e(\alpha n) \ll \frac{1}{|\alpha|}.
$$

**Proposition 2.** Let $P_l(x) = A_l x + B_l$, $l = 1, 2$, be linear polynomials with integer coefficients and non-negative leading terms $A_l$ which are co-prime to $q_l$. Set $N_l = \lfloor \log_{q_l} x \rfloor$, $l = 1, 2$, let $\lambda, \eta > 0$ be arbitrary constants and let $h_1, h_2$ be positive integers. Then for integers

$$
N_l^\eta \leq k_1^{(l)} < k_2^{(l)} < \ldots < k_{h_l}^{(l)} \leq N_l - N_l^\eta \quad (l = 1, 2)
$$

we have, as $x \to \infty$,

$$
\frac{1}{x} \# \{ n < x \mid a_{q_l, k_j^{(l)}}(A_l n + B_l) = b_j^{(l)}, \ 0 \leq j \leq h_l, \ l = 1, 2 \} = \frac{1}{q_1^{h_1} q_2^{h_2}} + O((\log x)^{-\lambda})
$$

uniformly for $b_j^{(l)} \in E_{q_l}$ and $k_j^{(l)}$ in the given range, where the implicit constant of the error term may depend on $q_l, h_l$ and $\lambda$.

**Proof.** The proof runs along the same lines as the proof of Proposition 1. The only problem is to estimate the sum

$$
\sum_{(m_1, m_2) \neq 0} |T_{m_1, m_2}| \cdot \left| \frac{1}{x} \sum_{n<x} e((A_1 m_1 \cdot v_1 + A_2 m_2 \cdot v_2) n) \right|,
$$

where $m_l = (m_1^{(l)}, \ldots, m_{h_l}^{(l)})$ and $v_l = (q_l^{-k_1^{(l)}-1}, \ldots, q_l^{-k_{h_l}^{(l)}-1})$, $l = 1, 2$, such that the integers $k_j^{(l)}$ are in the given range (4.1).

First we fix $\Delta = (\log x)^{-\lambda_0}$ with an arbitrary (but fixed) constant $\lambda_0 > 0$. Furthermore, since

$$
\sum_{\exists l \exists j: |m_j^{(l)}| > (\log x)^{2\lambda_0}} |T_{m_1, m_2}| \ll (\log x)^{-\lambda_0}
$$

we can restrict to those $m_l \neq 0$ for which $|m_j^{(l)}| \leq (\log x)^{2\lambda_0}$ for all $l, j$ and $m_j^{(l)} \neq 0 \mod q_l$ if $m_j^{(l)} \neq 0$.

We also note that it is also sufficient to consider just the case where $m_j^{(l)} \neq 0$ for all $j$ and $l = 1, 2$. (Otherwise we just reduce $h_1$ resp. $h_2$ to a smaller value and use the same arguments.)

Set $\delta = \eta/(h_1 + h_2 - 1)$. Then there exists an integer $k$ with $0 \leq k \leq h_1 + h_2 - 2$ such that for all $j$ and $l = 1, 2$

$$
k_{j+1}^{(l)} - k_j^{(l)} \not\in [(\log x)^{k\delta}, (\log x)^{(k+1)\delta})].$$
So fix \( k \) with this property. Before discussing the general case, let us consider two extremal ones.

First suppose that
\[
k^{(l)}_{j+1} - k^{(l)}_j < (\log x)^{k\delta}
\]
for all \( j \) and \( l = 1, 2 \). Set
\[
\overline{m}_l = A_l \sum_{j=1}^{h_l} m^{(l)}_j q^{k^{(l)}_j - k^{(l)}_j}_l \quad (l = 1, 2).
\]
Then we have \( \overline{m}_l \not\equiv 0 \mod q_l \) and \( \log |\overline{m}_l| \ll (\log x)^{k\delta} \). Hence, we can apply Corollary 5 to
\[
A_1 m_1 \cdot v_1 + A_2 m_2 \cdot v_2 = \frac{\overline{m}_1}{q_1} \frac{k^{(1)}_{h_1} - 1}{k^{(1)}_{h_1} + 1} + \frac{\overline{m}_2}{q_2} \frac{k^{(2)}_{h_2} - 1}{k^{(2)}_{h_2} + 1}
\]
and obtain
\[
|A_1 m_1 \cdot v_1 + A_2 m_2 \cdot v_2| \geq \max(q_1^{k^{(1)}_{h_1} - 1}, q_2^{k^{(2)}_{h_2} - 1})e^{-C \log \log x (\log x)^{k\delta}}
\]
for some constant \( C > 0 \). Since \( |A_1 m_1 \cdot v_1 + A_2 m_2 \cdot v_2| \leq 1/2 \), from Lemma 6 we get
\[
\left| \frac{1}{x} \sum_{n < x} e((A_1 m_1 \cdot v_1 + A_2 m_2 \cdot v_2)n) \right| < \frac{1}{x} q^{\log_q x - (\log x)^{(h_1 + h_2 - 1)\delta}} e^{-C \log \log x (\log x)^{k\delta}}
\]
\[
= e^{-(\log x)^{(h_1 + h_2 - 1)\delta} / \log q + C \log \log x (\log x)^{k\delta}} \ll (\log x)^{-\lambda}
\]
for any given \( \lambda > 0 \).

Next suppose that
\[
k^{(l)}_{j+1} - k^{(l)}_j \geq (\log x)^{(k+1)\delta}
\]
for all \( j \) and \( l = 1, 2 \). Here we set \( \overline{m}_l = A_l m^{(l)}_1 \) \( (l = 1, 2) \) and obtain
\[
|A_1 m_1 \cdot v_1 + A_2 m_2 \cdot v_2| \geq \left| \frac{\overline{m}_1}{q_1} \frac{k^{(1)}_{h_1} - 1}{k^{(1)}_{h_1} + 1} - \sum_{j_1=2}^{h_1} \frac{m^{(1)}_{j_1}}{q_1} \frac{k^{(1)}_{j_1} - 1}{k^{(1)}_{j_1} + 1} - \sum_{j_2=2}^{h_2} \frac{m^{(2)}_{j_2}}{q_2} \frac{k^{(2)}_{j_2} - 1}{k^{(2)}_{j_2} + 1} \right|
\]
\[
\geq \max(q_1^{k^{(1)}_{h_1} - 1}, q_2^{k^{(2)}_{h_2} - 1})e^{-C(\log \log x)^2}
\]
\[
- O((\log x)^{2\lambda_0} \max(q_1^{k^{(1)}_{h_1} - 1}, q_2^{k^{(2)}_{h_2} - 1})e^{-(\log x)^{(k+1)\delta}})
\]
\[
\gg \max(q_1^{k^{(1)}_{h_1} - 1}, q_2^{k^{(2)}_{h_2} - 1})e^{-C(\log \log x)^2}.
\]
Thus, we again have
\[(4.3) \quad \left| \frac{1}{x} \sum_{n<x} e((A_1 m_1 \cdot v_1 + A_2 m_2 \cdot v_2)n) \right| \ll (\log x)^{-\lambda}\]
for any given \(\lambda > 0\).

In general, we assume that for some \(s_l\) (\(l = 1, 2\)),
\[k^{(l)}_{j+1} - k^{(l)}_j < (\log x)^{k\delta} \quad (j < s_l)\]
and
\[k^{(l)}_{s_l+1} - k^{(l)}_{s_l} \geq (\log x)^{(k+1)\delta}.\]

Here we set
\[\overline{m}_l = A_l \sum_{j=1}^{s_l} m^{(l)}_j q^{k^{(l)}_{s_l} - k^{(l)}_j}_{l+1} \quad (l = 1, 2).\]

Then we have (as in the first case) \(\overline{m}_l \not\equiv 0 \text{ mod } q_l\) and \(\log |\overline{m}_l| \ll (\log x)^{k\delta}\).
Furthermore, we can estimate the sums
\[\sum_{j=s_l+1}^{h_l} \frac{m^{(l)}_j}{q^{k^{(l)}_{j+1}+1}} = O((\log x)^{2\lambda_0} q_l^{-(\log x)^{(k+1)\delta}}).\]
Thus we get
\[|A_1 m_1 \cdot v_1 + A_2 m_2 \cdot v_2| \geq \left| \frac{\overline{m}_1}{q^{k^{(1)}_{s_1}+1}_1} + \frac{\overline{m}_2}{q^{k^{(2)}_{s_2}+1}_2} \right| - \left| \sum_{j_1=s_1+1}^{h_1} \frac{m^{(1)}_{j_1}}{q^{k^{(1)}_{j_1}+1}_1} \right| - \left| \sum_{j_2=s_2+1}^{h_2} \frac{m^{(2)}_{j_2}}{q^{k^{(2)}_{j_2}+1}_2} \right| \geq \max(q^{k^{(1)}_{s_1}-1}_1, q^{k^{(2)}_{s_2}-1}_2) e^{-C \log \log x (\log x)^{k\delta} - O((\log x)^{2\lambda_0} \max(q^{k^{(1)}_{s_1}-1}_1, q^{k^{(2)}_{s_2}-1}_2)) e^{-(\log x)^{(k+1)\delta}}} \geq \max(q^{k^{(1)}_{s_1}-1}_1, q^{k^{(2)}_{s_2}-1}_2) e^{-C \log \log x (\log x)^{k\delta}},\]
which again implies (4.3).

Hence, we finally get
\[\sum_{(m_1, m_2) \neq 0} |T_{m_1, m_2}| \cdot \left| \frac{1}{x} \sum_{n<x} e((A_1 m_1 \cdot v_1 + A_2 m_2 \cdot v_2)n) \right| = O((\log x)^{-\lambda_0}) + O((\log x)^{4\lambda_0-\lambda}),\]
which completes the proof of Proposition 2. ■

5. Proof of Theorem 4. The proof of Theorem 4 relies on a direct application of proper saddle point approximations.
Set
\[ a_{k_1k_2} = \#\{n < x \mid s_{q_1}(n) = k_1, s_{q_2}(n) = k_2\}. \]

Then the empirical characteristic function is given by
\[ \varphi_x(t_1, t_2) = \frac{1}{x} \sum_{n < x} e^{it_1s_{q_1}(n)+it_2s_{q_2}(n)} = \frac{1}{x} \sum_{k_1,k_2 \geq 0} a_{k_1k_2}e^{it_1k_1+it_2k_2}, \]
which implies that the numbers \( a_{k_1k_2} \) can be determined by
\[ a_{k_1k_2} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \varphi_x(t_1, t_2)e^{-it_1k_1-it_2k_2} \, dt_1 \, dt_2. \]

We first use Theorem 2 to extract the asymptotic leading term of \( a_{k_1k_2} \). In fact, we need a little bit more general property.

**Lemma 7.** Set
\[ M_l(x) := \frac{q_l - 1}{2} \log q_l, \quad x \quad \text{and} \quad D_l(x) := \frac{q_l^2 - 1}{12} \log q_l, x \]
and let \( P(x) \) denote the linear polynomial \( P(x) = \text{lcm}(q_1 - 1, q_2 - 1)x + B \) for some integer \( B \) with \( 0 \leq B < \text{lcm}(q_1 - 1, q_2 - 1) \). Then, for every \( \varepsilon > 0 \) there exists \( x_0 = x_0(\varepsilon) \) such that
\[ \left| \frac{1}{x} \sum_{n < x} e^{it_1s_{q_1}(P(n))+it_2s_{q_2}(P(n))} \right| < \varepsilon \]
for all \( x \geq x_0 \) and for all \( t_1, t_2 \) real.

**Proof.** First we notice that Theorem 2 cannot be directly applied. It may occur that the leading term \( A = \text{lcm}(q_1 - 1, q_2 - 1) \) of \( P(x) \) is not coprime to \( q_1 \) resp. to \( q_2 \). However, if \( A = q_i^K_l \overline{A}_l \) (for some \( K_l > 0 \) and \( A_l \) coprime to \( q_l \)) and if \( B_l \) has \( q_l \)-ary expansion \( B_l = B_0 + B_1q_l + \ldots + B_{L_l}q_l^{L_l} \) then
\[ s_{q_l}(An + B) = s_{q_l}(q_i^{K_l} \overline{A}_ln + B_0 + B_1q_l + \ldots + B_{L_l}q_l^{L_l}) \]
\[ = s_{q_l}(q_i^{K_l-1} \overline{A}_ln + B_1 + B_2q_l + \ldots + B_{L_l}q_l^{L_l-1}) + B_0 \]
\[ = s_{q_l}(q_i^{K_l-2} \overline{A}_ln + B_2 + B_3q_l + \ldots + B_{L_l}q_l^{L_l-2}) + B_0 + B_1 \]
\[ \vdots \]
\[ = s_{q_l}(\overline{A}_ln + B_l) + C_l \]
for some integers \( B_l, C_l \). Therefore, the joint (normalized) limiting distribution of \( (s_{q_1}(An + B), s_{q_2}(An + B)) \) is the same as that of \( (s_{q_1}(\overline{A}_ln + B_1), s_{q_2}(\overline{A}_2n + B_2)) \), and \( \overline{A}_l \) is coprime to \( q_l, l = 1, 2 \). Hence, we can always apply Theorem 2 for properly chosen linear polynomials \( P_l(x), l = 1, 2 \).
By Levi’s theorem it now follows from Theorem 2 (and the above remark) that for every fixed $t_1, t_2$ we have, as $x \to \infty$,

\[
(5.1) \quad \frac{1}{x} \sum_{n<x} e^{i(t_1 s_{q_1}(P(n)) + t_2 s_{q_2}(P(n)))} \frac{1}{\sqrt{\log x}} - e^{i(t_1 M_1(x) + t_2 M_2(x))} \frac{1}{\sqrt{\log x}} - \frac{1}{2} (t_1^2 D_1^2(x) + t_2^2 D_2^2(x)) / (\log x) \to 0.
\]

Moreover, we can show that this convergence is uniform for all $t_1, t_2$. Since $\Phi(y_1) \Phi(y_2)$ is continuous we know that the normalized empirical distribution function

\[
\widetilde{F}_x(y_1, y_2) := \frac{1}{x} \# \{ n < x \mid s_{q_l}(n) \leq M_l(n) + y_l D_l(x), \; l = 1, 2 \}
\]

converges uniformly to $\Phi(y_1) \Phi(y_2)$. Furthermore, the variances

\[
\frac{1}{x} \sum_{n<x} \frac{(s_{q_l}(n) - M_l(n))^2}{D_l^2(x)}
\]

are bounded (compare with (1.1)). Hence we get

\[
\max_{\{ |y_1|, |y_2| \geq A \}} \int d\widetilde{F}_x(y_1, y_2) \ll \frac{1}{A}.
\]

Thus it follows by elementary means (and by using the definition of the characteristic function) that the convergence in (5.1) is uniform. ■

The proof of Theorem 2 will also make use of the following estimate on exponential sums.

**Proposition 3.** Let $q_1, \ldots, q_d > 1$ be pairwise coprime integers. Then there exists a constant $c > 0$ such that for all real numbers $t_1, \ldots, t_d$,

\[
\left| \frac{1}{x} \sum_{n<x} e(t_1 s_{q_1}(n) + t_2 s_{q_2}(n) + \ldots + t_d s_{q_d}(n)) \right| \ll e^{-c \log x \sum_{l=1}^d \| (q_l - 1) t_l \| ^2},
\]

where $\| t \| = \min_{k \in \mathbb{Z}} | t - k |$ denotes the distance to the integers.

A proof of Proposition 3 can be found in [7]. It is, more or less, a slight generalization of a corresponding estimate of exponential sums presented by Kim [18].

Now we can start with the proof of Theorem 4.

**Proof.** For any $K > 0$ and integers $s_1, s_2$ set

\[
C_K(s_1, s_2) := \left\{ (t_1, t_2) \in [-\pi, \pi]^2 : \left| t_l - \frac{2\pi s_l}{q_l - 1} \mod 2\pi \right| \leq \frac{K}{\sqrt{\log x}}, \; l = 1, 2 \right\}.
\]
Furthermore set
\[ A_K := [-\pi, \pi]^2 \setminus \bigcup_{s_1=0}^{q_1-2} \bigcup_{s_2=0}^{q_2-2} C_K(s_1, s_2). \]

By Proposition 3 for every \( \varepsilon > 0 \) there exists \( K = K(\varepsilon) \) such that
\[ \frac{1}{(2\pi)^2} \int_{A_K} |\varphi_x(t_1, t_2)| \, dt_1 \, dt_2 \leq \frac{\varepsilon}{\log x}. \]

Furthermore, we can choose \( K \leq c'(\log \varepsilon)^{1/2} \) (for some constant \( c' > 0 \)). So it remains to consider the integrals
\[ I_K(s_1, s_2) := \frac{1}{(2\pi)^2} \int_{C_K(s_1, s_2)} \left( \frac{1}{x} \sum_{n < x} e^{it_1(s_1(n) - k_1) + it_2(s_2(n) - k_2)} \right) \, dt_1 \, dt_2 \]
\[ = e^{-2\pi i(k_1 s_1/q - t_1 + k_2 s_2/q - t_2)} \int_{C_K(0,0)} \left( \frac{1}{x} \sum_{n < x} e^{it_1'(s_1(n) - k_1) + it_2'(s_2(n) - k_2)} \right) e^{2\pi i((s_1/q - t_1) + (s_2/q - t_2))} \, dt_1' \, dt_2'. \]

By Lemma 7 it is easy to evaluate \( I_K(0,0) \) asymptotically. For sufficiently large \( x \geq x_0(\varepsilon) \) we have
\[ |\varphi_x(t_1, t_2) - e^{i(t_1 M_1(x) + t_2 M_2(x)) - \frac{1}{2}(t_1^2 D_1^2(x) + t_2^2 D_2^2(x))}| < \varepsilon \]
for all real \( t_1, t_2 \), and consequently
\[ I_K(0,0) \]
\[ = \frac{1}{(2\pi)^2} \int_{C_K(0,0)} e^{it_1(M_1(x) - k_1) + it_2(M_2(x) - k_2) - \frac{1}{2}(t_1^2 D_1^2(x) + t_2^2 D_2^2(x))} \, dt_1 \, dt_2 \]
\[ + O \left( \frac{\varepsilon K^2}{\log x} \right) \]
\[ = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{it_1(M_1(x) - k_1) + it_2(M_2(x) - k_2) - \frac{1}{2}(t_1^2 D_1^2(x) + t_2^2 D_2^2(x))} \, dt_1 \, dt_2 \]
\[ + O \left( \frac{\varepsilon (-\log \varepsilon)}{\log x} \right) \]
\[ = 2 \prod_{t=1}^{2} \left( \frac{1}{\sqrt{2\pi D^2_{q_t}(x)}} \exp \left( -\frac{(k_t - M_{q_t}(x))^2}{2D^2_{q_t}(x)} \right) \right) + O \left( \frac{\varepsilon (-\log \varepsilon)}{\log x} \right). \]

In order to treat the remaining integrals \( I_K(s_1, s_2) \) we recall that \( d \) and \( A \) denote \( d = \gcd(q_1 - 1, q_2 - 1) \) and \( A = \text{lcm}(q_1 - 1, q_2 - 1) \). We represent
Joint distribution of $q$-additive functions

$s_1, s_2$ by
\[ s_l = m_l \frac{q_l - 1}{d} + r_l \quad (0 \leq m_l < d, \ 0 \leq r_l < (q_l - 1)/d, \ l = 1, 2) \]
and observe that
\[ \frac{s_1}{q_1 - 1} + \frac{s_2}{q_2 - 1} = \frac{m_1 + m_2}{d} + \frac{r_1}{q_1 - 1} + \frac{r_2}{q_2 - 1} = \frac{m_1 + m_2}{d} + \frac{r_1 \frac{q_2 - 1}{d}}{A} + \frac{r_2 \frac{q_1 - 1}{d}}{A}. \]
Thus, \( \zeta := e^{2\pi i \left( \frac{s_1}{q_1 - 1} + \frac{s_2}{q_2 - 1} \right)} \) is always an $A$th root of unity and \( \zeta = 1 \) if and only if
\[ (5.3) \quad m_1 + m_2 = d, \quad r_1 = 0 \text{ and } r_2 = 0. \]
Thus, if (5.3) is satisfied, i.e., \( s_1 = m_1 \frac{q_1 - 1}{d} \) and \( s_2 = (d - m_1) \frac{q_2 - 1}{d} \), we have (recall that \( k_1 \equiv k_2 \mod d \))
\[ I_K(s_1, s_2) = e^{-2\pi i \frac{m_1}{d}(k_1 - k_2)} I_K(0, 0) = I_K(0, 0). \]
Hence
\[ \sum_{m=0}^{d-1} I_K \left( m_1 \frac{q_1 - 1}{d}, (d - m_1) \frac{q_2 - 1}{d} \right) = dI_K(0, 0) \]
which fits (by (5.2)) the asymptotic leading term of \( a_{k_1 k_2} \).

Finally we have to consider the case where
\[ \zeta = e^{2\pi i \left( \frac{s_1}{q_1 - 1} + \frac{s_2}{q_2 - 1} \right)} \neq 1. \]
Here we have
\[ I_K(s_1, s_2) = e^{-2\pi i (k_1 \frac{s_1}{q_1 - 1} + k_2 \frac{s_2}{q_2 - 1})} \]
\[ \times \sum_{B=0}^{A-1} \zeta^B \int_{C_K(0,0)} \left( \frac{1}{x} \sum_{n' < (x-B)/A} e^{it_1'(s_{q_1}(An'+B)-k_1) + it_2'(s_{q_2}(An'+B)-k_2)} \right) dt_1' dt_2'. \]
As above, it follows by Lemma 7 that for sufficiently large $x \geq x_1(\varepsilon)$ (and of course uniformly for all $B = 0, 1, \ldots, A - 1$)
\[ \int_{C_K(0,0)} \left( \frac{1}{x} \sum_{n' < (x-B)/A} e^{it_1'(s_{q_1}(An'+B)-k_1) + it_2'(s_{q_2}(An'+B)-k_2)} \right) dt_1' dt_2' \]
\[ = \frac{1}{A} \prod_{l=1}^{2} \left( \frac{1}{\sqrt{2\pi D_{q_l}(x)}} \exp \left( - \frac{(k_l - M_{q_l}(x))^2}{2D_{q_l}^2(x)} \right) \right) + O \left( \frac{\varepsilon \log(-\varepsilon)}{\log x} \right). \]
Thus
\[ I_K(s_1, s_2) = O \left( \frac{\varepsilon (-\log \varepsilon)}{\log x} \right). \]
This completes the proof of Theorem 4. ■
Acknowledgements. The author is indebted to Cecile Dartyge for pointing out the possible use of [1] to describe the joint distribution of \( q \)-additive functions. This hint was the key to all major results of this paper. The author also wants to thank Adolf J. Hildebrand for several discussions on this topic.

References


Department of Geometry
Technische Universität Wien
Wiedner Hauptstraße 8-10/113
A-1040 Wien, Austria
E-mail: michael.drmota@tuwien.ac.at

Received on 18.2.2000
and in revised form on 1.2.2001