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Multiplicative functions with difference tending to zero

by

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1. Introduction. Denote by Δ and Q the operators on arithmetic functions defined by

$$\Delta f(n) := f(n+1) - f(n), \qquad Qf(n) := f(n+1)/f(n).$$

There are two objectives to the present note. The first is related to a result of the first named author, published jointly with Tao Yuan-Sheng and Shao Pin-Tsung [3]:

THEOREM 1. *If an additive function* $f : \mathbb{N} \to \mathbb{R}/\mathbb{Z}$ *has the property* $\Delta f(n) \to 0$ *as* $n \to \infty$, *then*

$$f(n) = c\log n + \mathbb{Z} \qquad \text{with some constant } c \in \mathbb{R}.$$

We will show that the proof of this theorem as given there, although already quite short, can still be shortened and made clearer by a certain rearrangement of the main arguments.

Theorem 1 has an obvious translation to the multiplicative setting: If a multiplicative function $f : \mathbb{N} \to \mathbb{C}$ with modulus 1 has the property $Qf(n) \to 1$, then it is of the form n^s with s a purely imaginary complex number. In fact this statement remains true even if we drop the condition of unimodularity, except that now of course the exponent can be arbitrary:

THEOREM 2. *If* $f : \mathbb{N} \to \mathbb{C}$ *is multiplicative and* $Qf(n) \to 1$ *as* $n \to \infty$, *then* $f(n) = n^s$ *with some* $s \in \mathbb{C}$.

On the other hand, for $f : \mathbb{N} \to \mathbb{C}$ of modulus 1 the conditions "$Qf(n) \to 1$" and "$\Delta f(n) \to 0$" are equivalent, so a different strengthening of Theorem 1 is given by the following result, which is our second main objective:

THEOREM 3. *If* $f : \mathbb{N} \to \mathbb{C}$ *is multiplicative and* $\Delta f(n) \to 0$, *then either* $f(n) = n^s$ *with* $s \in \mathbb{C}$, $0 \le \operatorname{Re} s < 1$, *or else* $f(n) \to 0$ *as* $n \to \infty$.

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Theorem 3, which confirms an older conjecture of Kátai, was stated as a consequence of Theorem 1 by the first named author in a letter to Kátai in 1984, and is quoted and applied as "a result of Wirsing from 1984" in a paper of Kátai and Phong from 1996 [1]. Unfortunately it was never published. We shall supply here a proof of Theorem 3 via Theorem 2.

The results of this paper extend—and use—a well known theorem of Erdős on additive functions, of which we append a short proof for the reader's convenience.

2. Proof of Theorem 1. We denote by $\|\cdot\|$ the norm in $\mathbb{R}/\mathbb{Z}$, defined by $\|\kappa x\| = |x - x'|$, where κ is the canonical mapping from $\mathbb{R}$ to $\mathbb{R}/\mathbb{Z}$ and x' the integer nearest to x.

I. There is a function $F : \mathbb{N} \to \mathbb{R}$ such that $\kappa \circ F = f$ and $|\Delta F(n)| = \|\Delta f(n)\|$. Just choose each $F(n+1)$ from $(F(n) - 1/2, F(n) + 1/2]$. Consequently, $\Delta F(n) \to 0$.

II. In terms of F the additivity of f is expressed by stating that

$$\gamma(a,b) := F(ab) - F(a) - F(b) \quad \text{is in } \mathbb{Z} \text{ if } (a,b) = 1.$$

III. For given a and bounded gaps between n, n' we have $\gamma(a,n') - \gamma(a,n) = F(an') - F(an) - F(n') + F(n) = o(1)$. Thus the subsequence of the (integral!) $\gamma(a,n)$ with $(a,n) = 1$ stabilizes to some integer $\delta(a)$, and the whole sequence converges, i.e.

$$\lim_{n \in \mathbb{N}} \gamma(a,n) = \delta(a) \in \mathbb{Z}.$$

IV. Consider the easily checked identity:

$$\gamma(a,b) = \gamma(a,bc) + \gamma(b,c) - \gamma(ab,c).$$

If we send c to ∞, then in view of **III** we obtain

$$\gamma(a,b) = \delta(a) + \delta(b) - \delta(ab), \tag{1}$$

and here $b \to \infty$ yields $\delta(ab) - \delta(b) \to 0$, that is,

$$\delta(ab) = \delta(b) \quad \text{for all } b \geq n_a,\ n_a \text{ suitable.} \tag{2}$$

The first of these relations is best expressed if we introduce the new function

$$G(n) := F(n) + \delta(n).$$

Then (1) states that G is completely additive.

V. Let us look at (2). In particular it implies that if $b \geq n_2$ then $\delta(b) = \delta(2b)$. Now also $2b \geq n_2$ etc., hence $\delta(2b) = \delta(4b)$ etc., $\delta(b) = \delta(2^k b)$ for all k. Furthermore for all large k (as soon as $2^k \geq n_b$) another application of (2) gives $\delta(b) = \delta(2^k)$. Since this is independent of b we have:

The function δ is constant from some point ($= n_2$) on.

VI. From this and **I** we obtain $\Delta G(n) \to 0$ for the (completely) additive function G. Then by Erdős's Theorem (cf. §4) it follows that $G(n) = c \log n$ for some constant $c \in \mathbb{R}$ and finally, since $\delta(n) \in \mathbb{Z}$ and $f = \kappa \circ F = \kappa \circ G$,

$$f(n) = c \log n + \mathbb{Z}. \blacksquare$$

3. Proof of Theorems 2 and 3

Proof of Theorem 2. If we write $f(n) = |f(n)|\, e^{2\pi i g(n)}$ then under the given assumptions $\log|f| : \mathbb{N} \to \mathbb{R}$ and $g = (2\pi)^{-1} \arg f : \mathbb{N} \to \mathbb{R}/\mathbb{Z}$ are additive and $\Delta \log|f(n)| = \log|Qf(n)| \to \log 1 = 0$, $\Delta g(n) = (2\pi)^{-1} \arg Qf(n) \to (2\pi)^{-1} \arg 1 = 0$, so $\log|f(n)| = \sigma \log n$ and $g(n) = \tau(2\pi)^{-1} \log n + \mathbb{Z}$ by Erdős's Theorem and Theorem 1 respectively. Thus, as claimed, $f(n) = n^{\sigma + i\tau}$. $\blacksquare$

Proof of Theorem 3. Note that $\Delta f(n) \to 0$ implies $Qf(n) \to 1$, provided $|f(n)|$ is bounded below by some positive constant μ:

$$|Qf(n) - 1| = \left| \frac{\Delta f(n)}{f(n)} \right| \le \frac{|\Delta f(n)|}{\mu}.$$

So in this case Theorem 2 applies and gives $f(n) = n^s$. Obviously $\operatorname{Re} s < 1$, since $\Delta f(n) \to 0$ implies $f(n) = o(n)$.

It remains to show $f(n) \to 0$ if f is not bounded in this way. In fact a weaker assumption suffices and Theorem 3 will follow immediately from

LEMMA. *If $f : \mathbb{N} \to \mathbb{C}$ is multiplicative, $\Delta f(n) \to 0$, and there is an $a \in \mathbb{N}$ such that $|f(a)| < 1$, then $f(n) \to 0$ as $n \to \infty$.*

Proof. Let a be a fixed number such that $|f(a)| =: q < 1$. By assumption, we have $|\Delta f(n)| \le \varepsilon / a^2$ for $n \ge n_0(\varepsilon)$. To each $n \in \mathbb{N}$ we attach a sequence $n, n', n'', \ldots, n^{(k)}$ by the modified division algorithm $n^{(i-1)} = an^{(i)} + r_i$ that allows $0 \le r_i < a^2$ but requires that a and $n^{(i)}$ be coprime. We stop at the first index k for which $n^{(k)} < n_0(\varepsilon) + a^2$. Breaking the gaps up into r_i steps of length 1 we see $|f(n^{(i-1)}) - f(a) f(n^{(i)})| < \varepsilon$ for $i = 1, \ldots, k$. Thus

$$|f(n^{(i-1)})| < q|f(n^{(i)})| + \varepsilon.$$

The total result from these inequalities is

$$\begin{aligned} |f(n)| &< \varepsilon + q(\varepsilon + q(\varepsilon + \ldots + q(\varepsilon + q|f(n^{(k)})|) \ldots)) \\ &< \varepsilon(1-q)^{-1} + q^k |f(n^{(k)})|. \end{aligned}$$

Since ε is arbitrary, $f(n^{(k)})$ stays bounded once ε is fixed, and k tends to infinity as n does, we see that indeed $f(n) \to 0$. $\blacksquare$ $\blacksquare$

4. Appendix: Erdős's Theorem

THEOREM 4 (Erdős, 1946). *If an additive function* $f : \mathbb{N} \to \mathbb{R}$ *has the property* $\Delta f(n) \to 0$ *as* $n \to \infty$, *then*

$$f(n) = c \log n \quad \textit{with some constant } c \in \mathbb{R}.$$

The following proof is taken from [2]. For convenience, assume first that f is completely additive, which is all that is needed for our application. Choose an integer $g \geq 2$ and attach to each $n \in \mathbb{N}$ a sequence $n, n', n'', \ldots, n^{(k)}$ by the g-adic division algorithm $n^{(i-1)} = gn^{(i)} + r_i$ where $0 \leq r_i < g$, which terminates when $n^{(k)} < g$. Breaking the gaps up into steps of length 1 we see $|f(n^{(i-1)}) - f(n^{(i)}) - f(g)| < \varepsilon$, for any ε, as long as $n^{(i)}$ is sufficiently large, and bounded for the remaining i. The total result from these inequalities is

$$|f(n) - kf(g)| < k\varepsilon + O_\varepsilon(1),$$

in other words,

$$f(n) = kf(g) + o(k).$$

Since $k \sim \log n / \log g$ as $n \to \infty$ we have

$$f(n) = \frac{\log n}{\log g} f(g) + o(\log n).$$

The very first impression is disappointment: An asymptotic relation rather than the expected identity? But the asymptotic behavior is independent of the choice of g. Therefore $f(g)/\log g$ is a constant! The same proof works with restricted additivity if one uses the modified division algorithm as in the lemma. ■

References

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