A note on the Diophantine equation
\[(a^n x^m \pm 1)/(a^n x \pm 1) = y^n + 1\]

by

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1. Introduction. Let \(\mathbb{Z}, \mathbb{N}\) denote the sets of integers and positive integers respectively. Le Mao Hua [4] has proved that the following equation has no solution \((x, y, m, n)\):

\[(1) \quad \frac{x^m - 1}{x - 1} = y^n + 1, \quad x, y, m, n \in \mathbb{N}, \quad x > 1, \quad y > 1, \quad m > 2, \quad n \text{ an odd prime.}\]

In this note, we investigate a more general equation applying another method. For

\[(2) \quad \frac{a^n x^m + \delta}{a^n x + \delta} = y^n + 1, \quad \delta \in \{1, -1\}, \quad x, y \in \mathbb{Z}, \quad a, m, n \in \mathbb{N}, \quad m > 2, \quad n > 1, \quad |x| > 1,\]

we prove the following results.

**Theorem 1.** For \(\delta = -1, x > 1\) all solutions of equation (2) are given by \((a, x, y, m, n) = (u^{m-2}, u^n, u^{m-1}, m, n)\) with \(u \in \mathbb{N} > 1\).

For \(\delta = -1, x < -1\) all solutions of equation (2) are given by \((a, x, y, m, n) = (u^{2k-2}, -u^{2l-1}, -u^{2k-1}, 2k, 2l - 1)\) with \(k, l, u \in \mathbb{N} > 1\).

**Theorem 2.** For \(\delta = 1, x > 1\) equation (2) has no solution.

For \(\delta = 1, x < -1\) all solutions of equation (2) are given by \((a, x, y, m, n) = (u^{2k-3}, -u^n, u^{2k-2}, 2k - 1, n)\) with \(k, n, u \in \mathbb{N} > 1\).

**Corollary 1.** The Diophantine equation

\[(3) \quad \frac{x^m - 1}{x - 1} = y^n + 1, \quad x, y \in \mathbb{Z}, \quad m, n \in \mathbb{N}, \quad |x| > 1, \quad m > 2, \quad n > 1,\]

has no solution \((x, y, m, n)\).

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Corollary 2. The Diophantine equation

\[
\frac{x^m + 1}{x + 1} = y^n + 1, \quad x, y \in \mathbb{Z}, \ m, n \in \mathbb{N}, \ |x| > 1, \ m > 2, \ n > 1,
\]
has no solution \((x, y, m, n)\).

Corollary 1 is a substantial generalization of Le’s result [4] for the equation (1).

2. Lemmas. Throughout this section, we assume that \(D\) and \(n\) are positive integers.

Lemma 1 ([3]). If \(n \geq 5\), then the equation

\[
X^n - DY^n = \pm 1, \quad X, Y \in \mathbb{N},
\]
has at most one solution \((X, Y)\) except possibly when \(D = 2\) or \(D = 2^n \pm 1\) and \(n \in \{5, 6\}\).

Lemma 2 ([5]). If \(D > 1\), then the equation

\[
X^3 + DY^3 = 1, \quad X, Y \in \mathbb{Z},
\]
has at most one solution \((X, Y)\) other than \(X = 1, Y = 0\).

Lemma 3. Let \(D\) be not a perfect square and \(x, y \in \mathbb{N}\) a solution of Pell’s equation

\[
x^2 - Dy^2 = 1, \quad x, y \in \mathbb{Z}.
\]

Let \((x_0, y_0)\) be the fundamental solution of (5) and \(\varepsilon = x_0 + y_0\sqrt{D}\). If \(x_0\)
is divisible by all prime divisors of \(x\), then \(x + y\sqrt{D} = \varepsilon\).

Proof. We may assume

\[
x + y\sqrt{D} = (x_0 + y_0\sqrt{D})^n, \quad n \geq 1.
\]
The result is clear for \(n = 1\). We assume \(n > 1\).

(1) If \(n\) is even, by (6) we get

\[
x = \sum_{j=0}^{n/2} \binom{n}{2j} x_0^{n-2j} (y_0^2 D)^j.
\]

Let \(p\) be a prime divisor of \(x\). Under our assumption \(p | x_0\). By (7) we get \(p | (Dy_0^2)^{n/2}\). This is impossible.

(2) If \(n\) is odd, we may write

\[
x + y\sqrt{D} = (x_0 + y_0\sqrt{D})^n = x_n + y_n\sqrt{D}.
\]

If \(r \mid n\), then \(x_r + y_r\sqrt{D} = (x_0 + y_0\sqrt{D})^r, x_r \mid x_n\). Therefore \(x_r\) also satisfies the assumption of the lemma. Now let \(p\) be any prime divisor of \(n\). We have

\[
x_p + y_p\sqrt{D} = (x_0 + y_0\sqrt{D})^p.
\]
Then
\[
x_p/x_0 = \sum_{j=0}^{(p-1)/2} \left(\frac{p}{2j}\right) x_0^{p-2j-1} (y_0^2 D)^j.
\]

Assume \( q \) is any prime divisor of \( x_p/x_0 \). Then \( q \mid x_0 \), by assumption. By (8), we get \( q \mid p(y_0^2 D)^{(p-1)/2} \). Since \( (q, y_0 D) = 1 \), we have \( q = p \). Furthermore we claim that \( x_p/x_0 \) is square-free. Otherwise, \( p^2 \mid x_p/x_0 \). By (8) we have \( p^2 \mid p(y_0^2 D)^{(p-1)/2} \). This is impossible. Therefore, \( x_p/x_0 = p \). On the other hand, when \( p > 3 \), we find from (8) that
\[
x_p/x_0 > p(p-1)/2 \geq p.
\]
This contradicts \( x_p/x_0 = p \). It shows that \( n \) has no prime divisor other than \( 3 \). Thus \( n = 3^f, f \geq 1 \). Therefore \( x_3 \mid x_n \), and we have \( 3 = x_3/x_0 = x_0^2 + 3Dy_0^2 = 4x_0^2 - 3 \), which is impossible.

Combining these results yields \( n = 1 \). The proof is complete.

**Definition.** Let \( p \) be a prime and \( n \) a nonzero integer. Then \( \text{ord}_p n \) is defined to be the unique nonnegative integer \( t \) such that \( p^t \mid n \) and \( p^{t+1} \nmid n \).

**Lemma 4.** Let \( n > 1 \), \( p \) be the largest prime divisor of \( n \), \( \text{ord}_p n = t \geq 1 \), \( r \in \mathbb{N} \), \( 1 \leq r \leq t \). Then \( (2^r - 1, (2^n - 1)/(2^r - 1)) = 1 \).

**Proof.** Let \( n = p^r s \). If \( (2^r - 1, (2^n - 1)/(2^r - 1)) > 1 \), then there is an odd prime \( q \) such that \( 2^r \equiv 1 \pmod{q} \), \( (2^n - 1)/(2^r - 1) \equiv 0 \pmod{q} \). We can find \( s \equiv 0 \pmod{q} \) and so \( q \mid n \). Since \( 2^r - 1 \equiv 1 \pmod{q} \), we have \( q \mid (2^r - 1, 2^q - 1) = 2^r(q-1) - 1 \) whence \( (p^r, q - 1) > 1 \). It follows that \( p \mid q - 1 \), whence \( q > p \), which is not true because \( p \) is the largest prime divisor of \( n \). The lemma is proved.

### 3. Proofs

**Proof of Theorem 1.** By (2), we have
\[
a^n x(x^{m-1} - 1) = (a^n x - 1)y^n.
\]
Assume first \( x > 1 \). Let \( x = p_1^{\alpha_1} \ldots p_r^{\alpha_r} \) be the prime decomposition of \( x \), \( r \geq 1 \), \( \alpha_i \geq 1 \) such that \( \text{ord}_{p_i} x = \alpha_i \). Since \( (x, x^{m-1} - 1) = (x, a^n x - 1) = 1 \), we have \( \text{ord}_{p_i} (x^{m-1} - 1) = \text{ord}_{p_i} (a^n x - 1) = 0 \). From (9) we get
\[
n \text{ord}_{p_i} a + \alpha_i = n \text{ord}_{p_i} y. \quad \text{Then } \alpha_i = n(\text{ord}_{p_i} y - \text{ord}_{p_i} a) = nu_i \text{ for all } i.
\]
Put \( u = p_1^{v_1} \ldots p_r^{v_r} \). We obtain \( x = u^n \). Replacing \( x \) by \( u^n \) in (9), we get
\[
(au)^n (u^n(m-1) - 1) = ((au)^n - 1) y^n.
\]
Since \( ((au)^n, (au)^n - 1) = 1 \), we get \( au \mid y \) from (10). Replace \( y \) by \( auy_1 \) in (10). Then
\[
u^{n(m-1)} - ((au)^n - 1)y_1^n = 1, \quad u > 1, y_1 > 0.
\]
(1) If $2 \mid n$, we see from (11) that $(u^{n(m-1)/2}, y_1^{n/2})$ is a solution of Pell’s equation $X^2 - ((au)^n - 1)Y^2 = 1$. But $((au)^n/2, 1)$ is its fundamental solution. By Lemma 3 we get $u^{n(m-1)/2} = (au)^{n/2}, y_1 = 1$. It follows that $a = u^{m-2}$, $y = u^{m-1}$. Thus in this case, the assertion of Theorem 1 is true.

(2) If $n = 3$, by (11) we see that $(u^{m-1}, -y_1)$ is a solution of the equation $X^3 + ((au)^3 - 1)Y^3 = 1$. But $(au, -1)$ also is a solution of this equation. By Lemma 2, we get $u^{m-2} = a, y_1 = 1$. Hence in this case, the assertion of Theorem 1 is also true.

(3) If $2 \nmid n$ and $n \geq 5$, we find from (11) that $(u^{m-1}, y_1)$ is a solution of the equation

\begin{equation}
X^n - ((au)^n - 1)Y^n = 1, \quad X, Y \in \mathbb{N}, \quad n \geq 5, \quad 2 \nmid n.
\end{equation}

By Lemma 1, (12) has at most one solution $(X, Y)$ except possibly when $(au)^n - 1 = 2^n \pm 1$ and $n = 5$ or $(au)^n - 1 = 2$. The latter is impossible.

When $n = 5$, $(au)^n - 1 = 2^n + 1$ is clearly not true.

When $n = 5$ and $(au)^n - 1 = 2^n - 1$, we have $a = 1, u = 2$. Thus, by (11),

\begin{equation}
2^{5(m-1)} - 1 = 31y_1^5.
\end{equation}

Put $s = m - 1$. Let $p$ be the largest prime divisor of $s$. Let $p \neq 5$. By Lemma 4, $(2^p - 1, (2^s - 1)/(2^p - 1)) = 1$. Notice that $(2^{5s} - 1)/(2^s - 1) \equiv 2^4 + 2^3 + 2^3 + 2^2 + 2 + 1 \equiv 5 \pmod{2^p - 1}$ and $(2^p - 1, 5) = 1$. We get $(2^p - 1, (2^{5s} - 1)/(2^p - 1)) = 1$. Therefore we find from (13) that $2^p - 1 = z_1^5$, which is impossible by [2]. Let $p = 5$. By Lemma 4, $(2^{25} - 1, (2^{5s} - 1)/(2^{25} - 1)) = 1$. We find from (13) that $2^{25} - 1 = 31z_1^5$. Then $z_1^5 = 1082401$, which is impossible. We conclude that $(au, 1)$ is the only solution of (12). Hence $u^{m-1} = au, y_1 = 1$. Thus $a = u^{m-2}, y = u^{m-1}$.

Combining the above results, Theorem 1 is proved in the case of $x > 1$.

Consider the case of $x < -1$. When $2 \mid m$ and $2 \nmid n$, the equation (2) clearly has no solution. When $2 \nmid m$, putting $x = -x_1$, the problem is changed into the case $x > 1$ of Theorem 2 and we refer to the proof of Theorem 2. When $2 \mid m$ and $2 \nmid n$, i.e. $m = 2k$ and $n = 2l - 1$ with $k, l \in \mathbb{N} > 1$, as at the beginning of the proof, we may write $x = -u^n, y = -auy_1$ with $u > 1, y_1, u \in \mathbb{N}$. By (2), we get

\begin{equation}
u^{n(m-1)} - ((au)^n + 1)y_1^n = -1.
\end{equation}

If $n = 3$, we find from (14) that $(-u^{m-1}, y_1)$ is a solution of the equation $X^3 + ((au)^3 + 1)Y^3 = 1$. But so is $(-au, 1)$. By Lemma 2, $a = u^{m-2}$, $y = -u^{m-1}$.

If $n \geq 5$, we find from (14) that $(u^{m-1}, y_1)$ is a solution of the equation

\begin{equation}
X^n - ((au)^n + 1)Y^n = -1.
\end{equation}

By Lemma 1, (15) has at most one solution except possibly when $n = 5$ and $(au)^n + 1 = 2^n \pm 1$ or $(au)^n + 1 = 2$. The latter is impossible.
Of course \((au, 1)\) is a solution of (15). This yields \(u^{m-1} = au, y_1 = 1\), whence \(a = u^{m-2} = u^{2k-2}, y = -u^{m-1} = -u^{2k-1}\). Since \((au)^n + 1 = 2^n - 1\) is clearly not true, there can only be an additional solution if \(n = 5\) and \((au)^n + 1 = 2^n + 1\). This implies \(a = 1, u = 2\) whence \(x = -32\). So (2) reduces to the equation

\[
\frac{(-32)^m - 1}{-32 - 1} = (-2y_1)^5 + 1 \quad \text{with } m \text{ even}, \ y_1 > 0
\]

which we rewrite as \(33y_1^5 - 32(2^{m-2})^5 = 1\). According to [1] the equation \(33X^5 - 32Y^5 = 1\) has only the solution \(X = Y = 1\). Thus \(y_1 = 1, m = 2,\) contrary to the assumption \(m > 2\). This completes the proof of Theorem 1.

Proof of Theorem 2. Suppose \(x > 1\). As in the proof of Theorem 1, we may put \(x = u^n, y = au^y_1, u > 1, y_1 > 0\). By (2), we get

\[(16) \quad u^{n(m-1)} - ((au)^n + 1)y_1^s = 1.\]

If \(2 \mid n\), we find from (16) that \((u^{n(m-1)/2}, y_1^{n/2})\) is a solution of Pell’s equation \(X^2 - ((au)^n + 1)Y^2 = 1\). But \((2(au)^n + 1, 2(au)^{n/2})\) is its fundamental solution. So we get

\[u^{n(m-1)/2} + y_1^{n/2}\sqrt{(au)^n + 1} = [2(au)^n + 1 + 2(au)^{n/2}\sqrt{(au)^n + 1}]^s, \quad s \geq 1.\]

When \(2 \nmid s\), we find that

\[u^{n(m-1)/2} = \sum_{j=0}^{(s-1)/2} \binom{s}{2j}(2(au)^n + 1)^{s-2j}(4(au)^n((au)^n + 1))^j.\]

Thus \(u \mid [2(au)^n + 1]^s\), which is impossible.

When \(2 \mid s\), we find that

\[u^{n(m-1)/2} = \sum_{j=0}^{s/2} \binom{s}{2j}(2(au)^n + 1)^{s-2j}(4(au)^n((au)^n + 1))^j.\]

Thus \(u \mid [2(au)^n + 1]^s\), which is impossible. In this case, the assertion of Theorem 2 is true.

If \(n = 3\), we find from (16) that \((u^{m-1}, -y_1)\) is a solution of the equation \(X^3 + ((au)^3 + 1)Y^3 = 1\). But so is \((-au, 1)\). By Lemma 2, \(y_1 = -1\), which is impossible.

If \(n \geq 5\) and \(2 \nmid n\), we find from (16) that \((u^{m-1}, y_1)\) is a solution of the equation

\[(17) \quad X^n - ((au)^n + 1)Y^n = 1.\]

Notice that \((au, 1)\) is a solution of the equation \(X^n - ((au)^n + 1)Y^n = -1\). By Lemma 1, we see that either \(n = 5\) and \((au)^n + 1 = 2^n \pm 1\) or \((au)^n + 1 = 2\). The latter is impossible.

When \(n = 5\), \((au)^n + 1 = 2^n - 1\) is clearly not true.
When \( n = 5 \) and \((au)^n + 1 = 2^n + 1\), we have \( a = 1, u = 2\).

Thus, by (16),
\[
2^{5(m-1)} - 1 = 33y_1^5.
\]

If \( 2 \mid m \), we find from (18) that \( \left( \frac{2}{3} \right) = 1 \), which is not true because \( \left( \frac{2}{3} \right) = -1 \).

If \( 2 \nmid m \), put \( 2s = 5(m-1) \). Since \((2^s - 1, 2^s + 1) = 1 \) and \( 33 \mid 2^s + 1 \), we find from (18) that \( 2^s - 1 = z_1^5, 2^s + 1 = 33z_2^5 \). However \( 2^s - 1 = z_1^5 \) is not true by [4]. We find that (16) is not true if \( n \geq 5 \) and \( 2 \nmid n \).

Combining the above results, Theorem 2 is proved in the case \( x > 1 \).

Consider the case \( x < 1 \). When \( 2 \nmid m \), putting \( x = -x_1 \), the problem is changed into the case \( x > 1 \) of Theorem 1 and we refer to the proof of Theorem 1. When \( 2 \mid m \) and \( 2 \nmid n \), the equation (2) clearly has no solution.

When \( 2 \mid m \) and \( 2 \mid n \), as at the beginning of the proof we may write \( x = -u^n, y = -auy_1, u > 1, y_1 > 0 \). We see from (2) that
\[
(n(m-1)) - ((au)^n - 1)y_1^n = -1.
\]

If \( n = 3 \), we find from (19) that \((-u^{-1}, y_1)\) is a solution of the equation \( X^3 + ((au)^n - 1)Y^3 = 1 \). But so is \((au, -1)\). By Lemma 2, \( y_1 = -1 \), which contradicts \( y_1 > 0 \).

If \( n \geq 5 \), we find from (19) that \((u^{-1}, y_1)\) is a solution of the equation
\[
X^n - ((au)^n - 1)Y^n = -1.
\]

Notice that \((au, 1)\) is a solution of the equation \( X^n - ((au)^n - 1)Y^n = 1 \). By Lemma 1, we deduce that either \( n = 5 \) and \((au)^n - 1 = 2^n \pm 1\) or \((au)^n - 1 = 2\). The latter is impossible.

When \( n = 5, (au)^n - 1 = 2^n + 1 \) is clearly not true.

When \( n = 5 \) and \((au)^n - 1 = 2^n - 1 \), we have \( a = 1 \) and \( u = 2 \).

We see from (19) that
\[
2^{5(m-1)} + 1 = 31y_1^5.
\]

Therefore \( 1 = \left( \frac{2}{31} \right) = \left( \frac{2^{5(m-1)}}{31} \right) = \left( \frac{-1}{31} \right) = -1 \), which is impossible.

This completes the proof of Theorem 2.

By putting \( a = 1 \) in Theorem 1 and in Theorem 2, we obtain Corollary 1 and Corollary 2, respectively.

**RemarK 1.** By using the same method, it can be proved that the equation
\[
\frac{a^n x^m - 1}{a^n x - 1} = y^n, \quad a, m, n \in \mathbb{N}, \ x, y \in \mathbb{Z}, \ |x| > 1, \ m > 2, \ n > 1,
\]
has no solution \((a, x, y, m, n)\) which makes \( x \) an \( n \)th perfect power.
2. Also, it can be proved that the equation
\[ a^n x \frac{a^n x^{m-1} - 1}{a^n x - 1} = y^n, \quad a, m, n \in \mathbb{N}, \; x, y \in \mathbb{Z}, \; |x| > 1, \; m > 2, \; n > 1, \]
has no solution.

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