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Optimal bound for the discrepancies of lacunary sequences

by

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1. Introduction. The discrepancies of a sequence $\{a_k\}$ of real numbers are defined by

$$D_N\{a_k\} = \sup_{0\le a<b<1}\left|\frac{1}{N}\#\{k\le N \mid \langle a_k\rangle \in [\,a,b)\} - (b-a)\right|,$$
$$D_N^*\{a_k\} = \sup_{0\le a<1}\left|\frac{1}{N}\#\{k\le N \mid \langle a_k\rangle \in [\,0,a)\} - a\right|,$$

where $\langle x\rangle$ denotes the fractional part $x-[\,x\,]$ of x. They are used to measure deviation of the distribution of the fractional parts of a_k from the uniform distribution. One can find a detailed survey on the theory of uniform distribution in [12].

The celebrated Chung–Smirnov Theorem [11, 28] states the following law of the iterated logarithm for a uniformly distributed i.i.d. sequence $\{U_k\}$:

$$\limsup_{N\to\infty}\frac{ND_N^*\{U_k\}}{\sqrt{2N\log\log N}} = \limsup_{N\to\infty}\frac{ND_N\{U_k\}}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.s.}$$

For a sequence $\{n_k\}$ of positive integers satisfying the Hadamard gap condition

$$n_{k+1}/n_k \ge q > 1, \tag{1.1}$$

Philipp [27] proved the following bounded law of the iterated logarithm by modifying the method due to Takahashi [30]: for almost every x,

$$\frac{1}{4\sqrt{2}} \le \limsup_{N\to\infty}\frac{ND_N^*\{n_kx\}}{\sqrt{2N\log\log N}} \le \limsup_{N\to\infty}\frac{ND_N\{n_kx\}}{\sqrt{2N\log\log N}} \le \frac{166}{\sqrt{2}} + \frac{664}{\sqrt{2}(q^{1/2}-1)}.$$

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Aistleitner [1] improved the estimates and replaced the lower bound and the upper bound by $1/2 - 8/q^{1/4}$ and $1/2 + 6/q^{1/4}$ when $q \geq 2$.

Recently, it was proved in [13] that these limsups with respect to the sequence $\{\theta^k x\}$ are equal to a constant for almost every x if $\theta > 1$. The constant is equal to the Chung–Smirnov constant $1/2$ when θ is not a power root of a rational number, and is greater than $1/2$ otherwise (cf. [16]). In the latter case, the constant can be concretely evaluated under some arithmetic condition. For example, when $\theta = q \geq 3$ is an odd integer, the constant is equal to $\frac{1}{2}\sqrt{(q+1)/(q-1)}$. Other sequences for which limsups have been concretely calculated can be found in [17–19, 23–25].

Aistleitner [1] gave a nearly optimal Diophantine condition on the sequence $\{n_k\}$ to have the Chung–Smirnov type result below. For positive integers N and d, and for a non-negative integer u, we denote the cardinality of

$$\big\{(j,j',k,k') \in [1,d]^2 \times [1,N]^2 \;\big|\; jn_k - j'n_{k'} = u\big\} \cap \big\{(j,j,k,k) \;\big|\; j,k \in \mathbb{N}\big\}^c$$

by $L_{N,d,u}$, and we put $L^*_{N,d} = \sup_{u\in\mathbb{N}} L_{N,d,u}$.

Theorem 1 (Aistleitner [1]). *Let $\{n_k\}$ be a sequence of positive integers satisfying the Hadamard gap condition* (1.1)*. Suppose that there exists an $\varepsilon > 0$ such that for any $d \in \mathbb{N}$,*

$$L_{N,d,0} \vee L^*_{N,d} = O(N/(\log N)^{1+\varepsilon}).$$

Then

$$\limsup_{N\to\infty} \frac{ND^*_N\{n_k x\}}{\sqrt{2N\log\log N}} = \limsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \frac{1}{2} \qquad a.e.$$

As Aistleitner [2, 3] constructed lacunary sequences for which the limsups are not constant a.e., and we can also find related examples in [15, 22], we are interested in giving a condition to have constant limsups. Since all limsups so far determined for lacunary sequences with (1.1) belong to

$$I_q = \left[\frac{1}{2}, \frac{1}{2}\sqrt{\frac{q+1}{q-1}}\,\right],$$

it is natural to expect the same bound for all lacunary sequences. Now we state our result.

Theorem 2. *Let $\{n_k\}$ be a sequence of positive integers satisfying the Hadamard gap condition* (1.1)*. Suppose that there exists an $\varepsilon \in (0,1)$ such* *that for all* $d \in \mathbb{N}$,

$$L^*_{N,d} = O\big(N/(\log N)^{1+\varepsilon}\big). \tag{1.2}$$

Then there exists a constant $\Sigma_{\{n_k\}}$ *such that*

$$\limsup_{N\to\infty} \frac{ND^*_N\{n_k x\}}{\sqrt{2N\log\log N}} = \limsup_{N\to\infty} \frac{ND_N\{n_k x\}}{\sqrt{2N\log\log N}} = \Sigma_{\{n_k\}} \in I_q \qquad a.e. \tag{1.3}$$

Moreover, if we assume

$$L_{N,d,0} = o(N) \qquad (N \to \infty) \tag{1.4}$$

together with (1.2) *for all* d, *then*

$$\Sigma_{\{n_k\}} = 1/2. \tag{1.5}$$

The estimate $\Sigma_{\{n_k\}} \in I_q$ in (1.3) is best possible when $q \geq 3$ is odd, since $\Sigma_{\{q^k\}}$ attains its upper bound and $\Sigma_{\{q^{k(k+1)}\}}$ attains its lower bound (see [13, 14]). It is also proved in [20] that the set of constants $\Sigma_{\{q^{m(k)}\}}$ for all subsequences $\{q^{m(k)}\}$ of $\{q^k\}$ coincides with I_q. Note that our condition to have (1.5) is weaker than that in the previous theorem.

At least $L_{N,d,u} = o(N)$ is necessary to have constant limsups, since limsup for star discrepancy is not constant for $\{2^k - 1\}$ and we have $N \ll L_{N,d,u}$ (see [22]). Our condition (1.2) is stronger than this, and it is open if it is necessary or not.

The condition (1.4) is necessary to have (1.5), since we have $\Sigma_{\{q^k\}} > 1/2$ and $L_{N,d,0} \gg N$ in this case.

To end the introduction, we mention a result of [21]. Suppose that $\{n_k\}$ is a sequence of non-zero real numbers such that $\{|n_k|\}$ satisfies the Hadamard gap condition (1.1). Then for any permutation ϖ of $\mathbb{N}$ (i.e. bijection $\mathbb{N} \to \mathbb{N}$), we have the bounded law of the iterated logarithm for the discrepancies of $\{n_{\varpi(k)} x\}$ with upper bound constant $\frac{1}{2}\sqrt{(q-1+4/\sqrt{3})/(q-1)}$, slightly greater than $\frac{1}{2}\sqrt{(q+1)/(q-1)}$. For other recent developments and studies on permuted sequences, see papers by Aistleitner, Berkes, and Tichy [4–9].

2. Proof. Let $\mathbf{1}_{[a,b)}$ be the indicator function of $[a,b)$, put

$$\widetilde{\mathbf{1}}_{[a,b)}(x) = \mathbf{1}_{[a,b)}(\langle x\rangle) - (b-a),$$

and denote by $\widetilde{\mathbf{1}}_{[a,b);d}$ the dth subsum of the Fourier series of $\widetilde{\mathbf{1}}_{[a,b)}$. Put

$$\rho^2_{q,d} = \frac{4}{d}\left(\log_q d + \frac{2q-1}{q-1}\right), \qquad \tau^2_{q,d} = \frac{1}{4}\,\frac{q+1}{q-1} + \frac{1}{2}\rho^2_{q,d}, \qquad \zeta^2_{q,d} = \frac{1}{4} - \frac{1}{2}\rho^2_{q,d}.$$

We first prove the following key inequalities:

$$\Big\|\sum_{k=M+1}^{M+N}\widetilde{\mathbf{1}}_{[a,b);d}(n_k\,\cdot\,)\Big\|_2^2 \le \Big\|\sum_{k=M+1}^{M+N}\widetilde{\mathbf{1}}_{[0,b-a);d}(n_k\,\cdot\,)\Big\|_2^2 + \rho_{q,d}^2 N, \tag{2.1}$$

$$\Big\|\sum_{k=M+1}^{M+N}\widetilde{\mathbf{1}}_{[a,b);d}(n_k\,\cdot\,)\Big\|_2^2 \le \tau_{q,d}^2 N, \qquad \Big\|\sum_{k=M+1}^{M+N}\widetilde{\mathbf{1}}_{[0,1/2);d}(n_k\,\cdot\,)\Big\|_2^2 \ge \zeta_{q,d}^2 N, \tag{2.2}$$

$$\Big|\Big\|\sum_{k=M+1}^{M+N}\widetilde{\mathbf{1}}_{[a,b);d}(n_k\,\cdot\,)\Big\|_2^2 - N\|\widetilde{\mathbf{1}}_{[a,b);d}\|_2^2\Big| \le L_{M+N,d,0} - L_{M,d,0}. \tag{2.3}$$

For $k \le k'$, by putting $P = n_k/\gcd(n_k, n_{k'})$ and $Q = n_{k'}/\gcd(n_k, n_{k'})$, we have $\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\widetilde{\mathbf{1}}_{[a,b)}(n_{k'} x)\,dx = \int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(Px)\widetilde{\mathbf{1}}_{[a,b)}(Qx)\,dx$. For coprime integers P and Q, we have (Lemma 1 of [13])

$$\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(Px)\widetilde{\mathbf{1}}_{[a,b)}(Qx)\,dx = \frac{\widetilde{V}(\langle Pa\rangle, \langle Pb\rangle, \langle Qa\rangle, \langle Qb\rangle)}{PQ},$$
$$\widetilde{V}(\langle Pa\rangle, \langle Pb\rangle, \langle Qa\rangle, \langle Qb\rangle) \le \widetilde{V}(0, \langle P(a-b)\rangle, 0, \langle Q(a-b)\rangle),$$
$$0 \le \widetilde{V}(0, \langle P/2\rangle, 0, \langle Q/2\rangle),$$

where $\widetilde{V}(x,y,\xi,\eta) = V(x,\xi) + V(y,\eta) - V(x,\eta) - V(y,\xi) \le 1/4$ and $V(x,\xi) = x\wedge\xi - x\xi$ for $0 \le x,\ y,\ \xi,\ \eta < 1$. Hence

$$\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\widetilde{\mathbf{1}}_{[a,b)}(n_{k'} x)\,dx \le \frac{1}{4PQ} \le \frac{P}{4Q} = \frac{n_k}{4n_{k'}} \le \frac{1}{4q^{k'-k}}, \tag{2.4}$$

$$\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\widetilde{\mathbf{1}}_{[a,b)}(n_{k'} x)\,dx \le \int_0^1 \widetilde{\mathbf{1}}_{[0,b-a)}(n_k x)\widetilde{\mathbf{1}}_{[0,b-a)}(n_{k'} x)\,dx, \tag{2.5}$$

$$\int_0^1 \widetilde{\mathbf{1}}_{[0,1/2)}(n_k x)\widetilde{\mathbf{1}}_{[0,1/2)}(n_{k'} x)\,dx \ge 0, \tag{2.6}$$

$$\int_0^1 \widetilde{\mathbf{1}}_{[0,1/2)}(n_k x)\widetilde{\mathbf{1}}_{[0,1/2)}(n_k x)\,dx = \widetilde{V}(0,1/2,0,1/2) = \frac{1}{4}. \tag{2.7}$$

Since

$$\Big\|\sum_{k=M+1}^{M+N}\widetilde{\mathbf{1}}_{[a,b)}(n_k\,\cdot\,)\Big\|_2^2 = \sum\nolimits^* (2-\delta_{k,k'})\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\widetilde{\mathbf{1}}_{[a,b)}(n_{k'} x)\,dx$$

where $\sum^*$ stands for the summation over k and k' satisfying $M+1 \le k \le k' \le M+N$, by applying (2.4) and $\sum^*(2-\delta_{k,k'})/4q^{k'-k} \le N\frac{1}{4}\frac{q+1}{q-1}$, we have the first inequality of

$$(2.8)\quad \Big\| \sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[a,b)}(n_k\,\cdot\,)\Big\|_2^2 \le N\,\frac{1}{4}\,\frac{q+1}{q-1}, \qquad \Big\| \sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[0,1/2)}(n_k\,\cdot\,)\Big\|_2^2 \ge \frac{N}{4},$$

while the second follows from (2.6) and (2.7). By (2.5), we can verify

$$(2.9)\qquad \Big\| \sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[a,b)}(n_k\,\cdot\,)\Big\|_2^2 \le \Big\| \sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[0,b-a)}(n_k\,\cdot\,)\Big\|_2^2.$$

From $\int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(Px)\widetilde{\mathbf{1}}_{[a,b);d}(Qx)\,dx = \int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(Px)\widetilde{\mathbf{1}}_{[a,b)}(Qx)\,dx$, we have

$$\begin{aligned} h_{k,k'} &:= \Big|\int_0^1 \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\widetilde{\mathbf{1}}_{[a,b)}(n_{k'} x)\,dx - \int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\widetilde{\mathbf{1}}_{[a,b);d}(n_{k'} x)\,dx\Big| \\ &= \Big|\int_0^1 (\widetilde{\mathbf{1}}_{[a,b)} - \widetilde{\mathbf{1}}_{[a,b);d})(Px)\widetilde{\mathbf{1}}_{[a,b)}(Qx)\,dx\Big| \le \sum_{|\lambda|\ge d/Q} \big|\widehat{\widetilde{\mathbf{1}}}_{[a,b)}(Q\lambda)\widehat{\widetilde{\mathbf{1}}}_{[a,b)}(-P\lambda)\big| \\ &\le \frac{2}{\pi^2 PQ}\sum_{\lambda\ge d/Q}\frac{1}{\lambda^2} \le \frac{2}{\pi^2 PQ}\Big(2\wedge\frac{2Q}{d}\Big) \le \frac{P}{Q}\wedge\frac{1}{d} = \frac{n_k}{n_{k'}}\wedge\frac{1}{d} \le \frac{1}{q^{k'-k}}\wedge\frac{1}{d}. \end{aligned}$$

Here we used $|\widehat{\widetilde{\mathbf{1}}}_{[a,b)}(j)| \le 1/\pi|j|$. Hence

$$\begin{aligned} &\bigg|\Big\| \sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k\,\cdot\,)\Big\|_2^2 - \Big\| \sum_{k=M+1}^{M+N} \widetilde{\mathbf{1}}_{[a,b)}(n_k\,\cdot\,)\Big\|_2^2\bigg| \le 2\sum\nolimits^* h_{k,k'} \\ &\qquad \le 2\sum\nolimits^* \frac{1}{q^{k'-k}}\wedge\frac{1}{d} \le 2N\sum_{l=0}^{\infty}\frac{1}{q^l}\wedge\frac{1}{d} = 2N\Big(\frac{l_0+1}{d} + q^{-(l_0+1)}\frac{q}{q-1}\Big) \\ &\qquad \le 2N\Big(\frac{\log_q d+1}{d} + \frac{1}{d}\,\frac{q}{q-1}\Big) \le \frac{\rho_{q,d}^2}{2}N, \end{aligned}$$

where l_0 is the largest integer satisfying $q^{-l_0} \ge 1/d$. By combining this with (2.8), we have (2.2), and with (2.9), we obtain (2.1). By summing

$$\begin{aligned} \Big|\int_0^1 \widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\widetilde{\mathbf{1}}_{[a,b);d}(n_{k'} x)\,dx\Big| &\le \sum_{0<|j|\le d}\ \sum_{0<|j'|\le d} \big|\widehat{\widetilde{\mathbf{1}}}_{[a,b)}(j)\widehat{\widetilde{\mathbf{1}}}_{[a,b)}(j')\big|\delta_{jn_k+j'n_k,0} \\ &\le \frac{2}{\pi^2}\sum_{j=1}^{d}\sum_{j'=1}^{d}\delta_{jn_k-j'n_k,0} \end{aligned}$$

over $M+1 \le k' < k \le M+N$, we see that the left hand side of (2.3) is bounded by $\#\{(j,j',k,k')\in[1,d]^2\times[M+1,M+N]^2 \mid jn_k - j'n_{k'} = 0,\ k<k'\}$ $\le L_{M+N,d,0} - L_{M,d,0}$.

Now we use a method of martingale approximation, which is a slight modification of the proof given in [1] and originated in Berkes–Philipp [10]. We regard $[0,1)$ equipped with the Borel field and the Lebesgue measure as a probability space.

First we recall two lemmas. The proofs can be found in [10], [13].

LEMMA 3. *If g is a bounded measurable function with period 1 satisfying $\int_0^1 g = 0$, then*

$$\Big|\int_a^b g(\lambda x)\,dx\Big| \le \|g\|_\infty/\lambda \quad \textit{for all } a<b \textit{ and } \lambda>0.$$

LEMMA 4. *Let g be a trigonometric polynomial with period 1 and degree d satisfying $\int_0^1 g = 0$. There exists a constant C_q depending only on q such that, for any sequence $\{n_k\}$ of positive integers satisfying the Hadamard gap condition* (1.1),

$$\int_0^1 \Big(\sum_{k=M+1}^{M+N} g(n_k x)\Big)^4 dx \le C_q \Big(\sum_{|\nu|\le d} |\widehat{g}(\nu)|\Big)^4 N^2.$$

Let us divide $\mathbb{N}$ into consecutive blocks $\Delta_1', \Delta_1, \Delta_2', \Delta_2, \ldots$ satisfying $\#\Delta_i' = [1 + 9\log_q i]$ and $\#\Delta_i = i$. Denote $i^- = \min \Delta_i$, $i^+ = \max \Delta_i$, and $l_M = \#\Delta_1 + \cdots + \#\Delta_M$. We have $M^- \sim M^+ \sim l_M = M(M+1)/2 \ll M^2$ and $n_{i^-}/n_{(i-1)^+} \ge q^{9\log_q i} = i^9$. Put

$$\mu(i) = [\log_2 i^4 n_{i^+}] + 1,$$
$$\mathcal{F}_i = \sigma\{[j2^{-\mu(i)}, (j+1)2^{-\mu(i)}) \mid j = 0, \ldots, 2^{\mu(i)} - 1\}.$$

Note that $i^4 n_{i^+} \le 2^{\mu(i)} \le 2i^4 n_{i^+}$. Denote $\widetilde{\mathbf{1}}_{[a,b);d}$ by f and put

$$T_i(x) = \sum_{k\in\Delta_i} f(n_k x), \quad T_i'(x) = \sum_{k\in\Delta_i'} f(n_k x), \quad Y_i = E(T_i \,|\, \mathcal{F}_i) - E(T_i \,|\, \mathcal{F}_{i-1}).$$

We also denote T_i and Y_i by $T_{[a,b);d;i}$ and $Y_{[a,b);d;i}$ to specify the parameters $[a,b)$ and d. Clearly $\{Y_i, \mathcal{F}_i\}$ forms a martingale difference sequence.

Let us prove

$$\|Y_i - T_i\|_\infty \ll 1/i^3, \tag{2.10}$$

$$\|Y_i^2 - T_i^2\|_\infty \ll 1/i^2, \tag{2.11}$$

$$\|Y_i^4 - T_i^4\|_\infty \ll 1. \tag{2.12}$$

Here and later, the constants implied by $\ll$ and O depend only on a, b, and d.

Suppose that $k \in \Delta_i$ and $x \in I = [\,j2^{-\mu(i)}, (j+1)2^{-\mu(i)}) \in \mathcal{F}_i$. In this case we have

$$\begin{aligned}|f(n_k x) - E(f(n_k\,\cdot\,)\,|\,\mathcal{F}_i)| &= |I|^{-1}\Big|\int_I (f(n_k x) - f(n_k y))\,dy\Big|\\ &\le \max_{y\in I}|f(n_k x) - f(n_k y))|\\ &\le \|f'\|_\infty n_k 2^{-\mu(i)} \le \|f'\|_\infty n_k/i^4 n_{i^+} \le \|f'\|_\infty/i^4.\end{aligned}$$

Hence $|T_i - E(T_i\,|\,\mathcal{F}_i)| \le \|f'\|_\infty \#\Delta_i/i^4 = \|f'\|_\infty/i^3$. Take $J = [j2^{-\mu(i-1)}, (j+1)2^{-\mu(i-1)}) \in \mathcal{F}_{i-1}$. Then by applying Lemma 3, we have

$$\begin{aligned}|E(f(n_k\,\cdot\,)\,|\,\mathcal{F}_{i-1})| = |J|^{-1}\Big|\int_J f(n_k y)\,dy\Big| &\le \|f\|_\infty 2^{\mu(i-1)}/n_k\\ &\le \|f\|_\infty 2(i-1)^4 n_{(i-1)^+}/n_{i^-} \le 2\|f\|_\infty/i^5.\end{aligned}$$

Therefore $|E(T_i\,|\,\mathcal{F}_{i-1})| \le 2\|f\|_\infty \#\Delta_i/i^5 = 2\|f\|_\infty/i^4$, and (2.10) is proved.

From $\|T_i\|_\infty \le i\|f\|_\infty$, we have $\|E(T_i\,|\,\mathcal{F}_i)\|_\infty, \|E(T_i\,|\,\mathcal{F}_{i-1})\|_\infty \le i\|f\|_\infty$. Hence $\|Y_i\|_\infty \le 2i\|f\|_\infty$, $\|Y_i + T_i\|_\infty \le 3i\|f\|_\infty$, $\|Y_i^2 + T_i^2\|_\infty \le 5i^2\|f\|_\infty^2$. Because $\|Y_i^2 - T_i^2\|_\infty \le \|Y_i - T_i\|_\infty \|Y_i + T_i\|_\infty$ and $\|Y_i^4 - T_i^4\|_\infty \le \|Y_i^2 - T_i^2\|_\infty \|Y_i^2 + T_i^2\|_\infty$, we obtain (2.11) and (2.12).

Put $\widetilde{\mathbf{1}}_{[a,b);d} = \sum_{j=1}^d (a_j \cos 2\pi jx + b_j \sin 2\pi jx)$, $v_i = v_{[a,b);d;i} = \int_0^1 T_{[a,b);d;i}^2$, $\beta_M = \beta_{[a,b);d;M} = v_{[a,b);d;1} + \cdots + v_{[a,b);d;M}$, and $V_M = \sum_{i=1}^M E(Y_i^2\,|\,\mathcal{F}_{i-1})$. Set

$$\begin{aligned}\Phi_i &= \{(k,k',j,j',\varsigma) \mid k,k' \in \Delta_i,\ j,j' = 1,\ldots,d,\ \varsigma = +1,-1\},\\ \Phi_i^v &= \{(k,k',j,j',\varsigma) \in \Phi_i \mid jn_k + \varsigma j' n_{k'} = 0\},\\ \Phi_i^U &= \{(k,k',j,j',\varsigma) \in \Phi_i \mid 0 < |jn_k + \varsigma j' n_{k'}| < n_{(i-1)^+}\},\\ \Phi_i^W &= \{(k,k',j,j',\varsigma) \in \Phi_i \mid n_{(i-1)^+} \le |jn_k + \varsigma j' n_{k'}| < n_{i^-}\},\\ \Phi_i^R &= \{(k,k',j,j',\varsigma) \in \Phi_i \mid n_{i^-} \le |jn_k + \varsigma j' n_{k'}|\}.\end{aligned}$$

For $\Psi \subset \Phi_i$, denote $\chi(\Psi) = \sum_{(k,k',j,j',\varsigma)\in\Psi} A_{k,k',j,j',\varsigma}$, where

$$\begin{aligned}2A_{k,k',j,j',\varsigma}(x) &= (a_j a_{j'} - \varsigma b_j b_{j'}) \cos 2\pi(jn_k + \varsigma j' n_{k'})x\\ &\quad + (\varsigma a_j b_{j'} + b_j a_{j'}) \sin 2\pi(jn_k + \varsigma j' n_{k'})x.\end{aligned}$$

We see $T_{[a,b);d;i}^2(x) = \chi(\Phi_i)$ and $v_{[a,b);d;i} = \chi(\Phi_i^v)$. Let $U_i = \chi(\Phi_i^U)$, $W_i = \chi(\Phi_i^W)$, and $R_i = \chi(\Phi_i^R)$. We can express Φ_i as a disjoint union $\Phi_i^v \cup \Phi_i^U \cup \Phi_i^W \cup \Phi_i^R$ and hence $T_i^2 = v_i + U_i + W_i + R_i$. We prove

$$\begin{aligned}(2.13)\qquad \|V_M - \beta_M\|_2 &\le \Big\|\sum_{i=1}^M E(Y_i^2 - T_i^2\,|\,\mathcal{F}_{i-1})\Big\|_2 + \Big\|\sum_{i=1}^M E(U_i\,|\,\mathcal{F}_{i-1})\Big\|_2\\ &\quad + \Big\|\sum_{i=1}^M E(W_i\,|\,\mathcal{F}_{i-1})\Big\|_2 + \Big\|\sum_{i=1}^M E(R_i\,|\,\mathcal{F}_{i-1})\Big\|_2 \ll M^2(\log M)^{-(1+\varepsilon)/2},\end{aligned}$$

where the first inequality is due to $Y_i^2 - v_i = (Y_i^2 - T_i^2) + U_i + W_i + R_i$.

By (2.11) we see $\| \sum_{i=1}^{M} E(Y_i^2 - T_i^2 \,|\, \mathcal{F}_{i-1})\|_2 = O(1)$.

From $\#\Phi_i^R \le \#\Phi_i \le 2d^2i^2$, $|a_ja_{j'} - \varsigma b_jb_{j'}|/2 \le 1$, and $|\varsigma a_jb_{j'} + b_ja_{j'}|/2 \le 1$, we obtain

$$|E(R_i \,|\, \mathcal{F}_{i-1})| \le 4d^2i^22^{\mu(i-1)}/n_{i^-} \le 8d^2/i^3$$

and $\| \sum_{i=1}^{M} E(R_i \,|\, \mathcal{F}_{i-1})\|_2 = O(1)$.

Let $k, k' \in \Delta_i$, $j, j' = 1, \dots, d$. Since $jn_k + j'n_{k'} \ge 2n_{i^-}$, we have $(k, k', j, j', +1) \notin \Phi_i^U \cup \Phi_i^W$. If $k \le k'$ and $n_{k'} > (d+1)n_k$, then $jn_k - j'n_{k'} \le dn_k - (d+1)n_k \le -n_{i^-}$. Hence $|jn_k - j'n_{k'}| < n_{i^-}$ implies $q^{k'-k} \le n_{k'}/n_k \le d+1$, that is, $k' - k \le \log_q(d+1)$. Therefore, if we fix k, j and j', then the number of k' such that $k \le k'$ and $|jn_k - j'n_{k'}| < n_{i^-}$ is at most $\log_q(d+1) + 1$. Therefore $\#(\Phi_i^U \cup \Phi_i^W) \le 2d^2(\log_q(d+1)+1)i$ and

$$\|U_i\|_\infty \ll i, \qquad \|W_i\|_\infty \ll i. \tag{2.14}$$

Hence $|E(W_i \,|\, \mathcal{F}_{i-1})| \le \|W_i\|_\infty \ll i$ and $\| \sum_{i=1}^{M} E(W_i \,|\, \mathcal{F}_{i-1})^2\|_\infty \ll M^3$. If $i < i'$, then

$$\begin{aligned} E\big(E(W_i \,|\, \mathcal{F}_{i-1})E(W_{i'} \,|\, \mathcal{F}_{i'-1}) \;\big|\; \mathcal{F}_{i-1}\big) &= E(W_i \,|\, \mathcal{F}_{i-1})E(W_{i'} \,|\, \mathcal{F}_{i-1}) \\ &= O(i)E(W_{i'} \,|\, \mathcal{F}_{i-1}) \end{aligned}$$

and $\big|E\big(E(W_i \,|\, \mathcal{F}_{i-1})E(W_{i'} \,\big|\, \mathcal{F}_{i'-1})\big)\big| \ll iE\big|E(W_{i'} \,\big|\, \mathcal{F}_{i-1})\big|$.

Since we can write

$$W_{i'}(x) = \sum_{u=n_{(i'-1)^+}}^{n_{(i')^-}} (c_u \cos 2\pi ux + d_u \sin 2\pi ux) \;\text{ with } \sum_{u=n_{(i'-1)^+}}^{n_{(i')^-}} (|c_u| + |d_u|) \ll i',$$

by Lemma 3 we obtain

$$\begin{aligned} \big|E(W_{i'} \,|\, \mathcal{F}_{i-1})\big| &\le \sum_u (|c_u| + |d_u|)2^{\mu(i-1)}/u \\ &\ll i'i^4n_{(i-1)^+}/n_{(i'-1)^+} \ll i'^5q^{(i-1)^+-(i'-1)^+} \ll i'^5q^{-i'}. \end{aligned}$$

Hence

$$\sum_{i<i'} \big|E\big(E(W_i \,|\, \mathcal{F}_{i-1})E(W_{i'} \,|\, \mathcal{F}_{i'-1})\big)\big| \ll \sum_{i<i'} ii'^5q^{-i'} \ll \sum_{i'} i'^7q^{-i'} \ll 1.$$

These imply $E(\sum_{i=1}^{M} E(W_i \,|\, \mathcal{F}_{i-1}))^2 \ll M^3$.

Since we can write

$$U_i(x) = \sum_{u=1}^{n_{(i-1)^+}} (c'_u \cos 2\pi ux + d'_u \sin 2\pi ux) \quad \text{with} \quad \sum_{u=1}^{n_{(i-1)^+}} (|c'_u| + |d'_u|) \ll i,$$

from $\big|E(\cos 2\pi u \cdot \,|\, \mathcal{F}_{i-1}) - \cos 2\pi ux\big| \le 2\pi u2^{-\mu(i-1)} \ll n_{(i-1)^+}/i^4n_{(i-1)^+} \ll$

$1/i^4$ and $|E(\sin 2\pi u \cdot \mid \mathcal{F}_{i-1}) - \sin 2\pi ux| \ll 1/i^4$, we have

$$\Big|\sum_{i=1}^{M} E(U_i \mid \mathcal{F}_{i-1}) - \sum_{i=1}^{M} U_i\Big| \ll \sum_{i=1}^{M} \sum_{u=1}^{n_{(i-1)^+}} (|c'_u| + |d'_u|)/i^4 \ll \sum_{i=1}^{M} \frac{1}{i^3} \ll 1.$$

We can write

$$\sum_{i=1}^{M} U_i(x) = \sum_{u=1}^{n_{(M-1)^+}} (c''_u \cos 2\pi ux + d''_u \sin 2\pi ux) \text{ with } \sum_{u=1}^{n_{(M-1)^+}} (|c''_u| + |d''_u|) \ll M^2,$$

and by (1.2) we have $|c''_u|, |d''_u| \le L_{M^+,d,u} \le L^*_{M^+,d} \ll M^2/(\log M)^{1+\varepsilon}$. Hence

$$\Big\|\sum_{i=1}^{M} U_i\Big\|_2^2 = \sum_{u=1}^{n_{(M-1)^+}} \frac{(c''_u)^2 + (d''_u)^2}{2} \ll \frac{M^2}{(\log M)^{1+\varepsilon}} \sum_{u=1}^{n_{(M-1)^+}} (|c''_u| + |d''_u|)$$
$$\ll \frac{M^4}{(\log M)^{1+\varepsilon}},$$

and $\|\sum_{i=1}^{M} E(U_i \mid \mathcal{F}_{i-1})\|_2 \ll M^2(\log M)^{-(1+\varepsilon)/2}$. Hence we obtain (2.13).

We choose another probability space on which a sequence $\{U, \xi_1, \xi_2, \ldots\}$ of independent random variables satisfying $P(\xi_k = 1) = P(\xi_k = -1) = 1/2$ and $P(U \in A) = |A \cap [0,1]|$ is defined. We take the product of $[0,1)$ on which $\{Y_i\}$ is defined and this new probability space, and regard Y_i, U, and $\Xi_i = \sum_{k\in\Delta_i} \xi_k$ as random variables on this product probability space. Take $m \in \mathbb{N}$ and define a martingale difference sequence $\{\widehat{Y}_i, \widehat{\mathcal{F}}_i\}$ on this space by putting $\widehat{\mathcal{F}}_i = \mathcal{F}_i \otimes \sigma\{\Xi_1, \ldots, \Xi_i\}$,

$$\widehat{Y}_i = \widehat{Y}_{[a,b);d;m;i} = Y_{[a,b);d;m;i} + \frac{1}{m}\Xi_i, \qquad \widehat{\beta}_M = \widehat{\beta}_{[a,b);d;m;M} = \beta_{[a,b);d;M} + \frac{1}{m^2} l_M.$$

By Lemma 4 and (2.12), we have $\|\widehat{Y}_i\|_4 \le \|Y_i\|_4 + \|\Xi_i\|_4 = \|T_i\|_4 + \|\Xi_i\|_4 + O(1) \ll i^{1/2}$, so $E\widehat{Y}_i^4 \ll i^2$. We have $E(\widehat{Y}_i^2 \mid \widehat{\mathcal{F}}_{i-1}) = E(Y_i^2 \mid \mathcal{F}_{i-1}) + m^{-2} i$ and hence

$$\widehat{V}_M := \sum_{i=1}^{M} E(\widehat{Y}_i^2 \mid \widehat{\mathcal{F}}_{i-1}) = V_M + \frac{1}{m^2} l_M \ge \frac{1}{m^2} l_M$$

and $\|\widehat{V}_M - \widehat{\beta}_M\|_2 \ll M^2(\log M)^{-(1+\varepsilon)/2}$. We now prove

$$\widehat{V}_M = \widehat{\beta}_M + o(\widehat{\beta}_M(\log \widehat{\beta}_M)^{-\varepsilon/4}) \quad \text{a.s.} \tag{2.15}$$

Note that $v_i \ll i$ by (2.2), so $\beta_M \ll M^2$, and hence $M^2 \ll \widehat{\beta}_M \ll M^2$. We also have $\beta_{M'} - \beta_M = \sum_{i=M+1}^{M'} v_i \ll M'(M' - M)$ and $\widehat{\beta}_{M'} - \widehat{\beta}_M \ll M'(M' - M)$. Put $\alpha = 1 - \varepsilon/2 + \varepsilon^2/4 < 1$ and $M_l = [\,2^{l^\alpha}\,]$. We have $(1 + \varepsilon/2)\alpha > 1$, $(\alpha - 1)/\alpha < \alpha - 1 < -\varepsilon/4$, and

$$M_{l+1}/M_l \sim 2^{\alpha l^{\alpha-1}} = 1 + O(l^{\alpha-1}) = 1 + O((\log M_l)^{(\alpha-1)/\alpha}) = 1 + o((\log M_l)^{-\varepsilon/4}),$$

so $M_{l+1} - M_l = o(M_l(\log M_l)^{-\varepsilon/4})$. Hence

$$\begin{aligned}0 \le \widehat{\beta}_{M_{l+1}} - \widehat{\beta}_{M_l} &\ll M_{l+1}(M_{l+1} - M_l)\\ &= o(M_l^2(\log M_l)^{-\varepsilon/4}) = o(\widehat{\beta}_{M_l}(\log \widehat{\beta}_{M_l})^{-\varepsilon/4}),\end{aligned}$$

that is, $\widehat{\beta}_{M_{l+1}}/\widehat{\beta}_{M_l} = 1 + o((\log M_l)^{-\varepsilon/4})$. Therefore

$$\sum_{l=1}^{\infty} E\bigg(\frac{\widehat{V}_{M_l} - \widehat{\beta}_{M_l}}{\widehat{\beta}_{M_l}(\log \widehat{\beta}_{M_l})^{-\varepsilon/4}}\bigg)^2 \ll \sum_{l=1}^{\infty}(\log M_l)^{-1-\varepsilon/2} \ll \sum_{l=1}^{\infty} l^{-\alpha(1+\varepsilon/2)} < \infty.$$

By Beppo Levi's theorem, we have $(\widehat{V}_{M_l} - \widehat{\beta}_{M_l})/\widehat{\beta}_{M_l}(\log \widehat{\beta}_{M_l})^{-\varepsilon/4} \to 0$ a.s., so $\widehat{V}_{M_l} - \widehat{\beta}_{M_l} = o(\widehat{\beta}_{M_l}(\log \widehat{\beta}_{M_l})^{-\varepsilon/4})$ a.s.

If $M_l \le M < M_{l+1}$, then

$$(\widehat{V}_{M_l} - \widehat{\beta}_{M_l}) + (\widehat{\beta}_{M_l} - \widehat{\beta}_{M_{l+1}}) \le \widehat{V}_M - \widehat{\beta}_M \le (\widehat{V}_{M_{l+1}} - \widehat{\beta}_{M_{l+1}}) + (\widehat{\beta}_{M_{l+1}} - \widehat{\beta}_{M_l})$$

and hence we have (2.15).

Now we use the following theorem by Monrad–Philipp [26] which is a modification of Strassen's theorem [29].

THEOREM 5. *Let $\{\widehat{Y}_i, \widehat{\mathcal{F}}_i\}$ be a square integrable martingale difference satisfying*

$$\widehat{V}_M = \sum_{i=1}^{M} E(\widehat{Y}_i^2 \,|\, \widehat{\mathcal{F}}_{i-1}) \to \infty \text{ a.s. and } \sum_{i=1}^{\infty} E\big(\widehat{Y}_i^2 \mathbf{1}_{\{\widehat{Y}_i^2 \ge \psi(\widehat{V}_i)\}}/\psi(\widehat{V}_i)\big) < \infty$$

for some non-decreasing ψ such that $\psi(\infty) = \infty$ and $\psi(x)(\log x)^\alpha/x$ is non-increasing for some $\alpha > 50$. If there exists a uniformly distributed random variable U which is independent of $\{\widehat{Y}_n\}$, then there exists a standard normal i.i.d. sequence $\{Z_i\}$ such that

$$\sum_{i\ge 1} \widehat{Y}_i \mathbf{1}_{\{\widehat{V}_i \le t\}} = \sum_{i \le t} Z_i + o\big(t^{1/2}(\psi(t)/t)^{1/50}\big) \quad (t \to \infty) \quad a.s.$$

Put $\psi(x) = x/(\log x)^{51}$. We can verify $\widehat{V}_M \ge m^{-2} l_M \to \infty$, and

$$\sum E\big(\widehat{Y}_i^2 \mathbf{1}_{\{\widehat{Y}_i^2 \ge \psi(\widehat{V}_i)\}}/\psi(\widehat{V}_i)\big) \le \sum \frac{E\widehat{Y}_i^4}{\psi^2(m^{-2}l_i)} \ll \sum i^2 (\log l_i)^{102}/l_i^2 < \infty.$$

Hence $\sum_{i=1}^{M} \widehat{Y}_i = \sum_{i \le \widehat{V}_M} Z_i + o\big(\widehat{V}_M^{1/2}(\log \widehat{V}_M)^{-51/50}\big)$ a.s. From (2.15) and

$$\sup_{0 \le |s| \le t(\log t)^{-\varepsilon/4}} |W_{t+s} - W_t| = O\big(t^{1/2}(\log t)^{-\varepsilon/8}(\log\log t)^{1/2}\big),$$

where $\{W_t\}$ is the Wiener process, we have

$$\sum_{i=1}^{M} \widehat{Y}_i = \sum_{i \le \widehat{\beta}_M} Z_i + O\big(\widehat{\beta}_M^{1/2}(\log \widehat{\beta}_M)^{-\varepsilon/9}\big) \quad \text{a.s.}$$

Hence by denoting $\phi(x) = \sqrt{2x \log\log x}$ and by applying the 0-1 law, we see that there exists a constant $C_{[a,b);d;m}$ such that

$$\limsup_{M\to\infty} \frac{1}{\phi(l_M)}\Big|\sum_{i=1}^{M} \widehat{Y}_{[a,b);d;m;i}\Big| = \limsup_{M\to\infty} \frac{1}{\phi(l_M)}\Big|\sum_{i\le\widehat{\beta}_{[a,b);d;m;M}} Z_i\Big| = C_{[a,b);d;m} \tag{2.16}$$

almost surely. Now we apply the following lemma with $\bar{v}_i = v_{[a,b);d;i} + i/m^2$, $\bar{v}'_i = v_{[0,b-a);d;i} + i/m^2$, $\bar{\beta}_M = \widehat{\beta}_{[a,b);d;M}$ and $\bar{\beta}'_M = \widehat{\beta}_{[0,b-a);d;M}$.

Lemma 6. *Let $\{Z_k\}$ and $\{Z'_k\}$ be standard normal i.i.d. sequences. Suppose that $\{\bar{v}_k\}$ and $\{\bar{v}'_k\}$ are sequences of positive numbers satisfying $c_1 i \le \bar{v}_i \le c_2 i$, $d_1 i \le \bar{v}'_i \le d_2 i$, and $\bar{v}_i \le \bar{v}'_i + \gamma i$ for some $0 < c_1 < c_2 < \infty$, $0 < d_1 < d_2 < \infty$, and $0 < \gamma < \infty$. Put $\bar{\beta}_M = \bar{v}_1 + \cdots + \bar{v}_M$ and $\bar{\beta}'_M = \bar{v}'_1 + \cdots + \bar{v}'_M$. Then*

$$\sqrt{c_1} \le \limsup_{M\to\infty} \frac{1}{\phi(l_M)}\Big|\sum_{k\le\bar{\beta}_M} Z_k\Big| \le \limsup_{M\to\infty} \frac{1}{\phi(l_M)}\Big|\sum_{k\le\bar{\beta}'_M} Z'_k\Big| + \sqrt{\gamma} \le \sqrt{d_2} + \sqrt{\gamma} \quad a.s.$$

By using conditions (2.1) and (2.2), we can verify the conditions of the lemma for $c_1 = d_1 = 1/m^2$, $c_2 = d_2 = \tau_{q,d}^2 + 1/m^2$, and $\gamma = \rho_{q,d}^2$, and we have

$$C_{[a,b);d;m} \le C_{[0,b-a);d;m} + \rho_{q,d} \le (\tau_{q,d}^2 + 1/m^2)^{1/2} + \rho_{q,d}.$$

Putting $\bar{v}_i = \bar{v}'_i = v_{[0,1/2);d;i} + i/m^2$ and $c_1 = c_2 = \zeta_{q,d}^2$, and d_1, d_2 as before, we obtain

$$C_{[0,1/2);d;m} \ge \zeta_{q,d}.$$

From

$$\left|\frac{1}{\phi(l_M)}\Big|\sum_{i=1}^{M} Y_{[a,b);d;i}\Big| - \frac{1}{\phi(l_M)}\Big|\sum_{i=1}^{M} \widehat{Y}_{[a,b);d;m;i}\Big|\right| \le \frac{1}{m\phi(l_M)}\Big|\sum_{i=1}^{M} \Xi_i\Big|,$$

we have

$$\left|\limsup_{M\to\infty} \frac{1}{\phi(l_M)}\Big|\sum_{i=1}^{M} Y_{[a,b);d;i}\Big| - C_{[a,b);d;m}\right| \le \frac{1}{m} \quad \text{a.s.}$$

Hence $C_{[a,b);d} = \lim_{m\to\infty} C_{[a,b);d;m}$ is a constant satisfying

$$\limsup_{M\to\infty} \frac{1}{\phi(l_M)}\Big|\sum_{i=1}^{M} Y_{[a,b);d;i}\Big| = C_{[a,b);d} \quad \text{a.s.}, \tag{2.17}$$

and

$$C_{[a,b);d} \le C_{[0,b-a);d} + \rho_{q,d} \le \tau_{q,d} + \rho_{q,d}, \qquad C_{[0,1/2);d} \ge \zeta_{q,d}.$$

Since Y_i is a function of x, by applying Fubini's theorem, we see that equality in (2.17) holds on $[0,1)$ and we can replace a.s. in (2.17) by a.e. By (2.10), we have $|\sum_{i=1}^{M} Y_{[a,b);d;i}| = |\sum_{i=1}^{M} T_{[a,b);d;i}| + O(1)$ and

$$\limsup_{M\to\infty} \frac{1}{\phi(l_M)} \Big|\sum_{i=1}^{M} T_{[a,b);d;i}\Big| = C_{[a,b);d} \quad \text{a.e.}$$

Because $\#\Delta'_1 + \cdots + \#\Delta'_M \ll M \log M$ and $l_M \sim M^+$, by applying the law of the iterated logarithm for lacunary trigonometric series, we have

$$\Big|\sum_{i=1}^{M} T'_{[a,b);d;i}\Big| \ll \sqrt{M \log M \log\log(M \log M)} = o(\phi(M^+)).$$

Therefore,

$$\limsup_{M\to\infty} \frac{1}{\phi(M^+)} \Big|\sum_{i=1}^{M} (T_{[a,b);d;i} + T'_{[a,b);d;i})\Big| = C_{[a,b);d} \quad \text{a.e.}$$

By noting $(M-1)^+ \sim M^+$ and $\max_{j=(M-1)^++1}^{M^+} |\sum_{k=(M-1)^++1}^{j} \widetilde{\mathbf{1}}_{[a,b);d}| \ll M = o(\phi(M^+))$, we have

$$\limsup_{N\to\infty} \frac{1}{\phi(N)} \Big|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\Big| = C_{[a,b);d} \quad \text{a.e.}$$

Now we apply the following proposition, essentially proved in [13]. The proof of the first part can be found in [16], and the full proof in [21].

Proposition 7. *Let $\{n_k\}$ be a sequence of positive numbers satisfying the Hadamard gap condition. Then for any dense countable set $S \subset [0,1)$, we have*

$$\text{(2.18)} \quad \begin{aligned} \limsup_{N\to\infty} \frac{N D_N\{n_k x\}}{\sqrt{2N \log\log N}} &= \sup_{S \ni a < b \in S} \limsup_{N\to\infty} \frac{1}{\phi(N)} \Big|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\Big|, \\ \limsup_{N\to\infty} \frac{N D_N^*\{n_k x\}}{\sqrt{2N \log\log N}} &= \sup_{a\in S} \limsup_{N\to\infty} \frac{1}{\phi(N)} \Big|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[0,a)}(n_k x)\Big|, \end{aligned}$$

and

$$\text{(2.19)} \quad \limsup_{N\to\infty} \frac{1}{\phi(N)} \Big|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}(n_k x)\Big| = \lim_{d\to\infty} \limsup_{N\to\infty} \frac{1}{\phi(N)} \Big|\sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b);d}(n_k x)\Big|,$$

for almost every $x \in \mathbb{R}$.

Put $S = [0,1) \cap \mathbb{Q}$. By applying (2.19), we have

$$\limsup_{N\to\infty} \frac{1}{\phi(N)} \Big| \sum_{k=1}^{N} \widetilde{\mathbf{1}}_{[a,b)}(n_k x) \Big| = C_{[a,b)} := \lim_{d\to\infty} C_{[a,b);d} \qquad \text{a.e.},$$

where $C_{[a,b)} \le C_{[0,b-a)} \le \frac{1}{2}\sqrt{(q+1)/(q-1)}$, and $1/2 \le C_{[0,1/2)}$. By (2.18), we have (1.3).

Suppose that the condition (1.4) is assumed. By (2.3) we obtain

$$\begin{aligned}\beta_{[a,b);d;M} = \sum_{i=1}^{M} ET^2_{[a,b);d;i} &= \|\widetilde{\mathbf{1}}_{[a,b);d}\|_2^2 l_M + O(L_{M^+,d,0}) \\ &= \|\widetilde{\mathbf{1}}_{[a,b);d}\|_2^2 l_M + o(l_M),\end{aligned}$$

and so $\widehat{\beta}_{[a,b);d;m;M} \sim (\|\widetilde{\mathbf{1}}_{[a,b);d}\|_2^2 + 1/m^2) l_M$. Hence, by (2.16) we directly have

$$\begin{aligned}C_{[a,b);d;m} &= \limsup_{M\to\infty} \frac{\phi(\widehat{\beta}_{[a,b);d;m;M})}{\phi(l_M)} \frac{1}{\phi(\widehat{\beta}_{[a,b);d;m;M})} \Big| \sum_{i=1}^{[\widehat{\beta}_{[a,b);d;m;M}]} Z_i \Big| \\ &= \sqrt{\|\widetilde{\mathbf{1}}_{[a,b);d}\|_2^2 + 1/m^2}.\end{aligned}$$

Therefore $C_{[a,b);d} = \|\widetilde{\mathbf{1}}_{[a,b);d}\|_2$ and $C_{[a,b)} = \|\widetilde{\mathbf{1}}_{[a,b)}\|_2 \le \|\widetilde{\mathbf{1}}_{[0,1/2)}\|_2 = 1/2 = C_{[0,1/2)}$.

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