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Approximate formulae for $L(1,\chi)$

by

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I. Results. For a primitive Dirichlet character χ of conductor $q \neq 1$, the size of the quantity $L(1,\chi) = \sum_{n\geq 1}\chi(n)/n$ has received considerable attention and measures the oscillatory nature of χ, its modulus being small whenever χ oscillates a lot and large otherwise. We address here the problem of the upper bound and are concerned with results valid for all χ (as opposed to asymptotically in q where much better results are available). For this reason, we shall restrict our attention to upper bounds of the shape $\frac{1}{2}\log q + C$ and seek the better C. Recall that χ is called even or odd according as to whether $\chi(-1) = 1$ or -1. Theoretically as well as numerically there seems to be a definite distinction between the even and the odd character case. We ran computations as described in the seventh section of this paper, on the basis of which one can formulate the following conjectures:

$$\begin{cases} \max\limits_{\chi\,\text{even}} \left\{|L(1,\chi)| - \frac{1}{2}\log q\right\} \stackrel{?}{=} -0.32404\ldots, \\ \max\limits_{\chi\,\text{odd}} \left\{|L(1,\chi)| - \frac{1}{2}\log q\right\} \stackrel{?}{=} 0.51482\ldots, \end{cases} \tag{1.1}$$

the first one being reached by a character modulo 241 and the second one by a character modulo 311. The author also believes that

$$|L(1,\chi)| \stackrel{?}{\leq} \tfrac{1}{2}\log q \quad \text{for } q \geq 4310. \tag{1.2}$$

Our approach goes through approximate formulae and will thus still be of use for computing upper bounds of the shape $|L(1,\chi)| \leq \left(\frac{1}{4}+\varepsilon\right)\log q + C$.

Recall the definition of the Gauss sum:

$$\tau(\chi) = \sum_{a \bmod q} \chi(a)e(a/q) \tag{1.3}$$

where $e(z)$ stands for $\exp(2i\pi z)$. The modulus of the Gauss sum is well known when χ is primitive and given by $|\tau(\chi)| = \sqrt{q}$. Two classical formulae

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(cf. [Washington, Theorem 4.9]) tell us that

$$L(1,\chi)=\sum_{n\geq 1}\frac{\chi(n)}{n} \tag{1.4}$$
$$=-\frac{\tau(\chi)}{q}\cdot\begin{cases} 2\sum_{1\leq m\leq q/2}\overline{\chi}(m)\log\left|\sin\frac{\pi m}{q}\right| & \text{if } \chi(-1)=1,\\ i\pi\sum_{1\leq m\leq q/2}\overline{\chi}(m)\Big(1-\frac{2m}{q}\Big) & \text{if } \chi(-1)=-1.\end{cases}$$

These formulae are usually useless due to the number of terms in the summation. We start with approximate formulae:

Theorem. *Let χ be a primitive Dirichlet character of conductor $q>1$. Let $F:\mathbb{R}\to\mathbb{R}$ be such that $f(t)=F(t)/t$ is C^2 over $\mathbb{R}$ (even at 0), vanishes at $t=\pm\infty$ and its first and second derivatives belong to $\mathcal{L}^1(\mathbb{R})$. Assume further that F is even if χ is odd and that F is odd if χ is even. Then for any $\delta>0$, we have*

$$L(1,\chi)=\sum_{n\geq 1}\chi(n)\frac{1-F(\delta n)}{n}$$
$$+\frac{\chi(-1)\tau(\chi)}{q}\sum_{m\geq 1}\overline{\chi}(m)\int_{-\infty}^{\infty}\frac{F(t)}{t}e(mt/(\delta q))\,dt.$$

This theorem contains the existence and/or convergence of the sums and integrals that occur, though in a somewhat restricted sense (see Section II for more details) and in particular the summation over n should be replaced by $\sum_{n\geq 1}^{w}$ while $\int_{-\infty}^{\infty}$ stands for $\lim_{T\to\infty}\int_{-T}^{T}$. In applications, however, these questions will receive a clear answer.

We now have to choose the weight F. We looked at entire functions having a bounded exponential type, so as to control the contribution of the summation over m in the above formula. Taking for F an approximation of $\mathrm{sgn}(x)$ in the even case and of 1 in the odd case, we get

Proposition 1. *Set*

$$F_1(t)=\frac{\sin(\pi t)}{\pi}\bigg(\log 4+\sum_{n\geq 1}(-1)^n\Big(\frac{2n}{t^2-n^2}+\frac{2}{n}\Big)\bigg), \tag{1.5}$$

$$F_2(t)=1-\frac{\sin(\pi t)}{\pi t}. \tag{1.6}$$

Let χ be a primitive Dirichlet character of conductor $q>1$. Then

$$L(1,\chi)=\begin{cases}\displaystyle\sum_{n\geq 1}\frac{(1-F_1(\delta n))\chi(n)}{n}-\frac{2\tau(\chi)}{q}\sum_{1\leq m\leq \delta q/2}\overline{\chi}(m)\log\left|\sin\frac{\pi m}{\delta q}\right| & \\ \hfill (\chi(-1)=1), & \\ \displaystyle\sum_{n\geq 1}\frac{(1-F_2(\delta n))\chi(n)}{n}-\frac{i\pi\tau(\chi)}{q}\sum_{1\leq m\leq \delta q/2}\overline{\chi}(m)\Big(1-\frac{2m}{\delta q}\Big) & \\ \hfill (\chi(-1)=-1). &\end{cases}$$

Whereas F_1 comes directly from [Vaaler], it is less obvious but no less true that our choice of F_2 has been inspired by [Vaaler, the use of (2.2)].

On taking $\delta=1$, we recover the formulae (1.4) above. Taking δ to be around $1/\sqrt{q}$, we can use these formulae to get explicit upper bounds for $L(1,\chi)$ as shown in Section VI and get

$$|L(1,\chi)|\leq \tfrac{1}{2}\log q+\begin{cases}0.006 & \text{if } \chi(-1)=1,\ q\gg 1,\\ 0.9 & \text{if } \chi(-1)=-1,\ q\gg 1,\end{cases}\tag{1.7}$$

which should be compared with [Louboutin, 1993, 1996, 1998] where the author proved similar upper bounds but with the constant 0.023 for even characters and the constant 0.716 for odd ones. Our smoothing is thus (slightly) better in the case of even characters but worse otherwise. Part of the loss is due to the fact that we consider $\sum_n|1-F(\delta n)|/n$ and not $\sum_n(1-F(\delta n))/n$. We can eliminate this loss by taking F such that $1-F$ is non-negative on $\mathbb{R}_+$, the cost being a doubling of the exponential type. We get more complicated formulae:

Proposition 2. *Set*

$$F_3(t)=\Big(\frac{\sin(\pi t)}{\pi}\Big)^2\Big(\frac{2}{t}+\sum_{m\in\mathbb{Z}}\frac{\operatorname{sgn}(m)}{(t-m)^2}\Big),\tag{1.8}$$

$$j(t)=2\int_{|t|}^{1}(\pi(1-u)\cot(\pi u)+1)\,du,\tag{1.9}$$

$$F_4(t)=1-\Big(\frac{\sin(\pi t)}{\pi t}\Big)^2.\tag{1.10}$$

Let χ be a primitive Dirichlet character of conductor $q>1$. Then

$$L(1,\chi)=\begin{cases}\displaystyle\sum_{n\geq 1}\frac{(1-F_3(\delta n))\chi(n)}{n}-\frac{\tau(\chi)}{q}\sum_{1\leq m\leq \delta q}\overline{\chi}(m)j\Big(\frac{m}{\delta q}\Big) & \\ \hfill (\chi(-1)=1), & \\ \displaystyle\sum_{n\geq 1}\frac{(1-F_4(\delta n))\chi(n)}{n}+\frac{i\pi\tau(\chi)}{q}\sum_{1\leq m\leq \delta q}\overline{\chi}(m)\Big(1-\frac{m}{\delta q}\Big)^2 & \\ \hfill (\chi(-1)=-1). &\end{cases}$$

Here again F_3 comes from [Vaaler] while F_4 has been guessed by looking at [Vaaler, Theorem 10].

As corollaries and after a good amount of work to control the error terms we get:

Corollary 1. *Let χ be a primitive character of conductor q. Then*
$$|L(1,\chi)| \le \tfrac{1}{2}\log q + \begin{cases} 0 & (\chi(-1)=1),\\ 5/2-\log 6 & (\chi(-1)=-1).\end{cases}$$

This indeed improves on S. Louboutin's bound, though by a (fairly) marginal factor since $5/2-\log 6 = 0.7082\ldots$

As was kindly shown to us by S. Louboutin, we do in fact prove a more precise result (see (6.4) and (6.5)) and this provides us with a simple proof of a result of [Le].

Corollary 2 (Le). *For every real quadratic field of discriminant $q \ge 16$,*
$$h(\mathbb{Q}(\sqrt{q})) \le \tfrac{1}{2}\sqrt{q}.$$

Some more work shows that we can remove $1/2-1/(5\log q)$ to this upper bound.

Corollary 3. *Let χ be a primitive character of even conductor q. Then*
$$|L(1,\chi)| \le \tfrac{1}{4}\log q + \begin{cases} \frac{1}{2}\log 2 & (\chi(-1)=1),\\ \frac{5}{4}-\frac{1}{2}\log 3 & (\chi(-1)=-1).\end{cases}$$

The best possible constants in the situation of this corollary and for conductors not more than 3000 were computed to be $0.28758\ldots$ in the even case (reached for a character modulo 184) and $0.68725\ldots$ in the odd case (reached for a character modulo 104) while $\frac{1}{2}\log 2 = 0.34657\ldots$ and $\frac{5}{4}-\frac{1}{2}\log 3 = 0.70069\ldots$ Comparatively, the best constants published are due to S. Louboutin and equal respectively to $0.358\ldots$ and $0.704\ldots$

In the second section of this paper, we describe a Poisson summation formula for characters valid for functions that behave like $1/t$ and which is the main tool for proving the Theorem above. We then prove the two propositions. To get Corollary 1 in the case of even characters, we first get a bound $\frac{1}{2}\log q+\mathcal{O}(q^{-1/2})$ and further work is required to show that this $\mathcal{O}$ is indeed non-positive, work that relies on the material presented in Section IV.

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Note: recently, S. Louboutin improved his 0.023 to 0.00886 by using yet another method.

II. A somewhat extended Poisson summation formula. Our Fourier transform is normalized as follows:

$$\widehat{\psi}(u)=\int_{-\infty}^{\infty}\psi(t)e(ut)\,dt \quad \text{so that} \quad \psi(t)=\int_{-\infty}^{\infty}\widehat{\psi}(u)e(-ut)\,du.$$

The usual setting. Let us start with the usual Poisson formula, adapted to primitive characters by recalling a specialization of a lemma of [Berndt, Theorem 2.3].

Lemma 1. *Let ϑ be a function of period q. Let $\psi:\mathbb{R}\to\mathbb{R}$ be a compactly supported C^0 function. Then*

$$\sum_{n\in\mathbb{Z}}\vartheta(n)\psi(n)=\sum_{m\in\mathbb{Z}}\widehat{\vartheta}(m)\widehat{\psi}(m/q)$$

where

$$\widehat{\vartheta}(m)=\frac{1}{q}\sum_{a \bmod q}\vartheta(a)e(-am/q).$$

Proof. We split the first sum in arithmetic progressions modulo q, then apply the Poisson summation formula to each resulting sum, getting

$$\begin{aligned}\sum_{n\in\mathbb{Z}}\vartheta(n)\psi(n)&=\sum_{a \bmod q}\vartheta(a)\sum_{n\in\mathbb{Z}}\psi(a+qn)\\&=\sum_{a \bmod q}\vartheta(a)\sum_{m\in\mathbb{Z}}\frac{1}{q}\widehat{\psi}(m/q)e(-ma/q)\\&=\sum_{m\in\mathbb{Z}}\widehat{\psi}(m/q)\frac{1}{q}\sum_{a \bmod q}\vartheta(a)e(-ma/q)\end{aligned}$$

and the result follows. ■

In the case when ϑ is a primitive character χ with conductor q, we have

$$\widehat{\vartheta}(m)=\frac{1}{q}\sum_{a \bmod q}\chi(a)e(ma/q)=\frac{\chi(-1)\tau(\chi)}{q}\overline{\chi}(m), \tag{2.1}$$

whose modulus is $1/\sqrt{q}$ or 0.

The extended setting. Let w be a C^2 function on $\mathbb{R}$ such that w is identically 1 on $[-1/2,1/2]$ and identically 0 outside $[-1,1]$. For any $T\ge 1$, we set $w_T(t)=w(t/T)$. For any sequence $(c(n))_{n\in\mathbb{Z}}$, we denote by $\sum_n^w c(n)$ the limit if it exists of $\sum_n c(n)w_T(n)$ as T goes to infinity.

By an *admissible function* f, we mean a C^2 function on $\mathbb{R}$ which satisfies:

- $f(t)$ tends to 0 as $|t|$ goes to ∞,
- f' and f'' are in $L^1(\mathbb{R})$.

LEMMA 2. *Let f be an admissible function. Then*

$$\int_{-\infty}^{\infty} f(t)w_T(t)e(ut)\,dt \to \int_{-\infty}^{\infty} f(t)e(ut)\,dt \qquad (T\to\infty)$$

uniformly over every compact subset (of u's) $\subset \mathbb{R}\setminus\{0\}$.

Note: $\int_{-\infty}^{\infty} f(t)e(ut)\,dt = \lim_{T\to\infty}\int_{-T}^{T} f(t)e(ut)\,dt$ *for* $u\neq 0$.

Proof. We have

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(t)w_T(t)e(ut)\,dt\\
&\quad= -\frac{1}{2i\pi u}\int_{-\infty}^{\infty}(fw_T)'(t)e(ut)\,dt\\
&\quad= -\frac{1}{2i\pi u}\int_{-\infty}^{\infty} f'(t)w(t/T)e(ut)\,dt - \frac{1}{2i\pi uT}\int_{-\infty}^{\infty} f(t)w'(t/T)e(ut)\,dt\\
&\quad= -\frac{1}{2i\pi u}\int_{-\infty}^{\infty} f'(t)e(ut)\,dt + \mathcal{O}\biggl(\frac{1}{u}\Bigl(\int_{|t|\geq T/2}|f'(t)|\,dt + \max_{|t|\geq T/2}|f(t)|\Bigr)\biggr)
\end{aligned}$$

and the result readily follows. ∎

LEMMA 3. *Let f be admissible and χ be a primitive character modulo $q>1$. Then $\sum_n^w f(n)\chi(n)$ exists and*

$$\sum_{n\in\mathbb{Z}}^{w} f(n)\chi(n) = \frac{\chi(-1)\tau(\chi)}{q}\sum_{m\in\mathbb{Z}\setminus\{0\}}\overline{\chi}(m)\int_{-\infty}^{\infty} f(t)e(mt/q)\,dt.$$

Note: $\int_{-\infty}^{\infty} f(t)e(ut)\,dt = \lim_{T\to\infty}\int_{-T}^{T} f(t)e(ut)\,dt$ *for* $u\neq 0$.

Note that the condition $q>1$ is used to say that $\chi(0)=0$.

Proof. By Lemma 1 and (2.1), we get

$$\sum_{n\in\mathbb{Z}} f(n)w_T(n)\chi(n) = \frac{\chi(-1)\tau(\chi)}{q}\sum_{m\in\mathbb{Z}}\overline{\chi}(m)\int_{-\infty}^{\infty} f(t)w_T(t)e\bigl(mt/q\bigr)\,dt.$$

Now, for $u\neq 0$, we have

$$\begin{aligned}(2i\pi u)^2 \int_{-\infty}^{\infty} f(t)w_T(t)e(ut)\,dt\\ &= \int_{-\infty}^{\infty} (fw_T)''(t)e(ut)\,dt\\ &\ll \int_{-\infty}^{\infty} \left| f''w(t/T) + \frac{2f'(t)w'(t/T)}{T} + \frac{f(t)w''(t/T)}{T^2}\right| dt\\ &\ll \|f''\|_1 + \|f'\|_1/T + \|f\|_\infty/T \ll 1\end{aligned}$$

whence

$$\begin{aligned}\sum_{n\in\mathbb{Z}} f(n)w_T(n)\chi(n)\\ &= \frac{\chi(-1)\tau(\chi)}{q} \sum_{|m|\le M} \overline{\chi}(m) \int_{-\infty}^{\infty} f(t)w_T(t)e(mt/q)\,dt + \mathcal{O}_{f,q}(M^{-1})\\ &= \frac{\chi(-1)\tau(\chi)}{q} \sum_{|m|\le M} \overline{\chi}(m) \int_{-\infty}^{\infty} f(t)e(mt/q)\,dt + \mathcal{O}_{f,q}(M^{-1}+\varepsilon)\end{aligned}$$

if $T \ge T_0(M,\varepsilon)$, from which we infer that the sum over m has a limit as M goes to infinity (apply Cauchy's criteria: take $M' \ge M$ and in the above equality, $\varepsilon = M^{-1}$ and a common $T = \max(T_0(M,\varepsilon), T_0(M',\varepsilon))$), which in turn enables us to prove that the above sum over n equals

$$\frac{\chi(-1)\tau(\chi)}{q} \sum_{m\in\mathbb{Z}} \overline{\chi}(m) \int_{-\infty}^{\infty} f(t)e(mt/q)\,dt + \mathcal{O}_{f,q}(M^{-1}+\varepsilon) \tag{2.2}$$

and the lemma readily follows. ■

The Theorem is an easy consequence of Lemma 3.

In the context of this extended setting, we shall require the following lemma:

Lemma 4. *Let f be an admissible function. Assume there exists a function $g \in L^1(\mathbb{R})$ such that for all t, we have*

$$f(t) = \int_{-\infty}^{\infty} g(u)e(tu)\,du.$$

Then for all t where g is continuous, we have

$$\int_{-\infty}^{\infty} f(u)e(tu)\,du = \lim_{U\to\infty} \int_{-U}^{U} f(u)\,e(tu)\,du = g(t).$$

Proof. Set

$$h(t)=\lim_{U\to\infty}\int_{-U}^{U} f(u)\,e(tu)\,du=\lim_{U\to\infty}\int_{-U}^{U}\int_{-\infty}^{\infty} g(v)\,e(u(t-v))\,dv\,du$$
$$=\lim_{U\to\infty}\int_{-\infty}^{\infty} g(v)\,\frac{\sin(2\pi U(t-v))}{\pi(t-v)}\,dv.$$

Given $\varepsilon>0$, write

$$\int_{-\infty}^{\infty} g(t-v)\frac{\sin(2\pi Uv)}{\pi v}\,dv=\Big(\int_{-\varepsilon}^{\varepsilon}+\int_{-\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\Big)g(t-v)\frac{\sin(2\pi Uv)}{\pi v}\,dv.$$

The last two integrals tend to 0 with U^{-1}, while in the first one, replace $g(t-v)$ by $g(t)$ up to an error term which is small uniformly in U; then use $\int_{-\infty}^{\infty}\frac{\sin(2\pi Uv)}{\pi v}\,dv=1$. ∎

III. Computations of some Fourier transforms. Proof of Propositions 1 and 2. For $i\in\{1,2,3,4\}$ we set

$$\varrho_i(t)=\int_{-\infty}^{\infty}\frac{F_i(u)}{u}e(tu)\,du\qquad(t\neq 0)\tag{3.1}$$

where the integral is to be understood in the sense of Lemma 3. We now turn to the task of computing these four functions.

We rely heavily on [Vaaler, Lemma 1–Theorem 4] and [Vaaler, Lemma 5–Corollary 7].

LEMMA 5. *For $t\neq 0$, we have*

$$\frac{F_1(t)}{t}=-2\int_{-1/2}^{1/2}\log|\sin(\pi u)|e(-tu)\,du,$$

and $\varrho_1(t)=-2\log|\sin(\pi t)|\cdot\mathbb{1}_{[-1/2,1/2]}(t)$.

Proof. An integration by parts tells us that

$$A=-2\int_{-1/2}^{1/2}\log|\sin(\pi u)|e(-tu)\,du=-\frac{2\pi}{2i\pi t}\,\mathrm{pp}\int_{-1/2}^{1/2}\cot(\pi u)e(-tu)\,du,$$

where

$$\mathrm{pp}\int_{-1/2}^{1/2}f(u)\,du=\lim_{\varepsilon\to 0^+}\Big(\int_{-1/2}^{-\varepsilon}f(u)\,du+\int_{\varepsilon}^{1/2}f(u)\,du\Big).\tag{3.2}$$

We now use the identity (cf. formula (2.12) of [Vaaler])

$$i\cot(\pi u)=\sum_{|n|\le N}\operatorname{sgn}(n)e(nu)+i\frac{\cos(\pi(2N+1)u)}{\sin(\pi u)} \tag{3.3}$$

and thus

$$\begin{aligned}tA&=\sum_{|n|\le N}\operatorname{sgn}(n)\int_{-1/2}^{1/2}e((n-t)u)\,du+i\,\mathrm{pp}\int_{-1/2}^{1/2}\frac{\cos(\pi(2N+1)u)}{\sin(\pi u)}e(-tu)\,du\\&=\sum_{|n|\le N}\operatorname{sgn}(n)\frac{\sin(\pi(n-t))}{\pi(n-t)}+o(1)\to F_1(t)\end{aligned}$$

as N goes to infinity (cf. formulae (2.12) and (2.6) of [Vaaler]). The $o(1)$ above is got by using the Riemann–Lebesgue lemma: we first map the interval $[-1/2,0]$ onto $[0,1/2]$ by using $u\mapsto -u$ and then we are left with Fourier coefficients of bounded functions. The second statement comes from Lemma 4. ■

Lemma 6. *For $t\neq 0$, we have*

$$\frac{F_2(t)}{t}=\int_{-1/2}^{1/2}\varphi(u)e(tu)\,du$$

where

$$\varphi(t)=\begin{cases}-i\pi(1-2t), & 0\le t\le 1/2,\\ -i\pi(-1-2t) & -1/2\le t\le 0,\\ 0 & \textit{otherwise.}\end{cases}$$

Moreover $\varphi(-t)=-\varphi(t)$, and $\varrho_2=\varphi$.

Proof. This lemma follows from a direct examination and from Lemma 4. ■

We also have

Lemma 7. *Extend the function j of Proposition 2 to $t\notin[-1,1]$ by setting $j(t)=0$ for such t. Then j is non-negative in $L^1(\mathbb{R})$ and*

$$\frac{F_3(t)}{t}=\int_{-1}^{1}j(u)e(tu)\,du.$$

Moreover $-2\log|t|-2(\log(2\pi)-1)\le j(t)\le -2\log|t|$, $\int_0^1 j(u)\,du=1$ and $\varrho_3=j$.

Proof. We first bound $j(t)$. Notice that from $\cot(\pi u)\le 1/(\pi u)$ for $u\in[0,1]$ (in fact $\cot(\pi u)$ is non-positive over $[1/2,1]$), we infer

$$j(t) = 2\int_{|t|}^{1}\left(\frac{1-u}{u}+1\right)du - 2\int_{|t|}^{1}\pi(1-u)\left(\frac{1}{\pi u}-\cot(\pi u)\right)du \le -2\log|t|.$$

The same expression also yields

$$j(t) \ge -2\log|t| - 2\int_{0}^{1}\pi(1-u)\left(\frac{1}{\pi u}-\cot(\pi u)\right)du = -2\log|t| - 2(\log(2\pi)-1).$$

By [Vaaler, Lemma 5], the integrand in (1.9) is non-negative, thus proving that j shares this property. This also proves that j is in $L^1(\mathbb{R})$. We get (using (3.3) and notation (3.1))

$$\begin{aligned}
&\int_{-1}^{1} j(u)e(tu)\,du \\
&\quad = \frac{1}{2i\pi t}\,\mathrm{pp}\int_{-1}^{1}\frac{2\widehat{J}(u)}{u}e(tu)\,du \\
&\quad = \frac{1}{it}\,\mathrm{pp}\int_{-1}^{1}(1-|u|)\cot(\pi u)e(tu)\,du + \frac{1}{i\pi t}\int_{-1}^{1}\mathrm{sgn}(u)e(zu)\,du \\
&\quad = -\frac{1}{t}\int_{-1}^{1}(1-|u|)\sum_{|n|\le N}\mathrm{sgn}(n)e(nu)e(tu)\,du + \frac{o(1)}{t} + 2\left(\frac{\sin(\pi t)}{\pi t}\right)^2 \\
&\quad = \frac{F_3(t)}{t} + \frac{o(1)}{t}
\end{aligned}$$

where $o(1)$ is a function that goes to 0 as N goes to infinity, and where we have used

$$\int_{-1}^{1}(1-|u|)e(xu)\,du = \left(\frac{\sin(\pi x)}{\pi x}\right)^2. \quad \blacksquare$$

Lemma 8. *For $t \neq 0$, we have*

$$\frac{F_4(t)}{t} = 2\pi\int_{0}^{1}(u-1)^2\sin(2\pi tu)\,du,$$

and $\varrho_4(t) = 2\pi(t-1)^2\mathbb{1}_{[-1,1]}(t)$.

Proof. This lemma follows from a direct examination and from Lemma 4. $\blacksquare$

IV. From a sum to an integral for convex non-increasing functions. This section is in essence due to S. Louboutin and replaces a weaker result of the author.

LEMMA 9. *Let k and $\beta > 0$ be two real numbers. Let f be a continuous, convex and non-increasing L^1 function on $]k-\theta, k+\beta]$. We have*

$$f(k) \le \frac{1}{\theta}\int_{k-\theta}^{k} f(t)\,dt - \theta\frac{f(k)-f(k+\beta)}{2\beta}.$$

Proof. We present two proofs.

Geometrical proof: draw the graph of f in the (t,y)-plane. The line L that goes through $A=(k,f(k))$ and $(k+\beta, f(k+\beta))$ cuts the line $t=k-\theta$ at B. The area of the triangle $(A,B,(k-\theta,f(k)))$ is $\theta^2(f(k)-f(k+\beta))/(2\beta)$ from which the reader will conclude easily.

We give a more didactical proof. For t belonging to $]k-\theta,k]$, we have

$$k = \frac{\beta}{k+\beta-t}t + \frac{k-t}{k+\beta-t}(k+\beta)$$

so that, by convexity, we infer

$$f(k) \le \frac{\beta}{k+\beta-t}f(t) + \frac{k-t}{k+\beta-t}f(k+\beta)$$

or equivalently

$$(k-t)(f(k)-f(k+\beta)) + \beta f(k) \le \beta f(t) \qquad (t\in\,]k-\theta,k]).$$

Integrating this inequality over $t\in[k-\theta,k]$ yields the lemma. ∎

LEMMA 10. *Let $\alpha \ge 1$. Let g be a continuous, convex, non-negative and non-increasing L^1 function on $]0,1]$. Then*

$$\sum_{1\le m\le\alpha} g(m/\alpha) \le \alpha\int_0^1 g(t)\,dt - \frac{g(1/\alpha)-g(1)}{2}.$$

Proof. Let N be the integer part of α. From Lemma 9 with $\beta=\theta=1$, we infer

$$\sum_{1\le m\le N-1} g(m/\alpha) \le \int_0^{N-1} g(t/\alpha)\,dt - \frac{g(1/\alpha)-g(N/\alpha)}{2}. \tag{4.1}$$

We again use Lemma 9 but this time with $k=N$, $\theta=1$ and $\beta=\alpha-N$, and get

$$g(N/\alpha) \le \int_{N-1}^{N} g(t/\alpha)\,dt - \frac{g(N/\alpha)-g(1)}{2\beta},$$

which together with $\beta\le 1$ and (4.1) yields

$$\sum_{1\le m\le N} g(m/\alpha) \le \int_0^N g(t/\alpha)\,dt - \frac{g(1/\alpha)-g(1)}{2}.$$

We complete the integral by using the non-negativity of g and the lemma follows. ∎

LEMMA 11. *Let $\alpha > 0$. Let g be a continuous, convex, non-negative and non-increasing L^1 function on* $]0,1]$. *Then*

$$\sum_{\substack{1\le m\le\alpha\\(m,2)=1}} g(m/\alpha) \le \frac{\alpha}{2}\int_0^1 g(t)\,dt + \frac{g(1)}{2}.$$

Proof. Let N be the largest odd integer less than or equal to α. From Lemma 9 with $\theta = 2$ and $\beta = 1$, we infer

$$\sum_{\substack{3\le m\le N-2\\(m,2)=1}} g(m/\alpha) \le \frac{1}{2}\int_1^{N-2} g(t/\alpha)\,dt - (g(1/\alpha) - g(N/\alpha)). \tag{4.2}$$

We again use Lemma 9 but this time with $k = N$, $\theta = 2$ and $\beta = \alpha - N$, and get

$$g(N/\alpha) \le \frac{1}{2}\int_{N-2}^{N} g(t/\alpha)\,dt - \frac{g(N/\alpha) - g(1)}{\beta},$$

which together with $\beta \le 2$ and (4.2) yields

$$\sum_{\substack{3\le m\le N\\(m,2)=1}} g(m/\alpha) \le \frac{1}{2}\int_1^{N} g(t/\alpha)\,dt - \frac{g(1/\alpha) - g(1)}{2}.$$

We complete the integral from N to α by using the non-negativity of g. As for the integral from 0 to 1, it is greater than $g(1/\alpha)$ and the lemma follows. ∎

V. Auxiliary lemmas. We start with preliminary lemmas.

LEMMA 12. *For $i \in \{1,2\}$, we have $0 \le \mathrm{sgn}(\sin(\pi u))(1 - F_i(u)) \le |1/(\pi u)|$ and $|F_i'(u)| \ll u^{-2}$ for u real.*

Proof. The case $i = 2$ is trivial. We thus restrict our attention to the case $i = 1$. The first inequality is proved in [Vaaler]. In this same paper, formula (2.14) (the next one is false), we find that

$$\begin{aligned}F_1'(u) &= 2\int_{-1/2}^{1/2} \pi v\cot(\pi v)e(uv)\,dv = \frac{2}{2i\pi u}\int_{-1/2}^{1/2} \psi'(v)e(uv)\,dv\\ &= -\frac{2}{(2\pi u)^2}\Big(\psi'(1/2)e(u/2) - \psi'(-1/2)e(-u/2) + \int_{-1/2}^{1/2} \psi''(v)e(uv)\,dv\Big)\end{aligned}$$

with $\psi(v) = \pi v\cot(\pi v)$ and a careful analysis yields $|F_1'(u)| \le 1/u^2$. ∎

LEMMA 13. *For $i \in \{1,2\}$ and $0 < \delta \le 1$, we have*

$$\sum_{n\ge 1} \frac{|1-F_i(\delta n)|}{n} = \log \delta^{-1} + c_i + \mathcal{O}(\delta)$$

where

$$c_i = -\int_0^1 \frac{F_i(t)}{t}\,dt + \int_1^\infty \frac{|1-F_i(t)|}{t}\,dt + \gamma.$$

Proof. We first compare this sum to an integral. For the integer $N = [\delta^{-1}+1]$, we have

$$\begin{aligned}\delta \sum_{n\ge N} &\frac{|1-F_i(\delta n)|}{\delta n} \\ &= \int_{\delta N}^{\infty} \frac{|1-F_i(t)|}{t}\,dt + \sum_{n\ge N} \int_{\delta n}^{\delta(n+1)} \left(\frac{|1-F_i(\delta n)|}{\delta n} - \frac{|1-F_i(t)|}{t}\right) dt\end{aligned}$$

and we use

$$\begin{aligned}\left|\frac{|1-F_i(T)|}{T} - \frac{|1-F_i(t)|}{t}\right| &\le \left|\frac{1-F_i(T)}{T} - \frac{1-F_i(t)}{t}\right| \\ &\le (t-T) \max_{T\le u\le t} \left|\frac{d}{du}\frac{1-F_i(u)}{u}\right|\end{aligned}$$

for $T \le t$, which together with Lemma 12 yields the bound

$$\sum_{n\ge N} \frac{|1-F_i(\delta n)|}{n} = \int_{\delta N}^{\infty} \frac{|1-F_i(t)|}{t}\,dt + \mathcal{O}(\delta). \tag{5.1}$$

Now $\delta n \le 1$ for the remaining n's and we get

$$\begin{aligned}\sum_{1\le n\le N-1} \frac{|1-F_i(\delta n)|}{n} &= \sum_{1\le n\le N-1} \frac{1-F_i(\delta n)}{n} \\ &= \sum_{1\le n\le N-1} \frac{1}{n} - \int_{\delta}^{\delta N} \frac{F_i(t)}{t}\,dt \\ &\quad + \sum_{1\le n\le N-1} \int_{\delta n}^{\delta(n+1)} \left(\frac{F_i(t)}{t} - \frac{F_i(\delta n)}{\delta n}\right) dt \\ &= -\int_{\delta}^{\delta N} \frac{F_i(t)}{t}\,dt + \gamma + \log N + \mathcal{O}(\delta) \\ &= \int_{\delta}^{\delta N} \frac{|1-F_i(t)|}{t}\,dt + \gamma + \mathcal{O}(\delta).\end{aligned}$$

Write

$$\int_{\delta}^{\infty} \frac{|1-F_i(t)|}{t}\,dt = -\int_{\delta}^{1} \frac{F_i(t)}{t}\,dt + \int_{1}^{\infty} \frac{|1-F_i(t)|}{t}\,dt + \log \delta^{-1}.$$

The lemma follows readily. The constants c_i are computed in the next lemma. ∎

LEMMA 14. *For $i \in \{1,2\}$, we have*

$$c_i = \log\frac{2}{\pi} + \gamma - \frac{1}{2\pi}\int_{-1/2}^{1/2} \frac{\tan(\pi t)}{t}\varrho_i(t)\,dt$$

($c_1 = -0.6280\ldots$ *and* $c_2 = \gamma + \log(2/\pi)$).

Proof. According to Lemma 12, we have

$$\begin{aligned} c_i &= \gamma - \int_0^1 \frac{F_i(t)}{t}\,dt + \lim_{K\to\infty}\sum_{k=1}^{4K}(-1)^k \int_k^{k+1} \frac{1-F_i(t)}{t}\,dt \\ &= \gamma + \sum_{k=1}^{4K}(-1)^k \log\frac{k+1}{k} + \sum_{k=0}^{4K}(-1)^{k+1}\int_k^{k+1}\frac{F_i(t)}{t}\,dt + o(1) \end{aligned}$$

where $o(1)$ goes to 0 as K goes to infinity. We have

$$\sum_{k=1}^{4K}(-1)^k \log\frac{k+1}{k} = \log\frac{(4K)!^2(4K+1)}{(2K)!^4 2^{8K}} = \log\frac{2}{\pi} + o(1)$$

thanks to Stirling's formula: $n! = (n/e)^n\sqrt{2\pi n}(1+o(1))$. Recalling Lemmas 5 and 6 according to which

$$\frac{F_i(t)}{t} = \int_{-1/2}^{1/2} \varrho_i(u)e(-ut)\,du,$$

we get

$$\begin{aligned} \sum_{k=0}^{4K}(-1)^{k+1}\int_k^{k+1}\frac{F_i(t)}{t}\,dt &= \sum_{k=0}^{4K}(-1)^{k+1}\int_{-1/2}^{1/2}\frac{e(-(k+1)t)-e(-kt)}{-2i\pi t}\varrho_i(t)\,dt \\ &= \sum_{k=0}^{4K}(-1)^{k+1}\frac{-1}{2i\pi}\int_{-1/2}^{1/2} e(-kt)\frac{e(-t)-1}{t}\varrho_i(t)\,dt \\ &= -\frac{1}{2i\pi}\int_{-1/2}^{1/2}\frac{1+e(-(4K+1)t)}{1+e(-t)}\cdot\frac{1-e(-t)}{t}\varrho_i(t)\,dt. \end{aligned}$$

By the Riemann–Lebesgue lemma the part depending on K goes to 0 from which the lemma follows readily. ∎

LEMMA 15. *j' is non-positive and non-decreasing over* $[0,1]$. *Also*

$$1-\left(\frac{\sin(\pi t)}{\pi t}\right)^2 \le F_3(t) \le 1.$$

Proof. Using [Vaaler, Theorem 6] we write $j'(t) = -\widehat{J}(t)/(2t)$. For $t>0$, the function $\widehat{J}$ is non-negative and non-increasing, and so is $t \mapsto 1/(2t)$, whence so is their product; the minus sign simply reverses the growth. The inequality for F_3 comes from [Vaaler, Lemma 5]. ∎

LEMMA 16. *For $\delta \in \,]0,1]$, we have*

$$\sum_{n\ge 1} \frac{1-F_3(\delta n)}{n} = -\log\delta - 1 + \delta.$$

Proof. Define

$$f(\delta) = \sum_{n\ge 1} \frac{1-F_3(\delta n)}{n} + \log\delta$$

where the sum is absolutely convergent, the convergence being also uniform on every subcompact of $]0,1]$. From [Vaaler, Theorem 6] (in Vaaler's notations, $J=\frac{1}{2}F_3'$), we take

$$F_3'(t) = 2\int_{-1}^{1} (\pi u(1-|u|)\cot(\pi u) + |u|)e(tu)\,du, \tag{5.2}$$

and we set $h(u) = \pi u(1-|u|)\cot(\pi u)+|u|$. We then have

$$f'(\delta) = -\sum_{n\ge 1} F_3'(\delta n) + \frac{1}{\delta} = \sum_{n\ge 1} \frac{2}{\pi n\delta}\int_0^1 h'(t)\sin(2\pi n\delta t)\,dt + \frac{1}{\delta}.$$

Note that $h'(t)$ is bounded on $[0,1]$. We have

$$f(\delta) = -\frac{2}{\delta}\int_0^1 h'(t)\sum_{n\ge 1}\frac{-\sin(2\pi n\delta t)}{\pi n}\,dt + \frac{1}{\delta} \tag{5.3}$$

by the Lebesgue theorem since the partial sums $\sum_{n\le N}\sin(nx)/n$ are uniformly bounded in N and x. The inner summation is the Bernoulli function $B_1(\delta t)$, i.e. $\delta t - 1/2$ here since $\delta t \in [0,1[$. We thus get

$$f'(\delta) = -\frac{2}{\delta}\int_0^1 h'(t)\left(\delta t - \frac{1}{2}\right)dt + \frac{1}{\delta} = 1,$$

the last inequality coming from the remark that $h(u) = uj'(u)$ for $u \geq 0$ followed by a use of Lemma 7. Integrating back we find $f(\delta) = c_3 + \delta$ and since $f(1) = 0$, we get $c_3 = -1$. ∎

LEMMA 17. *For $\delta > 0$, we have*

$$\sum_{n\geq 1} \frac{1 - F_4(\delta n)}{n} = -\log\delta + \frac{3}{2} - \log(2\pi) + 2\int_0^1 (1-t)\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right| dt.$$

Proof. Denote by $f(\delta)$ the sum to evaluate and compute its derivative:

$$f'(\delta) = -\sum_{n\geq 1} F_4'(\delta n).$$

We readily check that

$$-F_4'(z) = -4\pi\int_0^1 (1-t)t\sin(2\pi z t)\,dt \tag{5.4}$$

so that, for large N, we get

$$\begin{aligned} &-\sum_{1\leq n\leq N} F_4'(\delta n) \\ &\quad = -4\pi\int_0^1 (1-t)t\sum_{1\leq n\leq N}\sin(2\pi n\delta t)\,dt \\ &\quad = -4\pi\int_0^1 (1-t)t\frac{1}{2i}\left(e(N\delta t)\frac{e(\delta t)}{1-e(\delta t)} + e(-N\delta t)\frac{1}{1-e(\delta t)} - \frac{1+e(\delta t)}{1-e(\delta t)}\right)dt \end{aligned}$$

and the Riemann–Lebesgue lemma gives us

$$f'(\delta) = -2\pi\int_0^1 (1-t)t\tan(\pi\delta t)\,dt \tag{5.5}$$

from which we infer

$$f(\delta) = -2\int_0^1 (1-t)\log\sin(\pi\delta t)\,dt + c_4.$$

We compute c_4 by taking $\delta = 1$: since $f(1) = 0$, we get

$$c_4 = 2\int_0^1 (1-t)\log\sin(\pi t)\,dt = 2\int_0^1 t\log\sin(\pi t)\,dt$$

so that

$$2c_4 = 2\int_0^1 \log\sin(\pi t)\,dt = -2\log 2. \tag{5.6}$$

Thus

$$\begin{aligned} f(\delta) &= -2\int_0^1 (1-t)\log\sin(\pi\delta t)\,dt - \log 2 \\ &= -2\int_0^1 (1-t)\log(\pi\delta t)\,dt - \log 2 + 2\int_0^1 (1-t)\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right| dt \end{aligned}$$

and

$$-2\int_0^1 (1-t)\log(\pi\delta t)\,dt = -\log(\pi\delta) + 3/2,$$

thus concluding the proof. ∎

LEMMA 18. *For $0 < \delta \le 1/2$, we have*

$$2\int_0^1 (1-t)\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right| dt \le \pi^3\delta^2/12.$$

Proof. An integration by parts yields

$$\begin{aligned} 2\int_0^1 (1-t)\log\left|\frac{\pi\delta t}{\sin(\pi\delta t)}\right| dt &= \log\frac{\pi\delta}{\sin(\pi\delta)} - 2\int_0^1 (t-t^2/2)\frac{1-\pi\delta t\cot(\pi\delta t)}{t}\,dt \\ &\le \log\frac{\pi\delta}{\sin(\pi\delta)} \end{aligned}$$

since $\cot x = 1/\tan x \le 1/x$ for $x \in]0,\pi/2]$. Using $\log x \le x-1$, $x - \sin x \le x^3/6$ and $\sin x \ge 2x/\pi$ if $0 \le x \le \pi/2$, we see that the above is not more than $\pi^3\delta^2/12$ as required. ∎

VI. Bounding $|L(1,\chi)|$ from above

First smoothings. We use Proposition 1 for even χ. Let $\delta \in]0,1]$. By Lemma 13, we infer

$$|L(1,\chi)| \le -\frac{2}{\sqrt{q}}\sum_{m\le\delta q/2}\log\left|\sin\frac{\pi m}{q\delta}\right| - \log\delta + c_1 + \mathcal{O}(\delta).$$

The function $-\log\sin x$ is non-negative non-increasing on $[0,\pi/2]$, and thus by (5.6)

$$(6.1)\qquad -\frac{\pi}{q\delta}\sum_{m\le\delta q/2}\log\left|\sin\frac{\pi m}{q\delta}\right| \le -\int_0^{\pi/2}\log|\sin x|\,dx = -\frac{\pi}{2}\log 2.$$

This yields

$$|L(1,\chi)| \le \log\delta^{-1} + c_1 + \delta\sqrt{q}\log 2 + \mathcal{O}(\delta)$$

and the best value for δ is given by $\delta\sqrt{q}\log 2 = 1$ whence

$$(6.2)\qquad |L(1,\chi)| \le \tfrac{1}{2}\log q + 1 + c_1 + \log\log 2 + \mathcal{O}(q^{-1/2}) \qquad (\chi(-1)=1)$$

and $1 + c_1 + \log\log 2 = 0.0005\ldots$ by Lemma 14.

Let us now use Proposition 1 for odd χ. Let $\delta \in]0,1]$. Using Lemmas 13–15, we get

$$\begin{aligned}|L(1,\chi)| &\le -\log\delta + c_2 + \frac{\pi\delta\sqrt{q}}{2}\cdot\frac{2}{\delta q}\sum_{m\le\delta q/2}\left(1-\frac{2m}{\delta q}\right) + \mathcal{O}(\delta)\\ &\le -\log\delta + \gamma + \log(2/\pi) + \pi\delta\sqrt{q}/4 + \mathcal{O}(\delta).\end{aligned}$$

We take $\pi\delta\sqrt{q}/4 = 1$ and get

$$(6.3)\qquad |L(1,\chi)| \le \tfrac{1}{2}\log q - \log 2 + 1 + \gamma + \mathcal{O}(q^{-1/2}) \le \tfrac{1}{2}\log q + 0.8840684$$

for large enough q's.

Collecting (6.2) and (6.3), we find (1.7).

A second smoothing for even characters. Use Proposition 2 for even χ. Let $\delta \in]0,1]$. Lemma 10, together with Lemmas 7 and 15, gives us

$$\begin{aligned}\sum_{1\le m\le\delta q} j(m/(\delta q)) &\le \delta q\int_0^1 j(t)\,dt - j(1/(\delta q))/2\\ &\le \delta q - (\log(2\pi) - 1 + \log(\delta q)).\end{aligned}$$

Recall Lemma 16. We get

$$|L(1,\chi)| \le -\log\delta - 1 + \delta + \frac{1}{\sqrt{q}}\sum_{1\le m\le\delta q} j(m/(\delta q))$$

and thus

$$|L(1,\chi)| \le -\log\delta - 1 + \delta + \delta\sqrt{q} - \frac{\log(2\pi) - 1 + \log(\delta q)}{\sqrt{q}}.$$

The near-optimal δ is $1/\sqrt{q}$ and yields

$$(6.4)\qquad |L(1,\chi)| \le \frac{1}{2}\log q - \frac{\log(2\pi) - 2 + \frac{1}{2}\log q}{\sqrt{q}} \le \frac{1}{2}\log q - \frac{\log(5q/7)}{2\sqrt{q}},$$

as required.

A second smoothing for odd characters. Use Proposition 2 for odd χ. Let $\delta \in]0,1]$. By Lemmas 10, 17 and 18 we get

$$
\begin{aligned}
|L(1,\chi)| &\le -\log\delta + \frac{3}{2} - \log(2\pi) + \frac{\pi^3\delta^3}{12} + \frac{1}{\sqrt{q}} \sum_{1\le m\le \delta q} \pi(m/(\delta q)-1)^2 \\
&\le -\log\delta + \frac{3}{2} - \log(2\pi) + \frac{\pi}{\sqrt{q}}\left(\frac{\delta q}{3} - \frac{(1-1/(\delta q))^2}{2}\right) + \frac{\pi^3\delta^2}{12}.
\end{aligned}
$$

The near-optimal value for δ is given by

$$\frac{\pi\delta\sqrt{q}}{3} = 1,$$

which yields

$$
\begin{aligned}
(6.5)\qquad |L(1,\chi)| &\le \frac{1}{2}\log q + \frac{5}{2} - \log 6 - \frac{2\pi - (12+3\pi)/\sqrt{q} + 18/(\pi^2 q)}{4\sqrt{q}} \\
&\le \frac{1}{2}\log q + \frac{5}{2} - \log 6 - \frac{5}{4\sqrt{q}}
\end{aligned}
$$

if $q \ge 300$ which we extend to $q \ge 18$ by direct computations, and to all q if we remove the $-5/(4\sqrt{q})$.

Collecting (6.4) and (6.5), we get Corollary 1.

VII. Proof of Corollary 3. Since we assume that q is even, we have $\chi(n) = 0$ as soon as n is even which is the additional information we use in this section.

In the case of even characters, we have

$$
\begin{aligned}
(7.1)\qquad \sum_{\substack{n\ge 1\\ (n,2)=1}} \frac{1-F_3(\delta n)}{n} &= \sum_{n\ge 1} \frac{1-F_3(\delta n)}{n} - \sum_{n\ge 1} \frac{1-F_3(2\delta n)}{2n} \\
&= -\frac{1}{2}\log\delta - \frac{1}{2} + \frac{\log 2}{2}.
\end{aligned}
$$

Use Proposition 2 for an even χ. Let $\delta \in]0,1]$. By the preceding estimates, we get

$$|L(1,\chi)| \le -\frac{1}{2}\log\delta - \frac{1}{2} + \frac{\log 2}{2} + \frac{1}{\sqrt{q}} \sum_{\substack{m\ge 1\\ (m,2)=1}} j(m/(\delta q))$$

and we gather Lemmas 11, 7 and 15 to get

$$(7.2)\qquad |L(1,\chi)| \le -\frac{1}{2}\log\delta - \frac{1}{2} + \frac{\log 2}{2} + \frac{\delta\sqrt{q}}{2}$$

and the choice $\delta = 1/\sqrt{q}$ yields the estimate. A more careful proof would give an error term of the shape $-c/\sqrt{q}$ for some $c > 0$.

In the case of odd characters and after some manipulations, we find

$$\begin{aligned}\sum_{\substack{n\ge1\\(n,2)=1}}\frac{1-F_4(\delta n)}{n}&=-\frac12\log\delta+\frac{\log 2}{2}+\frac34-\frac{\log(2\pi)}{2}\\&\quad+\int_0^1(1-t)\left(\log|\pi\delta t|+\log\left|\frac{\cos(\pi\delta t)}{\sin(\pi\delta t)}\right|\right)dt\\&=-\frac12\log\delta+\frac{\log 2}{2}+\frac34-\frac{\log(2\pi)}{2}\\&\quad+\int_0^1(1-t)\log\left|\frac{\pi\delta t}{\tan(\pi\delta t)}\right|dt\end{aligned}$$

and this last summand is non-negative. Moreover, Lemma 11 implies

$$\sum_{\substack{1\le m\le\delta q\\(m,2)=1}}\left(\frac{m}{\delta q}-1\right)^2\le\frac{\delta q}{6}.$$

We thus reach

$$|L(1,\chi)|\le-\frac12\log\delta+\frac{\log 2}{2}+\frac34-\frac{\log(2\pi)}{2}+\frac{\pi}{\sqrt q}\cdot\frac{\delta q}{6}. \tag{7.3}$$

The choice $\delta=3/(\pi\sqrt q)$ concludes the proof.

VIII. Numerical verifications. We have computed $L(1,\chi)$ for all primitive χ of conductor ≤ 4500 by using the version 2.0.11/2.0.17 of the GP-calculator of the PARI system. The program used was very rudimentary and used the exact formulae (1.4). In particular, for $q\le 1000$, we computed the Gauss sums explicitly (though it is not required, only its absolute value being used) and checked that its modulus divided by $\sqrt q$ lied in $[1-10^{-10},1+10^{-10}]$. We found generators of the multiplicative groups modulo p^α for each p^α occurring in the prime decomposition of q and built a table of logarithms for integers $\le q/2$ and prime to q. With such a set of generators, it is very easy to build characters and to recognize which ones are primitive. We found that

$$|L(1,\chi)|\le\tfrac12\log q-0.324042\qquad(\chi(-1)=1,\ 3\le q\le 4500), \tag{8.1}$$

this bound being reached (within 10^{-6}) by a character modulo 241. Looking for when this constant would be close to $-1/2$, we found that

$$|L(1,\chi)|\le\tfrac12\log q-0.432088\qquad(\chi(-1)=1,\ 1203\le q\le 4500) \tag{8.2}$$

and there is a character modulo 1201 for which this constant is about $-0.396458\ldots$ and a character modulo 2641 for which this bound is reached (within 10^{-6}). We also checked that

$$|L(1,\chi)| \le \tfrac{1}{2}\log q + 0.514828 \qquad (\chi(-1) = -1,\ 3 \le q \le 4500), \tag{8.3}$$

this bound being reached (within 10^{-6}) by a character modulo 311. We found an odd character modulo 4309 for which $|L(1,\chi)| - \frac{1}{2}\log q > 0$. Such values are extremely rare, but to sustain (1.2) we clearly need to go beyond 4500. Since this conjecture is checked to be true for even characters by Corollary 1 and by odd characters to an even modulus by Corollary 3, it is enough to check what happens with odd characters to an odd conductor. This we did for all $q \le 5500$ and found no exception to (1.2).

This program was very lengthy since its complexity is $\mathcal{O}(q^2)$ for the modulus q (while a complexity of $\mathcal{O}(q^{3/2}\log q)$ can be attained), but it offers a verification of other programs. To this effect, we list below for some moduli the maxima of $|L(1,\chi)| - \frac{1}{2}\log q$ for odd and even χ. Results have been rounded up at the sixth decimal place. Full tables are available on request.

q	Even	Odd
7	−0.424995	0.214456
32	−0.697560	−0.281641
212	−1.229458	−0.738473
737	−0.736389	−0.100880
1009	−0.379913	−0.138504
1112	−1.702313	−1.533992

Computations regarding the upper bound $\frac{1}{4}\log q + C$ when q is even were of course conducted by the same program, up to some trivial modifications.

IX. An application. We prove Corollary 2. By the Dirichlet class number formula, we have

$$h(\mathbb{Q}(\sqrt{q})) = \frac{\sqrt{q}}{2\log\varepsilon_q} L(1,\chi_q)$$

where χ_q is the even primitive real character modulo q and ε_q is the fundamental unit. We assume $q \ge 5$. Classically (see [Le, Lemma 4] for instance), we have

Lemma 19. $\varepsilon_q \ge (\sqrt{q-4} + \sqrt{q})/2$.

Recalling (6.4), we thus reach

$$\frac{2}{\sqrt{q}} h(\mathbb{Q}(\sqrt{q})) \le \frac{\log q - \frac{\log(5q/7)}{\sqrt{q}}}{2\log((\sqrt{q-4} + \sqrt{q})/2)},$$

which is not more than 1 if $q \ge 6$. We end the proof by direct examination.

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