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Corrigendum to the paper "Limit theorems for the Mellin transform of the square of the Riemann zeta-function. I"

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by

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Let

$$\mathcal{Z}_1(s) = \int_1^\infty |\zeta(1/2+ix)|^2 x^{-s}\,dx, \qquad s = \sigma + it,$$

denote the modified Mellin transform of $|\zeta(1/2+ix)|^2$, where $\zeta(s)$ is the Riemann zeta-function.

In [4], a limit theorem in the sense of weak convergence of probability measures on the complex plane for the function $\mathcal{Z}_1(s)$ has been considered. Denote by $\operatorname{meas}\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}$, and let, for $T > 0$,

$$\nu_T(\ldots) = \frac{1}{T}\operatorname{meas}\{t \in [0,T] : \ldots\},$$

where in place of the dots a condition satisfied by t is to be written. Denote by $\mathcal{B}(S)$ the class of Borel subsets of the space S. Then the main result of [4] is the following statement.

Theorem 1. *Let $\sigma > 1/2$. Then on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ there exists a probability measure P_σ such that the probability measure*

$$\nu_T(\mathcal{Z}_1(\sigma+it) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to P_σ as $T \to \infty$.

However, the proof of Theorem 1 uses Theorem 2 of [4] whose proof is not correct. Let $a > 1$, $g(x)$ be an integrable function on $[1,a]$, and

$$\mathcal{Z}_{g,a}(s) = \int_1^a g(x)x^{-s}\,dx.$$

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Then Theorem 2 of [4] states that there exists a probability measure $P_{\sigma,a}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ such that the probability measure

$$\nu_T(\mathcal{Z}_{g,a}(\sigma+it) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to $P_{\sigma,a}$ as $T \to \infty$.

Denote by $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ the unit circle on the complex plane, and define

$$\Omega_a = \prod_{u \in [1,a]} \gamma_u,$$

where $\gamma_u = \gamma$ for each $u \in [1,a]$. With the product topology and pointwise multiplication Ω_a is a compact topological Abelian group. By Lemma 3 of [4], on $(\Omega_a, \mathcal{B}(\Omega_a))$ there exists a probability measure Q_a such that the probability measure

$$Q_{T,a}(A) = \nu_T(\{u^{-it} : u \in [1,a]\} \in A), \quad A \in \mathcal{B}(\Omega_a),$$

converges weakly to Q_a as $T \to \infty$.

For the proof of Theorem 2, Lemma 3 of [4] and Theorem 5.1 of [3] are applied. However, the function $h : \Omega_a \to \mathbb{C}$ defined by the formula

$$h(y_x) = \int_1^a g(x) x^{-\sigma} y_x^{-1} \, dx, \quad y_x \in \Omega_a,$$

and used in the proof of Theorem 2 is not continuous.

This gap can be corrected as follows. We start with a limit theorem for an integral sum. Divide the interval $[1,a]$ by points $1 = x_0 < x_1 < \cdots < x_n = a$ into n subintervals of the same length $(a-1)/n$. In each interval $[x_{j-1}, x_j]$, we take a point ξ_j and define the sum

$$S_{n,a}(s) = \sum_{j=1}^{n} g(\xi_j) \xi_j^{-s} \Delta x_j$$

with $\Delta x_j = x_j - x_{j-1}$. Define

$$Q_{T,n,\sigma,a}(A) = \nu_T(S_{n,a}(\sigma+it) \in A), \qquad A \in \mathcal{B}(\mathbb{C}).$$

Proposition 2. *On $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, there exists a probability measure $Q_{n,\sigma,a}$ such that the probability measure $Q_{T,n,\sigma,a}$ converges weakly to $Q_{n,\sigma,a}$ as $T \to \infty$.*

Proof. Let the function $h_{n,\sigma,a} : \Omega_a \to \mathbb{C}$ be given by the formula

$$h_{n,\sigma,a}(f) = \sum_{j=1}^{n} g(\xi_j) \xi_j^{-\sigma} f_{\xi_j} \Delta x_j, \qquad f \in \Omega_a.$$

Since in the definition of $h_{n,\sigma,a}(f)$ only a finite number of values of f occur, the function $h_{n,\sigma,a}$ is continuous in the product topology. Moreover,

$$h_{n,\sigma,a}(x^{-it}) = \sum_{j=1}^{n} g(\xi_j)\xi_j^{-\sigma-it}\Delta x_j = S_{n,a}(\sigma+it).$$

Hence from Theorem 5.1 of [3] and the weak convergence of the measure $Q_{T,a}$ we find that the probability measure $Q_{T,n,\sigma,a} = Q_{T,a}h_{n,\sigma,a}^{-1}$ converges weakly to the measure $Q_a h_{n,\sigma,a}^{-1}$ as $T\to\infty$. ∎

PROPOSITION 3. *We have the equality*

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T |S_{n,a}(\sigma+it)-\mathcal{Z}_{g,a}(\sigma+it)|\,dt = 0.$$

Proof. By the Cauchy–Schwarz inequality,

$$\begin{aligned}(1)\qquad &\frac{1}{T}\int_0^T |S_{n,a}(\sigma+it)-\mathcal{Z}_{g,a}(\sigma+it)|\,dt\\ &\ll \Big(\frac{1}{T}\int_0^T |S_{n,a}(\sigma+it)-\mathcal{Z}_{g,a}(\sigma+it)|^2\,dt\Big)^{1/2}.\end{aligned}$$

It is easily seen that

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T S_{n,a}(\sigma+it)\,\overline{S_{n,a}(\sigma+it)}\,dt = 0 \tag{2}$$

and

$$\limsup_{T\to\infty}\frac{1}{T}\int_0^T \mathcal{Z}_{g,a}(\sigma+it)\,\overline{\mathcal{Z}_{g,a}(\sigma+it)}\,dt = 0. \tag{3}$$

Since

$$\begin{aligned}(4)\qquad &\frac{1}{T}\int_0^T S_{n,a}(\sigma+it)\,\overline{\mathcal{Z}_{g,a}(\sigma+it)}\,dt\\ &\ll \Big(\frac{1}{T}\int_0^T |S_{n,a}(\sigma+it)|^2\,dt\Big)^{1/2}\Big(\frac{1}{T}\int_0^T |\mathcal{Z}_{g,a}(\sigma+it)|^2\,dt\Big)^{1/2},\end{aligned}$$

and the same estimate is true for the mean value of $\overline{S_{n,a}(\sigma+it)}\,\mathcal{Z}_{g,a}(\sigma+it)$, the proposition follows from (1)–(4). ∎

Proof of Theorem 2 of [4]. Let θ_T be the random variable defined in the proof of Theorem 5 of [4]. Define

$$U_{T,n,a}(\sigma) = S_{n,a}(\sigma+i\theta_T).$$

Then Proposition 2 implies the relation

$$U_{T,n,a}(\sigma) \xrightarrow[T\to\infty]{\mathcal{D}} U_{n,a}(\sigma), \tag{5}$$

where $U_{n,a}(\sigma)$ is a complex-valued random variable having the distribution $Q_{n,\sigma,a}$, and $\xrightarrow{\mathcal{D}}$ means convergence in distribution. We will show that the family $\{Q_{n,\sigma,a} : n \in \mathbb{N}\}$ of probability measures is tight.

We take an arbitrary $M > 0$. Then, by the Chebyshev inequality,

$$\mathbb{P}(|U_{T,n,a}(\sigma)| > M) \le \frac{1}{TM}\int_0^T |S_{n,a}(\sigma+it)|\,dt,$$

where θ_T is defined on the probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$. This together with Proposition 3 shows that

$$\begin{aligned}
\limsup_{T\to\infty} \mathbb{P}(|U_{T,n,a}(\sigma)| > M) &\le \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{TM}\int_0^T |S_{n,a}(\sigma+it)|\,dt \\
&\le \sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \frac{1}{TM}\int_0^T |S_{n,a}(\sigma+it) - \mathcal{Z}_{g,a}(\sigma+it)|\,dt \\
&\quad + \limsup_{T\to\infty} \frac{1}{TM}\int_0^T |\mathcal{Z}_{g,a}(\sigma+it)|\,dt \\
&\le \frac{1}{M} + \limsup_{T\to\infty} \frac{1}{TM}\int_0^T |\mathcal{Z}_{g,a}(\sigma+it)|\,dt \le \frac{R_{\sigma,a}}{M}
\end{aligned} \tag{6}$$

with $R_{\sigma,a} < \infty$. Let ε be an arbitrary positive number. Then, taking $M = M_\varepsilon = R_{\sigma,a}\varepsilon^{-1}$, we deduce from (5) and (6) that, for all $n \in \mathbb{N}$,

$$\mathbb{P}(|U_{n,a}(\sigma)| > M_\varepsilon) \le \varepsilon. \tag{7}$$

Define $K_\varepsilon = \{s \in \mathbb{C} : |s| \le M_\varepsilon\}$. Then K_ε is a compact subset of $\mathbb{C}$, and, in view of (7),

$$\mathbb{P}(U_{n,a}(\sigma) \in K_\varepsilon) \ge 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$,

$$Q_{n,\sigma,a}(K_\varepsilon) \ge 1 - \varepsilon.$$

Thus, we proved that the family $\{Q_{n,\sigma,a} : n \in \mathbb{N}\}$ is tight. Hence, by the Prokhorov theorem (see, for example, [3]), it is relatively compact. Therefore, there exists a sequence $\{Q_{n_k,\sigma,a}\} \subset \{Q_{n,\sigma,a}\}$ such that $Q_{n_k,\sigma,a}$ converges weakly to a probability measure $Q_{\sigma,a}$ on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ as $k \to \infty$. Thus,

$$U_{n_k,a}(\sigma) \xrightarrow[k\to\infty]{\mathcal{D}} Q_{\sigma,a}. \tag{8}$$

Define

$$X_{T,a} = \mathcal{Z}_{g,a}(\sigma + i\theta_T).$$

Then, taking into account Proposition 3, we find that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{T\to\infty} \mathbb{P}(|X_{T,a}(\sigma) - U_{T,n,a}(\sigma)| \geq \varepsilon)$$
$$\leq \lim_{n\to\infty} \limsup_{T\to\infty} \frac{1}{T\varepsilon} \int_0^T |S_{n,a}(\sigma + it) - \mathcal{Z}_{g,a}(\sigma + it)|\, dt = 0.$$

This, (5), (8) and Theorem 4.2 of [3] show that

$$X_{T,a}(\sigma) \xrightarrow[T\to\infty]{\mathcal{D}} Q_{\sigma,a}.$$

This is equivalent to the assertion of Theorem 2 from [4].

The further part of [4] remains without changes.

Analogous corrections must be made in the proof of Theorem 6 of [1], and in [2], [5], [6].

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