On the structure of compact subsets of $\mathbb{C}_p$

by

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Introduction. Let $\mathbb{Q}$ be the rational number field and let $p$ be a fixed prime integer. Let $v_p$ be the $p$-adic valuation on $\mathbb{Q}$ and let $\mathbb{Q}_p$ be the $p$-adic number field, i.e. the completion of $\mathbb{Q}$ with respect to $v_p$. Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ and let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\overline{\mathbb{Q}}_p$. Let $\overline{v}_p$ be the unique extension of $v_p$ to $\overline{\mathbb{Q}}_p$ and let $v$ be the restriction of $\overline{v}_p$ to $\overline{\mathbb{Q}}$. Let $G = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $G_p = \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Set $K_p = \overline{\mathbb{Q}} \cap \mathbb{Q}_p$ and $G_p' = \text{Gal}(\overline{\mathbb{Q}}/K_p)$. Since the restriction map from $G_p$ to $G_p'$ is injective and surjective ($\overline{\mathbb{Q}}$ is dense in $\overline{\mathbb{Q}}_p$) we can view $G_p$ as a subgroup of $G$. Here we used the fact that $v(\sigma(x)) = v(x)$ for every $x$ in $\overline{\mathbb{Q}}$ and for every $\sigma \in G_p$ ($\mathbb{Q}_p$ is a Henselian field).

For any subfield $L$ of $\overline{\mathbb{Q}}$ we denote by $\widetilde{L}$ the completion of $L$ with respect to the $p$-adic spectral norm

$$\|x\|_p = \max\{|\sigma(x)|_p | \sigma \in G\}$$

where $| \cdot |_p$ is the corresponding absolute value of $v$ (see also [P1], [PN], [PPV], [PPZ1]–[PPZ5]).

Denote by $\widetilde{\mathbb{Q}}_p$ the completion of $(\overline{\mathbb{Q}}, \| \cdot \|_p)$; we shall continue to use the same notation $\| \cdot \|_p$ for the unique extension of $\| \cdot \|_p$ to $\widetilde{\mathbb{Q}}_p$. This last completion is a regular commutative ring (a von Neumann regular ring). It has many other interesting properties (see [PPV]). An element in $\widetilde{\mathbb{Q}}_p$ is a class $\widehat{x}$ of Cauchy sequences, where $x = \{x_n\}_n$, $x_n \in \overline{\mathbb{Q}}$, $n = 1, 2, \ldots$, is a representative of $\widehat{x}$. It is easy to see that if $x = \{x_n\}_n$, $x_n \in \overline{\mathbb{Q}}$, is a Cauchy sequence relative to the $p$-adic spectral norm, then $\{x_n\}_n$ is a Cauchy sequence with

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respect to the absolute value $| \cdot |_{v_0} \sigma, \sigma \in G$, i.e. the sequence $\{ \sigma(x_n) \}_n$ has a limit in $\mathbb{C}_p$, the complex $p$-adic field (the completion of $\mathbb{Q}_p$ relative to $v_p$). Denote this limit by

$$x(\sigma) = \lim_{n \to \infty} \sigma(x_n).$$

We call $x(\sigma)$ the $\sigma$-component of $x$. Let $C(x)$ denote the set of all $\sigma$-components of $x$ and call it the pseudo-orbit of $x$.

Since $\{ \sigma(x_n) \}_n$ is also a Cauchy sequence relative to the $p$-adic spectral norm, we denote by $\sigma(x)$ its limit in $\tilde{\mathbb{Q}}_p$ for any $\sigma$ in $G$. The subset $O(x) = \{ \sigma(x) \mid \sigma \in G \}$ of $\tilde{\mathbb{Q}}_p$ is said to be the orbit of $x$ in $\tilde{\mathbb{Q}}_p$. By $(\sigma, x) \mapsto \sigma(x)$, $G$ acts continuously on $\tilde{\mathbb{Q}}_p$ if we consider the Krull topology on $G$ (see [PPV]). The same is true for the mapping $(\sigma, z) \mapsto z(\sigma)$ defined on $G \times \tilde{\mathbb{Q}}_p$ with values in $\mathbb{C}_p$. In general we have a homeomorphism $\sigma(x) \mapsto x(\sigma)$ from the orbit of $x$ onto the pseudo-orbit of the same $x$.

Three main results are proved relative to these completions:

1) Any compact subset $M$ of $\mathbb{C}_p$ which is invariant under the group $G_p$ ($= \text{Gal}_{cont}(\mathbb{C}_p/\mathbb{Q}_p)$) is of the form $M = C(x)$, where $x \in \tilde{\mathbb{Q}}_p$ and $C(x)$ is the pseudo-orbit of $x$ (Theorem 2.2).

2) The completion $\tilde{L}$ of a finite or infinite algebraic number field $L$, relative to the $p$-adic spectral norm, is a $\mathbb{Q}_p$-Banach algebra isomorphic to the $\mathbb{Q}_p$-Banach algebra of all the $G_p$-equivariant continuous functions $f : G/H_L \to \mathbb{C}_p$, where $H_L = \text{Fix} L$. Here $f$ is said to be $G_p$-equivariant if $f(\sigma \mu) = \sigma(f(\mu))$ for all $\mu \in G$ and $\sigma \in G_p$ (Theorem 2.4).

3) Any algebraic number field (finite or infinite) has a topological generic element $x$ in $\tilde{\mathbb{Q}}_p$ with respect to the $p$-adic spectral norm, i.e. $\tilde{L} = \mathbb{Q}[x]$ (Theorem 3.1). This result is a version of the “Primitive Element Theorem” for infinite algebraic number fields.

There is a nice connection between the topological generic elements $x \in \tilde{\mathbb{Q}}_p$ of an algebraic number field $L$ and the so-called Cantor compact subsets of $\mathbb{C}_p$ (Remark 2.1, Proposition 2.3, Theorem 3.3 and Theorem 3.4). At the end of the paper we give an explicit computation of a Galois action of $G$ on the compact set $\mathbb{Z}_p$, the $p$-adic integers, and we associate to it an algebraic number field, unique up to $\mathbb{Q}_p$-isomorphism (Section 4).

In a forthcoming paper we shall completely describe the structure of all compact subsets of $\mathbb{C}_p$ in connection with algebraic number fields and spectral norms.
1. Some general results. In this section we use the notations and definitions from the introduction. Now we recall a classical result in valuation theory (see for instance [Neu, pp. 161–167]):

**Theorem 1.1.** Let \( L/K \) be an algebraic extension of fields and let \( v \) be a fixed valuation on \( K \). Let \( K_v \) be the completion of \( K \) with respect to \( v \) and let \( \overline{K}_v \) be an algebraic closure of \( K_v \) which contains \( L \). Let \( \overline{v} \) be the unique extension of \( v \) to \( \overline{K}_v \). Let \( \overline{K} \) be the algebraic closure of \( K \) in \( \overline{K}_v \). Then:

(i) Any extension \( w \) of \( v \) to \( L \) is of the form \( w = \overline{v} \circ \tau \), where \( \tau \) is a \( K \)-embedding of \( L \) into \( \overline{K}_v \).

(ii) If \( \tau \) and \( \tau' \) are two \( K \)-embeddings of \( L \) into \( \overline{K}_v \), then \( \overline{v} \circ \tau = \overline{v} \circ \tau' \) if and only if \( \tau \) and \( \tau' \) are conjugate by a \( K_v \)-automorphism of \( \overline{K}_v \), i.e. \( \tau' = \sigma \circ \tau \) for some \( \sigma \in \text{Gal}(\overline{K}_v/K_v) \). In particular, if \( L/K \) is a Galois extension and if \( H = \text{Gal}(L/K) \), then any extension \( w' \) of \( v \) to \( L \) is of the form \( w' = w \circ \mu \), where \( w \) is a fixed extension of \( v \) to \( L \) and \( \mu \in H \). Moreover, \( w \circ \mu = w \circ \mu' \) for \( \mu, \mu' \in H \) if and only if \( \mu' = \varrho \circ \mu \) for some \( \varrho \in \text{Gal}(\overline{K}_v/K_v) \) = \( \text{Gal}(\overline{K}/\overline{K} \cap K_v) \).

We give here an elementary result which will be useful in the following (see [PPV]).

**Proposition 1.2.** Let \( v \) be the restriction of \( \overline{v}_p \) to \( \overline{Q} \) and let \( \sigma \) be an automorphism of \( G \). Then the following assertions are equivalent:

(i) \( v \) and \( v \circ \sigma \) are equivalent (they induce the same topology on \( \overline{Q} \)).

(ii) \( \sigma \in G_p \).

(iii) \( \sigma \) is a continuous mapping with respect to \( v \).

We need the following result, which partially appears in [PL].

**Proposition 1.3.** There exists a maximal extension \( L^{(p)} \) of \( Q \) in \( \overline{Q} \) such that \( v_p \) has only one extension \( w \) to \( L^{(p)} \) (for any finite extension \( K \) of \( L^{(p)} \), \( w \) has at least two distinct extensions to \( K \) ). This \( L^{(p)} \) is dense in \( C_p \). Moreover, any automorphism \( \mu \) of \( G \) can be uniquely written in the form \( \mu = \sigma \tau \), where \( \sigma \in G_p \) and \( \tau \in \text{Gal}(\overline{Q}/L^{(p)}) \).

**Proof.** According to [PL] we only have to prove the last statement. Since \( L^{(p)} \) is dense in \( \overline{Q}_p \) one can use Krasner’s lemma [Neu] to prove that \( L^{(p)} \overline{Q}_p = \overline{Q}_p \). Hence any embedding \( \lambda \) of \( L^{(p)} \) in \( \overline{Q} \) gives rise to a unique automorphism \( \overline{\lambda} \) of \( G_p = \text{Gal}(\overline{Q}_p/Q_p) \). If we start with a \( \mu \in G \), then \( \mu|_{L^{(p)}} \) is such a \( \lambda \). Hence \( \overline{\lambda}^{-1} \mu \in \text{Gal}(\overline{Q}/L^{(p)}) \). In the end we get \( \mu = \overline{\lambda} \tau \) with \( \overline{\lambda} \in G_p \) and \( \tau = \overline{\lambda}^{-1} \mu \in \text{Gal}(\overline{Q}/L^{(p)}) \). The unicity follows from the equality \( L^{(p)} \overline{Q}_p = \overline{Q}_p \).

**Remark 1.1.** For any natural number \( n \), it is not difficult to construct an algebraic extension \( T \) of \( L^{(p)} \) of degree \( n \) such that the valuation \( w \)
from the above proposition has exactly \( n \) extensions to \( T \). Namely, take an extension \( R \) of \( \mathbb{Q} \) of degree \( n \) such that the valuation \( v_p \) splits completely into \( n \) valuations on \( R \) (see the theorem of Hasse [R]). Then we can consider the compositum \( T = L^{(p)} R \), which is an extension of degree \( n \) over \( L^{(p)} \) and \( w \) splits exactly into \( n \) distinct valuations on \( T \).

2. \( G_p \)-equivariant compact subsets of \( \mathbb{C}_p \). Let \( G_p = \text{Gal}_{\text{cont}}(\mathbb{C}_p/\mathbb{Q}_p) \) denote the group of continuous automorphisms of the \( p \)-adic complex number field \( \mathbb{C}_p \) over \( \mathbb{Q}_p \). A compact subset \( M \) of \( \mathbb{C}_p \) is said to be \( G_p \)-equivariant if \( \sigma(x) \in M \) for any \( \sigma \in G_p \) and \( x \in M \).

**Proposition 2.1.** For any \( x \in \widetilde{\mathbb{Q}}_p \), the pseudo-orbit \( C(x) \) of \( x \) is a \( G_p \)-equivariant compact subset of \( \mathbb{C}_p \). Moreover, \( G_p \) acts continuously on \( C(x) \) by \( \sigma(x(\mu)) = x(\sigma \mu) \).

Let \( M \) be a \( G_p \)-equivariant compact subset of \( \mathbb{C}_p \). For any \( \rho > 0 \) we consider the covering of \( M \) with \( n_{(\rho)} \) disjoint closed balls of radius \( \rho \):

\[
S_{(\rho)} = \{ B[x_{\rho 1}, \rho], \ldots, B[x_{\rho n(\rho)}, \rho] \}
\]

where \( B[x, \rho] = \{ y \in \mathbb{C}_p \mid |x - y|_p \leq \rho \} \) and such that \( x_{\rho j} \in M \) for any \( j = 1, \ldots, n(\rho) \). For any fixed \( \rho \) the balls of \( S_{(\rho)} \) are uniquely determined. Since the mapping \( \rho \mapsto n_{(\rho)} \) has discrete values, the real interval \((0, \infty)\) can be written as a union

\[
(0, \infty) = (\infty, \varepsilon_1] \cup (\varepsilon_1, \varepsilon_2] \cup \cdots \cup (\varepsilon_{n-1}, \varepsilon_n] \cup \cdots
\]

where \( \{ \varepsilon_n \}_n \) is a decreasing sequence and \( \varepsilon_n \to 0 \). We briefly write \( S_n \) instead of \( S_{(\varepsilon_n)} \) and \( n_k \) for \( n_{\varepsilon_k} \). The two sequences \( \{ \varepsilon_k \}_k \) and \( \{ n_k \}_k \) are called the configuration numbers (sequences) of \( M \). They are invariants for \( M \). The set \( M \) is said to be a Cantor compact subset if all the balls from \( S_k \) contain the same number of balls from \( S_{k+1} \).

Let now \( M \) be a \( G_p \)-equivariant compact of \( \mathbb{C}_p \). We shall construct a new compact subset \( N \) of \( M \) and we shall call it a \( p \)-reduction of \( M \). It will be the projective limit of the following projective system of balls. Set \( S'_1 = S_1 \).

Assume we have constructed \( S'_k \). We now define \( S'_{k+1} \) to be a least subset of balls of \( S_{k+1} \) which are contained in \( S'_k \) and such that for any two balls of \( S_{k+1} \) no \( \sigma \) in \( G_p \) carries one ball into the other. Take now \( N = \lim S'_k \). This \( N \) can be obtained as the intersection of a tower of balls \( B'_{1i_1} \supset B'_{2i_2} \supset \cdots \), all of them from the initial configuration of \( M \). Briefly we say that \( N \) is a reduction of \( M \).

**Definition 2.1.** A \( G_p \)-equivariant Cantor compact subset of \( \mathbb{C}_p \) is said to be \( (p-) \) strong compact if it has a Cantor compact reduction \( N \subset M \).

**Theorem 2.2.** Let \( M \) be a \( G_p \)-equivariant compact subset of \( \mathbb{C}_p \). Then there exists an \( x \) in \( \widetilde{\mathbb{Q}}_p \) whose pseudo-orbit is exactly \( M \).
Proof. Let \( \{S_k\}_k \) and \( \{S'_k\}_k \) be the projective systems constructed above for \( M \) and for one of its reductions \( N \) respectively.

Let \( n'_1, n'_2, \ldots \), be the corresponding numbers of distinct balls which cover only the subset \( N \). Fix a \( k = 1, 2, \ldots \). If every ball \( B'[x_{kj}, \varepsilon_k] \in S'_k \), \( j = 1, \ldots, n'_k \), contains the same number of balls of radius \( \varepsilon_{k+1} \), namely \( n''_{k+1}/n'_k \), we put \( n''_{k+1} = n'_{k+1} \). If this last fraction is not a natural number, we denote by \( p(k, j) \) the number of balls of radius \( \varepsilon_{k+1} \) which are contained in \( B'[x_{kj}, \varepsilon_k] \) and put \( m_k = \text{l.c.m.}\{p(k, j)\}_j \). Finally, we change \( n'_{k+1} \) to \( n''_{k+1} = n''_k m_k \). In this way we must count some of the true balls of radius \( \varepsilon_{k+1} \) which are contained in \( B'[x_{kj}, \varepsilon_k] \) many times, i.e. we must consider them “with multiplicities”. We obtain inductively a new sequence of natural numbers, \( n''_1, n''_2, \ldots \), such that \( n''_k \) divides \( n''_{k+1} \) for any \( k = 1, 2, \ldots \). For every \( k = 1, 2, \ldots \), denote by \( S^*_k \) the set of all \( n''_k \) balls \( B'[x_{kj}, \varepsilon_k] \in N \) (for convenience we assume that only the first one, \( B'[x_{k1}, \varepsilon_k] \), may appear many times). It is now clear that the sets \( \{S^*_k\}_k \) can be organized as a projective system of balls and its projective limit is exactly \( N = \varprojlim S^*_k \), i.e. every element of \( N \) can be realized as the intersection of a tower of balls, one from every \( S^*_k, k = 1, 2, \ldots \).

We now want to associate to this projective system of balls in \( N \) a tower of algebraic fields:

\[
L^{(p)} = L_1 \subset L_2 \subset \cdots \subset \mathbb{Q}
\]

where \( L^{(p)} \) is the subfield considered in Proposition 1.3. For \( S^*_1 = \{B'[x_1, \varepsilon_1]\} \), \( x_1 \in N \), we take simply \( L_1 = L^{(p)} \). Consider now an extension \( L_2 \) of \( L_1 \) of degree \( n''_2 \) such that the unique extension of the \( p \)-adic valuation \( v_p \) to \( L_1 \) decomposes exactly into \( n''_2 \) distinct valuations \( v_{21}, v_{22}, \ldots, v_{2n''_2} \) on \( L_2 \) (this can be done as in Remark 1.1). Since \( L_2 \) is dense in \( \mathbb{C}_p \) (in fact \( L^{(p)} \) is dense in \( \mathbb{C}_p \) as we saw in Proposition 1.3) we can take \( z_{2j} \in B'[x_{2j}, \varepsilon_2] \) such that \( \sigma_{2j}^{-1}(z_{2j}) \in L_2 \) for every \( B'[x_{2j}, \varepsilon_2] \in S^*_2 \), where \( \{\sigma_{2j}\}_j \) are all the \( L^{(p)} \)-embeddings of \( L_2 \) into \( \overline{\mathbb{Q}} \) and \( v_{2j} = v \circ \sigma_{2j} \). We now use the Approximation Theorem to find an element \( w_2 \) in \( L_2 \) such that \( |w_2 - \sigma_{2j}^{-1}(z_{2j})|v_{2j} \leq \varepsilon_2 \) for every \( j = 1, \ldots, n''_2. \) This means that in every ball \( B'[x_{2j}, \varepsilon_2] \) from \( S^*_2 \) we have exactly one conjugate of \( w_2 \) over \( L^{(p)} \). It is easy to see that \( L_2 = L^{(p)}[w_2] \).

Assume that we have constructed the field \( L_k, n''_k \) distinct valuations \( v_{kj} = v \circ \sigma_{kj} \) on it and a generator \( w_k \) of it such that \( \sigma_{kj}(w_k) \in B'[x_{kj}, \varepsilon_k] \) for every \( j = 1, \ldots, n''_k. \) Here \( \sigma_{kj} \) are all the \( L^{(p)} \)-embeddings of \( L_k \) into \( \overline{\mathbb{Q}} \).

We now consider an extension \( L_{k+1} \) of \( L_k \) of degree \( q_k = n''_{k+1}/n''_k \) such that every valuation \( v_{kj} \) decomposes exactly into \( q_k \) valuations on \( L_{k+1} \) (Remark 1.1). Then \( v_{k+1,j} = v \circ \sigma_{k+1,j}, j = 1, \ldots, n''_{k+1} \), are all the distinct valuations on \( L_{k+1} \) which extend \( v_p \). Here \( \sigma_{k+1,j}, j = 1, \ldots, n''_{k+1} \), are all the \( L^{(p)} \)-embeddings of \( L_{k+1} \) into \( \overline{\mathbb{Q}} \). We must be careful with the notation of \( \sigma_{k+1,j} \). Namely, the restriction of \( \sigma_{k+1,j} \) to \( L_k \) must be \( \sigma_{k,j'} \) such that
\( \sigma_{k+1,j}(w_{k+1}) \) is in the ball \( B'[x_{k,j'}, \varepsilon_k] \) which also contains \( \sigma_{k,j'}(w_k) \). For any \( j = 1, \ldots, n_k' \), take \( z_{k+1,j} \in B'[x_{k+1,j}, \varepsilon_{k+1}] \) such that \( \sigma_{k+1,j}^{-1}(z_{k+1,j}) \in L_{k+1} \). Using the Approximation Theorem we find \( w_{k+1} \in L_{k+1} \) whose conjugates \( L^p(w_{k+1}) \) all belong to a ball of the form \( B'[x_{k+1,j}, \varepsilon_{k+1}] \). Hence \( L_{k+1} = L^p[w_{k+1}] = L_k[w_{k+1}] \).

Let now \( \mu \in G \). From Proposition 1.3 we can write \( \mu = \sigma \tau \), where \( \sigma \in G_p \) and \( \tau \in \text{Gal}(\overline{Q}/L^p) \). Therefore, every conjugate \( \mu(w_k) \) of \( w_k \) belongs to a ball from \( S_k \), where \( \{S_k\}_k \) is the projective system of balls which gives the whole compact subset \( M \). Moreover, any ball \( B_{kj} \) of \( S_k \) contains at least one such \( Q \)-conjugate of \( w_k \). We now prove that \( \{w_k\}_k \) is a Cauchy sequence relative to the \( p \)-adic spectral norm. Indeed,

\[
\|w_{k+n} - w_k\|_p = \max\{|\mu(w_{k+n} - w_k)|_p : \mu \in G\}.
\]

But \( |\mu(w_{k+n} - w_k)|_p = |\tau(w_{k+n} - w_k)|_p \), where \( \tau \in \text{Gal}(\overline{Q}/L^p) \). Since \( w_k, w_{k+n} \in L_{k+n}, \tau \) is one of the \( L^p \)-embeddings \( \sigma_{k+n,j} \) of \( L_{k+n} \) in \( \overline{Q} \) considered above. Because of the special choice of \( \sigma_{k+1,j}, \ldots, \sigma_{k+n,j} \), we see that \( \sigma_{k+n,j}(w_{k+n}) \) and \( \sigma_{k+n,j}(w_k) \) are in the same ball \( B'[x_{k,j}, \varepsilon_k] \), i.e.

\[
|\mu(w_{k+n} - w_k)|_p \leq \varepsilon_k
\]

for every \( n = 1, 2, \ldots \) and \( \mu \in G \). This means that

\[
\|w_{k+n} - w_k\|_p \leq \varepsilon_k
\]

for every \( n = 1, 2, \ldots \) and so \( \{w_k\}_k \) is a Cauchy sequence with respect to the \( p \)-adic spectral norm. Let

\[
x = \lim_{n \to \infty} w_n \quad \text{in} \quad \overline{Q}_p.
\]

It is not difficult to see that any element \( y \) of \( M \) is the intersection of a tower of balls of the form \( B[x_1, \varepsilon_1] \supset B[x_{2j_2}, \varepsilon_2] \supset \cdots \supset B[x_{k,j_k}, \varepsilon_k] \supset \cdots \) and each such ball contains an element of the form \( \mu(w_k) \in B[x_{k,j_k}, \varepsilon_k] \) for the same \( \mu \in G \) (see the construction of \( \sigma_{k+1,j} \) from \( \sigma_{k,j} \)). Hence

\[
x(\mu) = \lim_{n \to \infty} \mu(w_k),
\]

i.e. \( M = C(x) \) and the proof of the theorem is finished.

Remark 2.1. In the proof of Theorem 2.2 we have constructed an element \( x \in \tilde{L} \), the \( p \)-adic completion of \( L = \bigcup_{k=1}^\infty L_k \), such that \( M = C(x) \). Let \( M \) be a \( p \)-strong compact subset of \( \mathbb{C}_p \). Let \( \sigma, \mu \in G \) with \( \sigma(x) \neq \mu(x) \) (in \( \overline{Q}_p \)), i.e. \( x(\tau \sigma) \neq x(\tau \mu) \) for at least one \( \tau \in G \) (two elements in \( \overline{Q}_p \) are equal if and only if their components are equal). If \( \tau \in G_p \) then \( x(\tau \sigma) = \tau(x(\sigma)) \neq \tau(x(\mu)) = x(\tau \mu) \) if and only if \( x(\sigma) \neq x(\mu) \). If \( \tau \notin G_p \), then we can consider \( \tau, \sigma, \mu \) to be \( L^p \)-embeddings of \( L \) into \( \overline{Q} \) (see Proposition 1.3). In this last case, since \( N \) is a Cantor compact subset of \( \mathbb{C}_p \), \( x(\tau \sigma) \neq x(\tau \mu) \) means that the two towers of balls which define \( x(\sigma) \) and \( x(\mu) \) respectively do not
coincide, i.e. \( x_{(\sigma)} \neq x_{(\mu)} \). So we have proved that the continuous mapping \( \sigma(x) \mapsto x_{(\sigma)} \) from \( O(x) \) to \( C(x) \) is a homeomorphism.

**Proposition 2.3.** Let \( M \) be a \( p \)-strong compact subset of \( \mathbb{C}_p \). Then \( M \) is homeomorphic to a factor set of left cosets of the form \( G/H \), where \( H \) is a closed subgroup of the absolute Galois group of \( \mathbb{Q} \).

**Proof.** Let \( M = C(x) \) for \( x \in \overline{\mathbb{Q}}_p \) (Theorem 2.2). Let \( H_x = \{ \mu \in G \mid \mu(x) = x \text{ in } \overline{\mathbb{Q}}_p \} \). It is easy to see that \( H_x \) is a closed subgroup of \( G \). The orbit \( O(x) \) is homeomorphic to \( G/H_x \) through the mapping \( \sigma \mapsto \sigma(x) \). Take \( H = H_x \) and the proof is finished.

Let \( K \) be a subfield of \( \overline{\mathbb{Q}} \) and let \( H_K = \{ \sigma \in G \mid \sigma(x) = x \text{ for all } x \text{ in } K \} \) be the closed subgroup of \( G \) which fixes \( K \). Let \( G/H_K \) be the compact space of all left cosets of \( H_K \) in \( G \). A continuous function \( f : G/H_K \to \mathbb{C}_p \) is said to be \( G_p \)-equivariant if \( f(\mu\sigma H_K) = \mu(f(\sigma H_K)) \) for every \( \mu \in G_p \) and for all cosets \( \sigma H_K \) in \( G/H_K \). We denote by \( C_{G_p}(G/H_K, \mathbb{C}_p) \) the \( \mathbb{Q}_p \)-Banach algebra of all continuous \( G_p \)-equivariant functions \( f : G/H_K \to \mathbb{C}_p \).

**Theorem 2.4.** With the notations and the hypotheses above, let \( \tilde{K} \) be the completion of \( K \) relative to the \( p \)-adic spectral norm. Then the continuous mapping \( \varphi : K \to C_{G_p}(G/H_K, \mathbb{C}_p) \), defined by \( \varphi(x) = \varphi_x \), where \( \varphi_x(\sigma H_K) = \sigma(x) (= x_{(\sigma)}) \), can be uniquely extended to an isometric homomorphism of \( \mathbb{Q}_p \)-algebras, denoted also by \( \varphi : \tilde{K} \to C_{G_p}(G/H_K, \mathbb{C}_p) \), \( \varphi(z) = \varphi_z \), where \( \varphi_z(\sigma H_K) = z_{(\sigma)} \).

**Proof.** Since \( \|z\|_p = \sup_{\sigma \in G} |z_{(\sigma)}|_p \), the isometric property is clear (for \( f \in C_{G_p}(G/H_K, \mathbb{C}_p) \), \( \|f\| = \sup_{\sigma \in G} |f(\sigma H_K)|_p \), the usual supremum norm in a Banach algebra of continuous functions defined on a compact space). The continuity of \( \varphi \) comes from the continuity of the mapping \( \sigma \mapsto x_{(\sigma)} \) (see also [PPV]). It remains to prove the surjectivity of \( \varphi \). Let \( f \in C_{G_p}(G/H_K, \mathbb{C}_p) \) and let \( M \) be the \( G_p \)-equivariant compact subset \( f(G/H_K) \). Let \( \sim \) be the following equivalence relation on \( G/H_K \):

\[
\mu_1 H_K \sim \mu_2 H_K \quad \text{if} \quad \mu_2 H_K = \sigma \mu_1 H_K \text{ for some } \sigma \text{ in } G_p.
\]

Choose a representative \( \mu_i H_K \) in each equivalence class of this relation. Denote this set of representatives by \( \{\mu_i H_K\}_{t \in T} \). It is clear that the \( \{\mu_i\}_t \) give rise to a set of inequivalent independent absolute values on \( K \): \( |z|_{\mu_t} = |\mu_t(z)|_p \), \( t \in T \). Let now

\[
\mathbb{Q} = K_1 \subset K_2 \subset \cdots \subset K, \quad \bigcup_{n=1}^{\infty} K_n = K,
\]

be a tower of (finite) algebraic number fields which cover the whole \( K \).

Let \( \varepsilon_1 > \varepsilon_2 > \cdots > 0 \) be a sequence of real numbers convergent to zero and let \( n_k \) be the least number of balls \( B[x_{k_j}, \varepsilon_k], j = 1, \ldots, n_k \), which
cover $N$, a reduction of $M$ (see definition before Definition 2.1). We suppose that $n_1 = 1$ and $n_1 < n_2 < \cdots$. Consider now the next $n_2 > 1$ balls of radius $\varepsilon_2$ which cover $N$, and take an element $f(\mu_{2j}H_K)$, $j = 1, \ldots, n_2$, of $N$ in every such ball, where $\mu_{2j}$ is one of the above chosen $\{\mu_t\}_{t \in T}$. Since $|\cdot|_{\mu_{2j}}$, $j = 1, \ldots, n_2$, are independent absolute values on $K$, they are also independent on at least one field $K_{k_2}$ from the above tower. Choose the smallest $K_{k_2}$.

Now assume that we have already constructed $K_{k_2} \subset \cdots \subset K_{k_n}$ such that for any $i = 2, \ldots, n$ and any set of elements $\{f(\mu_{ij}H_K)\}$, $j = 1, \ldots, n_i$, in $N$ and, at the same time, in a ball $B[x_{ij}, \varepsilon_i]$, the corresponding absolute values $\{|\cdot|_{\mu_{ij}}\}$, $j = 1, \ldots, n_i$, are independent on the subfield $K_{k_i}$. If the set $N$ is finite, the above construction must stop at a subfield $K_{k_n}$, for an $m \in \mathbb{N}$. If $N$ is infinite, we consider the set $\{B[x_{n+1,j}, \varepsilon_n]\}_{j}$ of balls which cover $N$, some elements $f(\mu_{n+1,j}H_K)$ from $N$, each in one of these balls, and take a subfield $K_h$ with sufficiently large $h \in \mathbb{N}$ such that $K_{k_n} \subset K_h$ and the absolute values $\{|\cdot|_{\mu_{n+1,j}}\}$ are independent on $K_h$. Then we restrict the absolute values $\{|\cdot|_{\mu_{n+1,j}}\}$ to $K_{k_n}$. Let $k_{n+1}$ be the least $h$ with this property. Now we apply the Approximation Theorem on any $K_{k_n}$ and find an element $w_n \in K_{k_n}$ such that

$$|\mu_{n+j}(w_n) - f(\mu_{nj}H_K)|_p < \varepsilon_n$$

for any $j = 1, \ldots, n_{k_n}$ and $n = 2, 3, \ldots$. Since $f$ is $G_p$-equivariant we can extend these inequalities to the whole $M = f(G/H_K)$ and to the whole $G/H_K$.

Now it is easy to see that $\{w_n\}_n$ is a Cauchy sequence in $K$ relative to the $p$-adic spectral norm. Let $w = \lim_{n \to \infty} w_n$ be its limit in $\tilde{K}$. From the last inequality and from the way we have chosen the set $\{\mu_t\}_{t}$ one finds that $w(\tau) = f(\tau H_K)$, i.e. $f = \varphi_w$ and the proof is finished.

3. Topological generic elements in the $p$-adic case

**Theorem 3.1.** Any algebraic number field $L$ (finite or infinite) has a topological generic element $x \in \mathbb{Q}_p$, relative to the $p$-adic spectral norm, i.e. $\tilde{L} = \mathbb{Q}[x]$. Moreover, this $x$ is such that $\varphi_x : G/H_L \to \mathbb{C}_p$ is a topological embedding, where $H_L$ is the subgroup of $G$ which corresponds to $L$ (in the Galois correspondence).

**Proof.** From Theorem 2.4 we can work in $C_{G_p}(G/H_L, \mathbb{C}_p) \cong \tilde{L}$, where $H_L = \text{Fix } L = \{\sigma \in G \mid \sigma(z) = z \text{ for any } z \in L \}$. In order to find an element $x \in \tilde{L}$ with $\tilde{L} = \mathbb{Q}[x]$ it is enough to find $f \in C_{G_p}(G/H_L, \mathbb{C}_p)$ which separates the elements of $G/H_L$, i.e. $\sigma H_L \neq \mu H_L$ implies $f(\sigma H_L) \neq f(\mu H_L)$. Indeed, using the $p$-adic version of the Stone–Weierstrass theorem (see [Sch, Appendix]) for the $\mathbb{Q}_p$-subalgebra $\mathbb{Q}[f]$ of $C_{G_p}(G/H_L, \mathbb{C}_p)$ we will
then obtain $\widehat{f} = C_{G_p}(G/H_L, \mathbb{C}_p)$. Then the generic topological element of $\widetilde{L}$ will be the $x \in \widetilde{L}$ with $\varphi_x = f$.

Let us construct such an embedding $f : G/H_L \rightarrow \mathbb{C}_p$. Since $f$ must be a $G_p$-equivariant continuous function on $G/H_L$, first of all we take a subset $N$ of representatives $\{\tau_i\}_{i \in I}$ in $G/H_L$ such that for any $i \neq j$, $i, j \in I$, there is no $\sigma \in G_p$ with $\sigma \tau_i = \tau_j$. We construct $N$ exactly as in the case of a $G_p$-equivariant compact subset $M$ of $\mathbb{C}_p$. Namely, first of all let us organize $G/H_L$ as a profinite Cantor compact set, considering a tower of finite algebraic number fields:

$$Q = L_1 \subset L_2 \subset \cdots \subset L$$

where $L = \bigcup_{i=1}^{\infty} L_i$ and taking the corresponding tower of subgroups:

$$H_L \subset \cdots \subset H_2 \subset H_1 = G$$

where $\bigcap_{i=1}^{\infty} H_i = H_L$ and $H_n = H_{L_n} = \text{Fix } L_n$. Now $G/H_L = \varprojlim G/H_n$ and we construct the compact subset $N$ of $G/H_L$ as follows. Consider the partition $G = \mu_{21} H_2 \cup \cdots \cup \mu_{2n_2} H_2$. If $\mu_{22} H_2 = \sigma \mu_{21} H_2$ for some $\sigma \in G_p$ we remove $\mu_{22} H_2$ from this partition. We proceed in this way in order to obtain a “reduced” subset of $\{\mu_{2i} H_2\}_i$, $i = 1, \ldots, n_2$, with respect to $G_p$. Denote by

$$S^*_2 = \{\mu^*_{21} H_2, \mu^*_{2i_2} H_2, \ldots, \mu^*_{2i_{k_2}} H_2\}$$

this “reduced” subset. Consider now the partition $H_2 = \tau_{31} H_3 \cup \cdots \cup \tau_{3m_3} H_3$. Take $\mu^*_{21} H_2 \in S^*_2$ and find a corresponding partition:

$$\mu^*_{21} H_2 = \mu^*_{21} \tau_{31} H_3 \cup \mu^*_{21} \tau_{32} H_3 \cup \cdots \cup \mu^*_{21} \tau_{3m_3} H_3.$$

We now consider the “reduction” of the set $\{\mu^*_{21} \tau_{3j} H_3\}_j$ relative to $G_p$. We do the same with all $\mu^*_{2i_j} H_2$ of $S^*_2$ and finally obtain $S^*_3 = \{\mu^*_{31} H_3, \ldots, \mu^*_{3k_3} H_3\}$. We continue in this way and obtain $S^*_4, S^*_5, \ldots$. Since any set in $S^*_{n+1}$ is a subset of a set in $S^*_n$ we can organize $\{S^*_n\}_n$ into a projective system of finite sets. Let $N$ be its projective limit. It is clear that $N$ is a compact subset of $G/H_L$ and $\bigcup_{\sigma \in G_p} \sigma(N) = G/H_L$. The compact subset $N$ has a “configuration”

$$k_1 = 1 < k_2 < k_3 < \cdots$$

where $k_j = |S^*_j|$ for any $j = 1, 2, \ldots$. Let $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ be a sequence of positive real numbers which tends to zero. Let $Z$ be the following compact subset of $\mathbb{C}_p$ with the configuration $\{(\varepsilon_n)_n, (k_n)_n\}$. Take a collection $U_n = \{B_{n1}, \ldots, B_{nk_n}\}$ of disjoint balls such that any ball $B_{n+1, i}$ of $U_{n+1}$ is contained in one ball $B_{n, j}$ of $U_n$. Moreover we assume that for any $n = 1, 2, \ldots$ and $i \neq j$, $i, j \in \{1, \ldots, k_n\}$ there is no $\sigma \in G_p$, $\sigma \neq e$, such that $B_{nj} = \sigma(B_{ni})$. We also suppose that any two distinct towers of balls $B_{11} \supset B_{2i_2} \supset B_{3i_3} \supset \cdots$ and $B_{11} \supset B_{2j_2} \supset B_{3j_3} \supset \cdots$ have distinct intersection points. We consider the mapping $f_n : S^*_n \rightarrow U_n$, 


\[ f_n(\mu_{nj}^*H_n) = z_{nj}, \] where \( z_{nj} \) is a fixed point of \( B_{nj}, j = 1, \ldots, k_n \). If \( \sigma \in G_p \) we put \( f_n(\sigma \mu_{nj}^*H_n) = \sigma(z_{nj}). \)

In this way we have obtained a continuous function from \( G/H_n \) to \( \mathbb{C}_p \) which separates the elements of \( G/H_n \). The projective limit of \( \{f_n\}_n \) gives rise to a continuous function \( f \in C_{G_p}(G/H_L, \mathbb{C}_p) \), with \( \text{Im} f = Z \), which separates the elements of \( G/H_L \), and the proof of the theorem is finished. ■

In the course of the above proof we obtained in fact another important result.

**Corollary 3.2.** The element \( x \in \overline{\mathbb{Q}}_p \) is a generic element for \( L \) if and only if \( \varphi_x : G \to \mathbb{C}_p \) induces a continuous embedding \( \varphi_x : G/H_L \to \mathbb{C}_p \), i.e. \( \mu^{-1} \sigma \in H_L \) if and only if \( x(\sigma) = x(\mu) \).

**Remark 3.1.** An alternative proof for Theorem 3.1 can be given exactly as in the archimedean case (see [PPZ1]).

**Theorem 3.3.** Let \( L \) be a subfield of \( \overline{\mathbb{Q}} \). Assume that there exists a topological generic element \( x \) for \( L \), i.e. \( \widetilde{L} = \overline{\mathbb{Q}}[x] \). Then the pseudo-orbit \( C(x) \) of \( x \) is a Cantor compact subset of \( \mathbb{C}_p \).

**Proof.** We prove that the continuous surjection \( \sigma(x) \mapsto x(\sigma) \) from \( O(x) \) to \( C(x) \) is a bijection, i.e. \( C(x) \cong G/H_x \), where \( H_x = \{ \mu \in G \mid \mu(x) = x \} \). Let \( \sigma, \mu \in G \) be such that \( x(\sigma) = x(\mu) \), let \( z \in L \) and let \( \varepsilon > 0 \) be a small real number. Then \( z = \lim_{n \to \infty} P_n(x) \), \( P_n(x) \in \overline{\mathbb{Q}}[x] \). Let \( \{x_m\}_m \) be a Cauchy sequence in \( L \) which defines \( x \). Then, for fixed \( n \),

\[
\lim_{m \to \infty} P_n(\sigma(x_m)) = P_n(x(\sigma)) = P_n(x(\mu)) = \lim_{m \to \infty} P_n(\mu(x_m)).
\]

Choose \( n \) such that \( ||z - P_n(x)||_p < \varepsilon/6 \). Then

\[
||\sigma(z) - P_n(\sigma(x))||_p < \varepsilon/6, \quad ||\mu(z) - P_n(\mu(x))||_p < \varepsilon/6.
\]

For this \( n \) we choose \( m \) such that

\[
||P_n(\sigma(x)) - P_n(\sigma(x_m))||_p < \varepsilon/6, \quad ||P_n(\mu(x)) - P_n(\mu(x_m))||_p < \varepsilon/6.
\]

It follows that

\[
||\sigma(z) - P_n(\sigma(x_m))||_p < \varepsilon/3, \quad ||\mu(z) - P_n(\mu(x_m))||_p < \varepsilon/3.
\]

Possibly increasing \( m \) we have

\[
||P_n(\sigma(x_m)) - P_n(\mu(x_m))||_p < \varepsilon/3.
\]

Finally, we see that \( ||\sigma(z) - \mu(z)||_p < \varepsilon \) for any \( \varepsilon > 0 \). This means that \( \sigma(z) = \mu(z) \) for any \( z \in L \). Hence \( \sigma(x) = \mu(x) \), i.e. the mapping \( \sigma(x) \mapsto x(\sigma) \) is an injection and the proof is finished. ■

**Theorem 3.4.** Let \( x \) in \( \overline{\mathbb{Q}}_p \) be such that \( C(x) \) is a Cantor compact subset of \( \mathbb{C}_p \). Let \( H_x = \{ \sigma \in G \mid \sigma(x) = x \} \) and \( L = \{ y \in \overline{\mathbb{Q}} \mid \mu(y) = y \text{ for every } \mu \in H_x \} \). Then \( \widetilde{L} = \overline{\mathbb{Q}}[x] \), i.e. \( x \) is a topological generic element for \( \widetilde{L} \).
Proof. Since $\widetilde{L} \cong C_{G_p}(G/H_x, \mathbb{C}_p)$ and $x \in \widetilde{L}$ (Ker $\varphi_x = H_x$, where $\varphi_x : G \to \mathbb{C}_p$, $\varphi_x(\sigma) = x(\sigma)$). Since $C(x)$ is a Cantor compact subset, $\varphi_x$ separates the elements of $G/H_x$. Hence we can apply the $p$-adic version of the Stone–Weierstrass theorem (see [Sch]) for the subalgebra $\widetilde{Q}_p[x]$ of $C_{G_p}(G/H_x, \mathbb{C}_p)$ to conclude that $\widetilde{L} \cong \mathbb{Q}[x]$. \[ \]

Remark 3.2. If we start with a Cantor compact subset $M$ of $\mathbb{C}_p$, it is not difficult to find the least $G_p$-equivariant Cantor compact subset $M'$ which contains $M$, namely,

$$M' = \bigcup_{\sigma \in G_p} \sigma(M).$$

This is a consequence of a general observation. If $G$ is a compact group which acts continuously on a metric space $M$, that is, $(g, m) \mapsto g \cdot m$ is a continuous mapping, and if $N$ is a compact subset of $M$, then $\{g \cdot n \mid g \in G, n \in N\}$ is a compact subset of $M$.

Another remark is that an element $x$ can be a topological generic element only for one algebraic number field. Indeed, if $\widetilde{L} = \mathbb{Q}_p[x] = \widetilde{L}'$ then, according to [PPV], $L = \widetilde{L} \cap \overline{\mathbb{Q}} = \widetilde{L}' \cap \overline{\mathbb{Q}} = L'$.

If one puts together Theorems 3.1, 3.3, 3.4 and the method used in the proof of Theorem 3.1, one obtains the following basic result.

Theorem 3.5. Let $x \in \widetilde{\mathbb{Q}}_p$, let $H_x$ be its invariant subgroup in $G$ and $L = \text{Inv} H_x$. Then the following assertions are equivalent:

(i) $C(x)$ is a Cantor compact subset of $\widetilde{Q}_p$.
(ii) $x$ is a topological generic element for $\widetilde{L}$.
(iii) $x \overset{\|}{\Rightarrow} \lim_{n \to \infty} x_n$, where $x_n \in \overline{\mathbb{Q}}$, $\mathbb{Q}(x_n) \subset \mathbb{Q}(x_{n+1})$, any valuation on $\mathbb{Q}(x_n)$ which extends $v_p$ splits completely in $\mathbb{Q}(x_{n+1})$ for every $n = 1, 2, \ldots$, and $L = \bigcup_{n=1}^{\infty} \mathbb{Q}(x_n)$.

4. Unusual Galois actions on compact subsets of $\mathbb{C}_p$. In this section we use freely the notations and results of the previous sections.

Let $M \subset \mathbb{C}_p$ be a $p$-strong compact subset of $\mathbb{C}_p$ and let $x \in \overline{\mathbb{Q}}_p$ be such that $C(x) = M$ (Theorem 2.2). Since the continuous mapping $\sigma(x) \mapsto x(\sigma)$ from $O(x)$ to $C(x)$ is a homeomorphism (Remark 2.1), the map

$$(\sigma, x(\tau)) \mapsto \sigma \ast x(\tau) := x(\sigma \tau)$$

is a continuous group action of the absolute Galois group $G = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the compact subset $M$. We call such actions Galois actions on compact subsets of $\mathbb{C}_p$. It is easy to see that if the above defined function is a group action of $G$ on $M = C(x)$, then $M$ must be a Cantor compact subset. If $M$
is not $G_p$-equivariant or if $M$ is not a Cantor compact subset, we cannot define such a Galois action on it.

The usual compact subsets of $\mathbb{C}_p$ are the rings of integers of finite extensions of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$. The ring $\mathbb{Z}_p$ of $p$-adic integers is a $p$-strong compact subset of $\mathbb{C}_p$. Let us describe such a Galois action on $\mathbb{Z}_p$. $\mathbb{Z}$ itself is dense in $\mathbb{Z}_p$ (relative to the $p$-adic valuation). Consider a fixed tower of subfields of $\overline{\mathbb{Q}}$:

$$K_0 = \mathbb{Q} \subset K_1 \subset K_2 \subset \cdots \subset K,$$

where $K = \bigcup K_n \subset \overline{\mathbb{Q}}$, such that $[K_{n+1} : K_n] = p$ and the $p$-adic valuation splits completely in $K_n$ (see the theorem of Hasse [R]). Let $H_n = \{ \mu \in G \mid \mu(y) = y \text{ for every } y \in K_n \}$ be the corresponding closed subgroup of $G$. As in the proof of Theorem 2.2 we shall connect the natural profinite structure of $\mathbb{Z}_p$ to the profinite structure of $G$.

We denote by $S_1 = \{ B_{10}, B_{11}, \ldots, B_{1,p-1} \}$ the set of “closed” balls in $\mathbb{Z}_p$ of radius 1/p, with centres at 0, 1, 2, \ldots, $p - 1$, respectively. For instance $B_{1i} = B[i, 1/p] = \{ z \in \mathbb{Z}_p \mid |z-i|_p \leq 1/p \}$. It is clear that $\mathbb{Z}_p = \bigcup_{i=0}^{p-1} B_{1i}$ and this is a disjoint union. The ball $B_{1i}$ is the disjoint union of the following $p$ balls of radius 1/p$^2$: $B_{1i} = \bigcup_{j=0}^{p-1} B_{2j}^{(i)}$, where $B_{2j}^{(i)} = B[i+jp, 1/p^2]$, $0 \leq j < p$.

We put together all these balls of radius 1/p$^2$ for any $i = 0, 1, \ldots, p - 1$ and obtain $S_2 = \{ B_{20}, B_{21}, \ldots, B_{2,p^2-1} \}$; the first $p$ balls are contained in $B_{10}$, the next $p$ in $B_{11}$, etc. In this way we can construct $S_n$ from $S_{n-1}$ for every $n = 2, 3, \ldots$ and it is clear that $\mathbb{Z}_p = \varprojlim S_n$.

Let $| \cdot |_{10}, \ldots, | \cdot |_{1,p-1}$ be the $p$-adic absolute values on $K_1$, which extend the usual $p$-adic absolute value $| \cdot |_p$ on $\mathbb{Q}$.

Let $\sigma_{10}, \sigma_{11}, \ldots, \sigma_{1,p-1}$ be a fixed set of representatives of the left cosets in $G/H_1$ and we assume (after a suitable permutation of the above $p$ absolute values) that $|y_1|_{1j} = |\sigma_{1j}(y_1)|_v$ for any $y_1 \in K_1$ and $j = 0, 1, \ldots, p - 1$. Exactly as in the case of $S_2$, we consider a set of representatives $\sigma_{20}, \sigma_{21}, \ldots, \sigma_{2, p^2-1}$ of cosets in $G/H_2$, the first $p$ of which extend $\sigma_{10}$, the next $p$ extend $\sigma_{11}$, etc. At the same time we consider the $p^2$ absolute values: $|y_2|_{2j} = |\sigma_{2j}(y_2)|_v$ for any $y_2 \in K_2$ and $j = 0, 1, \ldots, p^2 - 1$.

We continue in this way for every $K_3, K_4, \ldots$. We obtain three “isomorphic” projective systems: of balls, $\{ S_n \}_n$, of automorphisms of $G$, and of absolute values. Using the Approximation Theorem on $K_n$ we can find $x_n \in K_n$ such that $|\sigma_{nj}(x_n) - j|_v \leq 1/p^n$ for every $j = 0, 1, \ldots, p^n - 1$. This means that $x_n$ has exactly $p^n = [K_n : \mathbb{Q}]$ conjugates (in particular $\mathbb{Q}(x_n) = K_n$) and each of them belongs to a ball from $S_n$. Since any automorphism $\sigma$ of $G$, when restricted to $K_n$, is one of the $\sigma_{nj}$, $j = 0, 1, \ldots, p^n - 1$, the sequence $\{x_n\}_n$ is a Cauchy sequence relative to the $p$-adic spectral norm on $\overline{\mathbb{Q}}$. Let $x = \lim_{n \to \infty} x_n$, $x \in \tilde{K}$. In fact we have a representation of the Cantor compact subset $\mathbb{Z}_p$ as the pseudo-orbit of this $x$: $\mathbb{Z}_p = C(x)$. 

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Now, the Galois action $\sigma \ast x(\mu) = x(\sigma \mu)$ of $G$ on $C(x) = \mathbb{Z}_p$ is easy to describe. Take a $p$-adic integer

$$\alpha = a_0 + a_1 p + \cdots, \quad a_i \in \{0, 1, \ldots, p - 1\} \text{ for all } i = 0, 1, \ldots$$

This $\alpha$ corresponds to a tower of balls $B_{1i_1} \supset B_{2i_2} \supset \cdots$, namely $B_{1i_1} = B[a_0, 1/p], B_{2i_2} = B[a_0 + a_1 p, 1/p^2], \ldots, B_{nin} = B[a_0 + a_1 p + \cdots + a_{n-1} p^{n-1}, 1/p^n].$ Moreover $B_{ni_n} \in S_n$ for $n = 1, 2, \ldots$ and $\{\alpha\} = \bigcap_n B_{ni_n}$. We associate to this $\alpha$ the unique $\mathbb{Q}$-embedding $\mu(\alpha)$ of $K = \bigcup_n K_n$ into $\overline{\mathbb{Q}}$, such that the restriction of $\mu(\alpha)$ to $K_n$ is exactly $\sigma_{n,j_{n,a}}$ (where $j_{n,a} = a_0 + a_1 p + \cdots + a_{n-1} p^{n-1}$) constructed above. It is easy to see that this assignment $\alpha \mapsto \mu(\alpha)$ is a one-to-one and onto correspondence between $\mathbb{Z}_p$ and the topological space $G/H$ (with its Krull topology), where $H = \text{Fix} K$. Moreover, this last mapping is a homeomorphism between $\mathbb{Z}_p$ and $G/H$. The above Galois action on $\mathbb{Z}_p$ is exactly $\sigma \ast \alpha = \beta \in \mathbb{Z}_p$, where $\beta$ corresponds to the embedding $\sigma \mu(\alpha)$ of $K$ into $\overline{\mathbb{Q}}$. This $\beta = b_0 + b_1 p + \cdots$ is the $p$-adic limit of the integers $k_n = b_0 + b_1 p + \cdots + b_{n-1} p^{n-1}$, where $k_n$ is the index which appears in $\sigma_{n,k_n}$, the restriction of $\sigma \mu(\alpha)$ to $K_n$, which in addition has the following property: $|\sigma_{n,k_n}(x_n) - k_n|_v \leq 1/p^n$ (see the above construction of $\{\sigma_{n,j}\}_{n,j}$, $j = 0, 1, \ldots, p^{n-1}$). This Galois action can also be described by using the above homeomorphism between $\mathbb{Z}_p$ and $G/H$. Let $\theta : \mathbb{Z}_p \rightarrow G/H$ be this homeomorphism. Then

$$\sigma \ast \alpha = \theta^{-1}(\sigma \theta(\alpha)).$$

This Galois action depends on the $p$-tower of fields

$$K_1 \subset K_2 \subset \cdots \subset K = \bigcup_n K_n$$

and on $x = \lim_{n \rightarrow \infty} x_n$.

**Remark 4.1.** The completion $\tilde{K}$ of the above infinite algebraic number field $K = \text{Inv} H_x$, with $\mathbb{Z}_p = C(x)$, is $\mathbb{Q}_p$-homeomorphic to $C\mathbb{C}_p(\mathbb{Z}_p, \mathbb{C}_p)$ (Theorem 3.1). But this last $\mathbb{Q}_p$-Banach algebra is in fact $C(\mathbb{Z}_p, \mathbb{Q}_p)$, the $\mathbb{Q}_p$-Banach algebra of all continuous functions from $\mathbb{Z}_p$ to $\mathbb{Q}_p$. The $p$-adic algebra and analysis of $C(\mathbb{Z}_p, \mathbb{Q}_p)$ can be sometimes more deeply understood if one uses the identification $C(\mathbb{Z}_p, \mathbb{Q}_p) = \tilde{K}$. For instance, instead of the well known orthogonal basis of Mahler for $C(\mathbb{Z}_p, \mathbb{Q}_p)$ (see [M]), we can use the image of the orthogonal basis $\{M_n(x)\}$, $n = 0, 1, \ldots$, constructed in [A]. This last basis has deep arithmetical roots (see also [APZ1], [APZ2], [P2]) and it will be studied in another paper.

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