Corrigendum to the paper “Limit theorems for the Mellin transform of the square of the Riemann zeta-function. I”


by

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Let

\[ Z_1(s) = \int_1^\infty |\zeta(1/2 + ix)|^2 x^{-s} \, dx, \quad s = \sigma + it, \]

denote the modified Mellin transform of \(|\zeta(1/2 + ix)|^2\), where \(\zeta(s)\) is the Riemann zeta-function.

In [4], a limit theorem in the sense of weak convergence of probability measures on the complex plane for the function \(Z_1(s)\) has been considered. Denote by \(\text{meas}\{A\}\) the Lebesgue measure of a measurable set \(A \subset \mathbb{R}\), and let, for \(T > 0\),

\[ \nu_T(\ldots) = \frac{1}{T} \text{meas}\{t \in [0,T] : \ldots\}, \]

where in place of the dots a condition satisfied by \(t\) is to be written. Denote by \(\mathcal{B}(S)\) the class of Borel subsets of the space \(S\). Then the main result of [4] is the following statement.

**Theorem 1.** Let \(\sigma > 1/2\). Then on \((\mathbb{C}, \mathcal{B}(\mathbb{C}))\) there exists a probability measure \(P_\sigma\) such that the probability measure

\[ \nu_T(Z_1(\sigma + it) \in A), \quad A \in \mathcal{B}(\mathbb{C}), \]

converges weakly to \(P_\sigma\) as \(T \to \infty\).

However, the proof of Theorem [1] uses Theorem 2 of [4] whose proof is not correct. Let \(a > 1\), \(g(x)\) be an integrable function on \([1, a]\), and

\[ Z_{g, a}(s) = \int_1^a g(x)x^{-s} \, dx. \]

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Then Theorem 2 of [4] states that there exists a probability measure $P_{\sigma,a}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that the probability measure
\[
\nu_T(Z_{g,a}(\sigma + it) \in A), \quad A \in \mathcal{B}(\mathbb{C}),
\]
converges weakly to $P_{\sigma,a}$ as $T \to \infty$.

Denote by $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ the unit circle on the complex plane, and define
\[
\Omega_a = \prod_{u \in [1,a]} \gamma_u,
\]
where $\gamma_u = \gamma$ for each $u \in [1,a]$. With the product topology and pointwise multiplication $\Omega_a$ is a compact topological Abelian group. By Lemma 3 of [4], on $(\Omega_a, \mathcal{B}(\Omega_a))$ there exists a probability measure $Q_a$ such that the probability measure
\[
Q_T,A(a) = \nu_T(\{u^{-it} : u \in [1,a]\} \in A), \quad A \in \mathcal{B}(\Omega_a),
\]
converges weakly to $Q_a$ as $T \to \infty$.

For the proof of Theorem 2, Lemma 3 of [4] and Theorem 5.1 of [3] are applied. However, the function $h : \Omega_a \to \mathbb{C}$ defined by the formula
\[
h(y_x) = \int_1^a g(x)x^{-\sigma}y_x^{-1} \, dx, \quad y_x \in \Omega_a,
\]
and used in the proof of Theorem 2 is not continuous. This gap can be corrected as follows. We start with a limit theorem for an integral sum. Divide the interval $[1,a]$ by points $1 = x_0 < x_1 < \cdots < x_n = a$ into $n$ subintervals of the same length $(a - 1)/n$. In each interval $[x_{j-1},x_j]$, we take a point $\xi_j$ and define the sum
\[
S_{n,a}(s) = \sum_{j=1}^n g(\xi_j)\xi_j^{-s} \Delta x_j
\]
with $\Delta x_j = x_j - x_{j-1}$. Define
\[
Q_T,n,\sigma,a(A) = \nu_T(S_{n,a}(\sigma + it) \in A), \quad A \in \mathcal{B}(\mathbb{C}).
\]

**Proposition 2.** On $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure $Q_{n,\sigma,a}$ such that the probability measure $Q_{T,n,\sigma,a}$ converges weakly to $Q_{n,\sigma,a}$ as $T \to \infty$.

**Proof.** Let the function $h_{n,\sigma,a} : \Omega_a \to \mathbb{C}$ be given by the formula
\[
h_{n,\sigma,a}(f) = \sum_{j=1}^n g(\xi_j)\xi_j^{-\sigma}f\xi_j \Delta x_j, \quad f \in \Omega_a.
\]
Since in the definition of $h_{n,\sigma,a}(f)$ only a finite number of values of $f$ occur,
the function $h_{n,\sigma,a}$ is continuous in the product topology. Moreover,

$$h_{n,\sigma,a}(x^{-it}) = \sum_{j=1}^{n} g(\xi_j) \xi_j^{-\sigma-it} \Delta x_j = S_{n,a}(\sigma + it).$$

Hence from Theorem 5.1 of [3] and the weak convergence of the measure $Q_{T,a}$ we find that the probability measure $Q_{T,n,\sigma,a} = Q_{T,a}h_{n,\sigma,a}^{-1}$ converges weakly to the measure $Q_{a}h_{n,\sigma,a}^{-1}$ as $T \to \infty$. $lacksquare$

**Proposition 3.** We have the equality

$$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} |S_{n,a}(\sigma + it) - Z_{g,a}(\sigma + it)| dt = 0.$$

**Proof.** By the Cauchy–Schwarz inequality,

(1) $$\frac{1}{T} \int_{0}^{T} |S_{n,a}(\sigma + it) - Z_{g,a}(\sigma + it)| dt \ll \left( \frac{1}{T} \int_{0}^{T} |S_{n,a}(\sigma + it) - Z_{g,a}(\sigma + it)|^2 dt \right)^{1/2}.$$

It is easily seen that

(2) $$\lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} S_{n,a}(\sigma + it) \overline{S_{n,a}(\sigma + it)} dt = 0$$

and

(3) $$\limsup_{T \to \infty} \frac{1}{T} \int_{0}^{T} Z_{g,a}(\sigma + it) \overline{Z_{g,a}(\sigma + it)} dt = 0.$$

Since

(4) $$\frac{1}{T} \int_{0}^{T} S_{n,a}(\sigma + it) \overline{Z_{g,a}(\sigma + it)} dt \ll \left( \frac{1}{T} \int_{0}^{T} |S_{n,a}(\sigma + it)|^2 dt \right)^{1/2} \left( \frac{1}{T} \int_{0}^{T} |Z_{g,a}(\sigma + it)|^2 dt \right)^{1/2},$$

and the same estimate is true for the mean value of $\overline{S_{n,a}(\sigma + it)} Z_{g,a}(\sigma + it)$, the proposition follows from (1)–(4). $lacksquare$

**Proof of Theorem 2 of [4].** Let $\theta_T$ be the random variable defined in the proof of Theorem 5 of [4]. Define

$$U_{T,n,a}(\sigma) = S_{n,a}(\sigma + i\theta_T).$$
Then Proposition 2 implies the relation

$$\lim_{T \to \infty} P\left(|U_{T,n,a}(\sigma)| > M\right) \leq \frac{1}{TM} \int_0^T |S_{n,a}(\sigma + it)| \, dt,$$

where $\theta_T$ is defined on the probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$. This together with Proposition 3 shows that

$$\limsup_{T \to \infty} P\left(|U_{n,a}(\sigma)| > M\right) \leq \sup_{n \in \mathbb{N}} \limsup_{T \to \infty} \frac{1}{TM} \int_0^T |S_{n,a}(\sigma + it)| \, dt \leq \frac{R_{\sigma,a}}{M},$$

with $R_{\sigma,a} < \infty$. Let $\varepsilon$ be an arbitrary positive number. Then, taking $M = M_\varepsilon = R_{\sigma,a}\varepsilon^{-1}$, we deduce from (5) and (6) that, for all $n \in \mathbb{N}$,

$$\mathbb{P}(|U_{n,a}(\sigma)| > M_\varepsilon) \leq \varepsilon.$$

Define $K_\varepsilon = \{s \in \mathbb{C} : |s| \leq M_\varepsilon\}$. Then $K_\varepsilon$ is a compact subset of $\mathbb{C}$, and, in view of (7),

$$\mathbb{P}(U_{n,a}(\sigma) \in K_\varepsilon) \geq 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$,

$$Q_{n,\sigma,a}(K_\varepsilon) \geq 1 - \varepsilon.$$

Thus, we proved that the family $\{Q_{n,\sigma,a} : n \in \mathbb{N}\}$ is tight. Hence, by the Prokhorov theorem (see, for example, [3]), it is relatively compact. Therefore, there exists a sequence $\{Q_{n_k,\sigma,a}\} \subset \{Q_{n,\sigma,a}\}$ such that $Q_{n_k,\sigma,a}$ converges weakly to a probability measure $Q_{\sigma,a}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $k \to \infty$. Thus,

$$U_{n_k,a}(\sigma) \xrightarrow{D} Q_{\sigma,a}.$$
Define
\[ X_{T,a} = Z_{g,a}(\sigma + i\theta_T). \]
Then, taking into account Proposition \[3\], we find that, for every \( \varepsilon > 0 \),
\[
\lim_{n \to \infty} \limsup_{T \to \infty} \mathbb{P}(\left| X_{T,a}(\sigma) - U_{T,n,a}(\sigma) \right| \geq \varepsilon)
\leq \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T\varepsilon} \int_{0}^{T} |S_{n,a}(\sigma + it) - Z_{g,a}(\sigma + it)| \, dt = 0.
\]
This, (5), (8) and Theorem 4.2 of \[3\] show that
\[ X_{T,a}(\sigma) \xrightarrow{D} T \to \infty Q_{\sigma,a}. \]
This is equivalent to the assertion of Theorem 2 from \[4\].
The further part of \[4\] remains without changes.
Analogous corrections must be made in the proof of Theorem 6 of \[1\], and in \[2\], \[5\], \[6\].

References


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