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A note on the Diophantine equation $(a^n x^m \pm 1)/(a^n x \pm 1) = y^n + 1$

by

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1. Introduction. Let $\mathbb{Z}, \mathbb{N}$ denote the sets of integers and positive integers respectively. Le Mao Hua [4] has proved that the following equation has no solution (x, y, m, n):

$$(1) \qquad \frac{x^m - 1}{x - 1} = y^n + 1, \quad x, y, m, n \in \mathbb{N},\ x > 1,\ y > 1,\ m > 2,\ n \text{ an odd prime.}$$

In this note, we investigate a more general equation applying another method. For

$$(2) \qquad \frac{a^n x^m + \delta}{a^n x + \delta} = y^n + 1, \quad \delta \in \{1, -1\},\ x, y \in \mathbb{Z},\ a, m, n \in \mathbb{N},\ m > 2,\ n > 1,\ |x| > 1,$$

we prove the following results.

THEOREM 1. *For $\delta = -1, x > 1$ all solutions of equation* (2) *are given by $(a, x, y, m, n) = (u^{m-2}, u^n, u^{m-1}, m, n)$ with $u \in \mathbb{N} > 1$.*

For $\delta = -1$, $x < -1$ all solutions of equation (2) *are given by $(a, x, y, m, n) = (u^{2k-2}, -u^{2l-1}, -u^{2k-1}, 2k, 2l-1)$ with $k, l, u \in \mathbb{N} > 1$.*

THEOREM 2. *For $\delta = 1, x > 1$ equation* (2) *has no solution.*

For $\delta = 1$, $x < -1$ all solutions of equation (2) *are given by $(a, x, y, m, n) = (u^{2k-3}, -u^n, u^{2k-2}, 2k-1, n)$ with $k, n, u \in \mathbb{N} > 1$.*

COROLLARY 1. *The Diophantine equation*

$$(3) \qquad \frac{x^m - 1}{x - 1} = y^n + 1, \quad x, y \in \mathbb{Z},\ m, n \in \mathbb{N},\ |x| > 1,\ m > 2,\ n > 1,$$

has no solution (x, y, m, n).

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Corollary 2. *The Diophantine equation*

$$\frac{x^m+1}{x+1} = y^n+1, \quad x,y\in\mathbb{Z},\ m,n\in\mathbb{N},\ |x|>1,\ m>2,\ n>1, \tag{4}$$

has no solution (x,y,m,n).

Corollary 1 is a substantial generalization of Le's result [4] for the equation (1).

2. Lemmas. Throughout this section, we assume that D and n are positive integers.

Lemma 1 ([3]). *If* $n\ge 5$, *then the equation*

$$X^n - DY^n = \pm 1, \quad X,Y\in\mathbb{N},$$

has at most one solution (X,Y) *except possibly when* $D=2$ *or* $D=2^n\pm 1$ *and* $n\in\{5,6\}$.

Lemma 2 ([5]). *If* $D>1$, *then the equation*

$$X^3+DY^3=1, \quad X,Y\in\mathbb{Z},$$

has at most one solution (X,Y) *other than* $X=1, Y=0$.

Lemma 3. *Let* D *be not a perfect square and* $x,y\in\mathbb{N}$ *a solution of Pell's equation*

$$x^2-Dy^2=1, \quad x,y\in\mathbb{Z}. \tag{5}$$

Let (x_0,y_0) *be the fundamental solution of* (5) *and* $\varepsilon = x_0+y_0\sqrt{D}$. *If* x_0 *is divisible by all prime divisors of* x, *then* $x+y\sqrt{D}=\varepsilon$.

Proof. We may assume

$$x+y\sqrt{D} = (x_0+y_0\sqrt{D})^n, \quad n\ge 1. \tag{6}$$

The result is clear for $n=1$. We assume $n>1$, then $x>1$.

(1) If n is even, by (6) we get

$$x=\sum_{j=0}^{n/2}\binom{n}{2j}x_0^{n-2j}(y_0^2D)^j. \tag{7}$$

Let p be a prime divisor of x. Under our assumption $p\mid x_0$. By (7) we get $p\mid (Dy_0^2)^{n/2}$. This is impossible.

(2) If n is odd, we may write

$$x+y\sqrt{D}=(x_0+y_0\sqrt{D})^n = x_n+y_n\sqrt{D}.$$

If $r\mid n$, then $x_r+y_r\sqrt{D}=(x_0+y_0\sqrt{D})^r, x_r\mid x_n$. Therefore x_r also satisfies the assumption of the lemma. Now let p be any prime divisor of n. We have

$$x_p+y_p\sqrt{D}=(x_0+y_0\sqrt{D})^p.$$

Then

$$x_p/x_0 = \sum_{j=0}^{(p-1)/2} \binom{p}{2j} x_0^{p-2j-1}(y_0^2 D)^j. \tag{8}$$

Assume q is any prime divisor of x_p/x_0. Then $q \mid x_0$, by assumption. By (8), we get $q \mid p(y_0^2 D)^{(p-1)/2}$. Since $(q, y_0 D) = 1$, we have $q = p$. Furthermore we claim that x_p/x_0 is square-free. Otherwise, $p^2 \mid x_p/x_0$. By (8) we have $p^2 \mid p(y_0^2 D)^{(p-1)/2}$. This is impossible. Therefore, $x_p/x_0 = p$. On the other hand, when $p > 3$, we find from (8) that

$$x_p/x_0 > p(p-1)/2 \geq p.$$

This contradicts $x_p/x_0 = p$. It shows that n has no prime divisor other than 3. Thus $n = 3^f, f \geq 1$. Therefore $x_3 \mid x_n$, and we have $3 = x_3/x_0 = x_0^2 + 3Dy_0^2 = 4x_0^2 - 3$, which is impossible.

Combining these results yields $n = 1$. The proof is complete.

Definition. Let p be a prime and n a nonzero integer. Then $\operatorname{ord}_p n$ is defined to be the unique nonnegative integer t such that $p^t \mid n$ and $p^{t+1} \nmid n$.

Lemma 4. *Let $n > 1$, p be the largest prime divisor of n, $\operatorname{ord}_p n = t \geq 1$, $r \in \mathbb{N}$, $1 \leq r \leq t$. Then $(2^{p^r} - 1, (2^n - 1)/(2^{p^r} - 1)) = 1$.*

Proof. Let $n = p^r s$. If $(2^{p^r} - 1, (2^n - 1)/(2^{p^r} - 1)) > 1$, then there is an odd prime q such that $2^{p^r} \equiv 1 \pmod q$, $(2^n - 1)/(2^{p^r} - 1) \equiv 0 \pmod q$. We can find $s \equiv 0 \pmod q$ and so $q \mid n$. Since $2^{q-1} \equiv 1 \pmod q$, we have $q \mid (2^{p^r} - 1, 2^{q-1} - 1) = 2^{(p^r, q-1)} - 1$ whence $(p^r, q-1) > 1$. It follows that $p \mid q - 1$, whence $q > p$, which is not true because p is the largest prime divisor of n. The lemma is proved.

3. Proofs

Proof of Theorem 1. By (2), we have

$$a^n x(x^{m-1} - 1) = (a^n x - 1)y^n. \tag{9}$$

Assume first $x > 1$. Let $x = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ be the prime decomposition of x, $r \geq 1$, $\alpha_i \geq 1$ such that $\operatorname{ord}_{p_i} x = \alpha_i$. Since $(x, x^{m-1} - 1) = (x, a^n x - 1) = 1$, we have $\operatorname{ord}_{p_i}(x^{m-1} - 1) = \operatorname{ord}_{p_i}(a^n x - 1) = 0$. From (9) we get $n \operatorname{ord}_{p_i} a + \alpha_i = n \operatorname{ord}_{p_i} y$. Then $\alpha_i = n(\operatorname{ord}_{p_i} y - \operatorname{ord}_{p_i} a) = n v_i$ for all i. Put $u = p_1^{v_1} \ldots p_r^{v_r}$. We obtain $x = u^n$. Replacing x by u^n in (9), we get

$$(au)^n (u^{n(m-1)} - 1) = ((au)^n - 1)y^n. \tag{10}$$

Since $((au)^n, (au)^n - 1) = 1$, we get $au \mid y$ from (10). Replace y by auy_1 in (10). Then

$$u^{n(m-1)} - ((au)^n - 1)y_1^n = 1, \quad u > 1,\ y_1 > 0. \tag{11}$$

(1) If $2\,|\,n$, we see from (11) that $(u^{n(m-1)/2}, y_1^{n/2})$ is a solution of Pell's equation $X^2-((au)^n-1)Y^2=1$. But $((au)^{n/2},1)$ is its fundamental solution. By Lemma 3 we get $u^{n(m-1)/2}=(au)^{n/2}$, $y_1=1$. It follows that $a=u^{m-2}$, $y=u^{m-1}$. Thus in this case, the assertion of Theorem 1 is true.

(2) If $n=3$, by (11) we see that $(u^{m-1},-y_1)$ is a solution of the equation $X^3+((au)^3-1)Y^3=1$. But $(au,-1)$ also is a solution of this equation. By Lemma 2, we get $u^{m-2}=a$, $y_1=1$. Hence in this case, the assertion of Theorem 1 is also true.

(3) If $2\nmid n$ and $n\ge 5$, we find from (11) that (u^{m-1},y_1) is a solution of the equation

$$X^n-((au)^n-1)Y^n=1,\qquad X,Y\in\mathbb{N},\ n\ge 5,\ 2\nmid n. \tag{12}$$

By Lemma 1, (12) has at most one solution (X,Y) except possibly when $(au)^n-1=2^n\pm 1$ and $n=5$ or $(au)^n-1=2$. The latter is impossible.

When $n=5$, $(au)^n-1=2^n+1$ is clearly not true.

When $n=5$ and $(au)^n-1=2^n-1$, we have $a=1$, $u=2$. Thus, by (11),

$$2^{5(m-1)}-1=31y_1^5. \tag{13}$$

Put $s=m-1$. Let p be the largest prime divisor of s. Let $p\ne 5$. By Lemma 4, $(2^p-1,(2^s-1)/(2^p-1))=1$. Notice that $(2^{5s}-1)/(2^s-1)\equiv 2^{4s}+2^{3s}+2^{2s}+2^s+1\equiv 5 \pmod{2^p-1}$ and $(2^p-1,5)=1$. We get $(2^p-1,(2^{5s}-1)/(2^p-1))=1$. Therefore we find from (13) that $2^p-1=z_1^5$, which is impossible by [2]. Let $p=5$. By Lemma 4, $(2^{25}-1,(2^{5s}-1)/(2^{25}-1))=1$. We find from (13) that $2^{25}-1=31z_1^5$. Then $z_1^5=1082401$, which is impossible. We conclude that $(au,1)$ is the only solution of (12). Hence $u^{m-1}=au$, $y_1=1$. Thus $a=u^{m-2}$, $y=u^{m-1}$.

Combining the above results, Theorem 1 is proved in the case of $x>1$.

Consider the case of $x<-1$. When $2\,|\,m$ and $2\,|\,n$, the equation (2) clearly has no solution. When $2\nmid m$, putting $x=-x_1$, the problem is changed into the case $x>1$ of Theorem 2 and we refer to the proof of Theorem 2. When $2\,|\,m$ and $2\nmid n$, i.e. $m=2k$ and $n=2l-1$ with $k,l\in\mathbb{N}>1$, as at the beginning of the proof, we may write $x=-u^n$, $y=-auy_1$ with $u>1$, $y_1,u\in\mathbb{N}$. By (2), we get

$$u^{n(m-1)}-((au)^n+1)y_1^n=-1. \tag{14}$$

If $n=3$, we find from (14) that $(-u^{m-1},y_1)$ is a solution of the equation $X^3+((au)^3+1)Y^3=1$. But so is $(-au,1)$. By Lemma 2, $a=u^{m-2}$, $y=-u^{m-1}$.

If $n\ge 5$, we find from (14) that (u^{m-1},y_1) is a solution of the equation

$$X^n-((au)^n+1)Y^n=-1. \tag{15}$$

By Lemma 1, (15) has at most one solution except possibly when $n=5$ and $(au)^n+1=2^n\pm 1$ or $(au)^n+1=2$. The latter is impossible.

Of course $(au, 1)$ is a solution of (15). This yields $u^{m-1} = au, y_1 = 1$, whence $a = u^{m-2} = u^{2k-2}$, $y = -u^{m-1} = -u^{2k-1}$. Since $(au)^n + 1 = 2^n - 1$ is clearly not true, there can only be an additional solution if $n = 5$ and $(au)^n + 1 = 2^n + 1$. This implies $a = 1$, $u = 2$ whence $x = -32$. So (2) reduces to the equation

$$\frac{(-32)^m - 1}{-32 - 1} = (-2y_1)^5 + 1 \quad \text{with } m \text{ even, } y_1 > 0$$

which we rewrite as $33y_1^5 - 32(2^{m-2})^5 = 1$. According to [1] the equation $33X^5 - 32Y^5 = 1$ has only the solution $X = Y = 1$. Thus $y_1 = 1$, $m = 2$, contrary to the assumption $m > 2$. This completes the proof of Theorem 1.

Proof of Theorem 2. Suppose $x > 1$. As in the proof of Theorem 1, we may put $x = u^n$, $y = auy_1$, $u > 1$, $y_1 > 0$. By (2), we get

$$u^{n(m-1)} - ((au)^n + 1)y_1^n = 1. \tag{16}$$

If $2 \mid n$, we find from (16) that $(u^{n(m-1)/2}, y_1^{n/2})$ is a solution of Pell's equation $X^2 - ((au)^n + 1)Y^2 = 1$. But $(2(au)^n + 1, 2(au)^{n/2})$ is its fundamental solution. So we get

$$u^{n(m-1)/2} + y_1^{n/2}\sqrt{(au)^n + 1} = [2(au)^n + 1 + 2(au)^{n/2}\sqrt{(au)^n + 1}]^s, \quad s \ge 1.$$

When $2 \nmid s$, we find that

$$u^{n(m-1)/2} = \sum_{j=0}^{(s-1)/2} \binom{s}{2j} (2(au)^n + 1)^{s-2j} (4(au)^n((au)^n + 1))^j.$$

Thus $u \mid [2(au)^n + 1]^s$, which is impossible.

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Thus $u \mid [2(au)^n + 1]^s$, which is impossible. In this case, the assertion of Theorem 2 is true.

If $n = 3$, we find from (16) that $(u^{m-1}, -y_1)$ is a solution of the equation $X^3 + ((au)^3 + 1)Y^3 = 1$. But so is $(-au, 1)$. By Lemma 2, $y_1 = -1$, which is impossible.

If $n \ge 5$ and $2 \nmid n$, we find from (16) that (u^{m-1}, y_1) is a solution of the equation

$$X^n - ((au)^n + 1)Y^n = 1. \tag{17}$$

Notice that $(au, 1)$ is a solution of the equation $X^n - ((au)^n + 1)Y^n = -1$. By Lemma 1, we see that either $n = 5$ and $(au)^n + 1 = 2^n \pm 1$ or $(au)^n + 1 = 2$. The latter is impossible.

When $n = 5, (au)^n + 1 = 2^n - 1$ is clearly not true.

When $n=5$ and $(au)^n+1=2^n+1$, we have $a=1$, $u=2$.

Thus, by (16),

$$2^{5(m-1)}-1=33y_1^5. \tag{18}$$

If $2\mid m$, we find from (18) that $\left(\frac{2}{3}\right)=1$, which is not true because $\left(\frac{2}{3}\right)=-1$. If $2\nmid m$, put $2s=5(m-1)$. Since $(2^s-1,2^s+1)=1$ and $33\mid 2^s+1$, we find from (18) that $2^s-1=z_1^5, 2^s+1=33z_2^5$. However $2^s-1=z_1^5$ is not true by [4]. We find that (16) is not true if $n\geq 5$ and $2\nmid n$.

Combining the above results, Theorem 2 is proved in the case $x>1$.

Consider the case $x<-1$. When $2\nmid m$, putting $x=-x_1$, the problem is changed into the case $x>1$ of Theorem 1 and we refer to the proof of Theorem 1. When $2\mid m$ and $2\mid n$, the equation (2) clearly has no solution. When $2\mid m$ and $2\nmid n$, as at the beginning of the proof we may write $x=-u^n$, $y=-auy_1$, $u>1$, $y_1>0$. We see from (2) that

$$u^{n(m-1)}-((au)^n-1)y_1^n=-1. \tag{19}$$

If $n=3$, we find from (19) that $(-u^{m-1},y_1)$ is a solution of the equation $X^3+((au)^n-1)Y^3=1$. But so is $(au,-1)$. By Lemma 2, $y_1=-1$, which contradicts $y_1>0$.

If $n\geq 5$, we find from (19) that (u^{m-1},y_1) is a solution of the equation

$$X^n-((au)^n-1)Y^n=-1. \tag{20}$$

Notice that $(au,1)$ is a solution of the equation $X^n-((au)^n-1)Y^n=1$. By Lemma 1, we deduce that either $n=5$ and $(au)^n-1=2^n\pm 1$ or $(au)^n-1=2$. The latter is impossible.

When $n=5, (au)^n-1=2^n+1$ is clearly not true.

When $n=5$ and $(au)^n-1=2^n-1$, we have $a=1$ and $u=2$.

We see from (19) that

$$2^{5(m-1)}+1=31y_1^5.$$

Therefore $1=\left(\frac{2}{31}\right)=\left(\frac{2^{5(m-1)}}{31}\right)=\left(\frac{-1}{31}\right)=-1$, which is impossible.

This completes the proof of Theorem 2.

By putting $a=1$ in Theorem 1 and in Theorem 2, we obtain Corollary 1 and Corollary 2, respectively.

Remark 1. By using the same method, it can be proved that the equation

$$\frac{a^nx^m-1}{a^nx-1}=y^n, \quad a,m,n\in\mathbb{N},\ x,y\in\mathbb{Z},\ |x|>1,\ m>2,\ n>1,$$

has no solution (a,x,y,m,n) which makes x an nth perfect power.

2. Also, it can be proved that the equation

$$a^n x \frac{a^n x^{m-1} - 1}{a^n x - 1} = y^n, \qquad a, m, n \in \mathbb{N},\ x, y \in \mathbb{Z},\ |x| > 1,\ m > 2,\ n > 1,$$

has no solution.

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