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A note on class number 1 criteria for totally real algebraic number fields

by

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1. Introduction. Generalizing the concept of the Euclidean algorithm, Rabinowitsch [7] proved the following theorem.

THEOREM 1.1. *Let* $K=\mathbb{Q}(\sqrt{1-4m})$, $m\in\mathbb{N}$, *be an imaginary quadratic field. Then the class number of* K *is equal to* 1 *if and only if* x^2-x+m *is a prime for any integer* x *such that* $1\le x\le m-2$.

Later, applying Rabinowitsch's method to real quadratic fields, Kutsuna [3] obtained various class number 1 criteria for real quadratic fields. One of them is the following:

THEOREM 1.2. *Let* $K=\mathbb{Q}(\sqrt{1+4m})$, $m\in\mathbb{N}$, *be a real quadratic field. If* $-x^2+x+m$ *is a prime for any integer* x *such that* $1\le x\le\sqrt{m}-1$, *then the class number of* K *is equal to* 1.

The aim of this paper is to extend Kutsuna's result to arbitrary totally real algebraic number fields by using Siegel's formula for the special values of Dedekind zeta functions attached to them.

In Section 2, we will state Siegel's formula for the special values of Dedekind zeta functions of totally real algebraic number fields and in Section 3, using the formula, we will give class number 1 criteria for them. In Section 4, as examples, we will treat a certain family of real quadratic fields, totally real cubic fields.

2. Siegel's formula. Let K be a totally real algebraic number field and δ be the different of K. Let A be an ideal class of K. To state Siegel's formula for the special value of Dedekind zeta function of K, we need some definitions. First, for any non-zero integral ideal $\mathbf{a}$ of K, we define

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$$\sigma(\mathbf{a}) = \sum_{\mathbf{b}|\mathbf{a}} N(\mathbf{b}),$$

where the sum is over all integral ideals $\mathbf{b}$ of K which divide $\mathbf{a}$ and $N(\mathbf{b})$ denotes the norm of $\mathbf{b}$. Similarly, for any non-zero integral ideal $\mathbf{a}$ of K, we define

$$\sigma_A(\mathbf{a}) = \sum_{\mathbf{b}\in A,\, \mathbf{b}|\mathbf{a}} N(\mathbf{b}),$$

where the sum is over all integral ideals $\mathbf{b}$ in A which divide $\mathbf{a}$. For a natural number l, let T_l be the set of all totally positive elements of K in δ^{-1} with given trace l, i.e.,

$$T_l = \{\nu \in K \mid \nu \in \delta^{-1},\ \nu \gg 0 \text{ and } \mathrm{Tr}_{K/\mathbb{Q}}(\nu) = l\}.$$

Then we know that T_l is a finite set for all $l \in \mathbb{N}$. Thus for a natural number l, we can define

$$S_l^K = \sum_{\nu\in T_l} \sigma((\nu)\delta) \quad \text{and} \quad S_l^A = \sum_{\nu\in T_l} \sigma_A((\nu)\delta).$$

Now we can state Siegel's formula for the value of the zeta function of K.

THEOREM 2.1 (Siegel [10] or [12]). *Let K be a totally real algebraic number field of degree $n > 1$ and A be an ideal class of K. Let $r = \dim_{\mathbb{C}} M_{2n}$, where M_{2n} is the space of modular forms of weight $2n$. Then*

$$\zeta_K(-1) = 2^n \sum_{l=1}^{r} b_l(2n) S_l^K \quad \textit{and} \quad \zeta_K(-1, A) = 2^n \sum_{l=1}^{r} b_l(2n) S_l^A,$$

where $b_1(2n), \ldots, b_r(2n)$ are rational and depend only on n.

REMARK. There is the following well known formula for r:

$$r = \begin{cases} [2n/12] & \text{if } 2n \equiv 2 \pmod{12}, \\ [2n/12] + 1 & \text{if } 2n \not\equiv 2 \pmod{12}, \end{cases}$$

where $[x]$ denotes the greatest integer $\le x$.

3. Class number 1 criteria. Let K be a totally real algebraic number field of degree n and δ the different of K. Set $T = \bigcup_{l=1}^{r} T_l$, where $r = \dim_{\mathbb{C}} M_{2n}$. From Siegel's formula in Theorem 2.1, we have the following class number 1 criterion for K.

THEOREM 3.1. *Let K be a totally real algebraic number field of degree $n > 1$ and δ be the different of K. Then the class number of K is equal to 1 if and only if the ideal $(\nu)\delta$ can be written as a product of powers of principal prime ideals of K for all $\nu \in T$.*

Proof. If the class number of K is 1, it is clear that $(\nu)\delta$ can be decomposed into principal prime ideals of K for all $\nu \in T$. Now we suppose that for all $\nu \in T$, $(\nu)\delta$ can be decomposed into principal prime ideals of K. Then we easily see that

$$\sigma((\nu)\delta) = \sigma_P((\nu)\delta) \quad \text{for all } \nu \in T,$$

where P is the principal ideal class of K. Thus from Theorem 2.1, we have

$$\zeta_K(-1) = \zeta_K(-1, P).$$

Note that for all ideal classes A of K, $\zeta_K(-1, A)$ have the same signs. Hence the class number of K is 1 and we have proved the theorem. ∎

Let $N_{K/\mathbb{Q}}$ denote the norm of elements in K from K to $\mathbb{Q}$. Then from Theorem 3.1, we have the following class number 1 criterion for K which is similar to Theorems 1.1 and 1.2.

Corollary 3.2. *Let K be a totally real algebraic number field of degree $n > 1$ whose different δ is (β) for some $\beta \in K$. Then if $N_{K/\mathbb{Q}}(\nu\beta)$ is ± 1 or a prime* (*or a power of a prime which is a value of $N_{K/\mathbb{Q}}(\nu\beta)$ for some $\nu \in T$, if K is Galois*) *for any $\nu \in T$, then the class number of K is* 1.

Proof. Suppose that $N_{K/\mathbb{Q}}(\nu\beta)$ is ± 1 or a prime for any $\nu \in T$. Then $(\nu)\delta$ should be a principal prime ideal for any $\nu \in T$. Thus from Theorem 3.1, we have the assertion. For the case that K is Galois, it can also be easily proved by the same reason. ∎

4. Examples

Example 1. Let m be a positive rational integer and $D = 4m + 1$. Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field. Then $\{1, (1 + \sqrt{D})/2\}$ is an integral basis of K and the different δ of K is $(\sqrt{D})$. Thus, if $\nu \in T = T_1$, then

$$(\nu)\delta = \left(x + \frac{1 + \sqrt{D}}{2}\right),$$

where x is a rational integer such that $\sqrt{D} > 2x + 1 > -\sqrt{D}$. So, from Corollary 3.2, we have the following class number 1 criterion for K, which is similar to Kutsuna's result in Theorem 1.2:

> *If $x^2 + x - m$ is a prime for any rational integer such that $0 \le x \le \sqrt{m} - 1$, then the class number of K is* 1.

Remark. Let $d = n^2 + r$, $d \ne 5$, be a positive square free integer satisfying $r \mid 4n$ and $-n < r \le n$. Then the real quadratic field $K = \mathbb{Q}(\sqrt{d})$ is called a real quadratic field of Richaud–Degert (R–D) type. For such a field, the fact that the condition in Corollary 3.2 is also necessary for the class number of K to be 1 was proved by many people with different methods. For examples, see [1], [5], [6], [11].

EXAMPLE 2. Let $m \geq -1$ be a rational integer such that $m^2 + 3m + 9$ is a prime. Let K_m be the cubic field defined by the irreducible polynomial $f(x) = x^3 + mx^2 - (m+3)x + 1$ over $\mathbb{Q}$. Let α be the negative root of $f(x)$. Then $\alpha' = 1/(1-\alpha)$ and $\alpha'' = 1 - 1/\alpha$ are the other two roots, so K_m is a totally real cyclic cubic field. We let K_m be the simplest cubic field [9]. It is well known that $\{\alpha, \alpha'\}$ is a system of fundamental units of K_m and $\{1, \alpha, \alpha^2\}$ is a basis of K_m. Thus from [8, p. 207], the different δ of K_m is $(f'(\alpha))$. By easy computation, we have

$$\begin{aligned} S &:= \{\nu f'(\alpha) \mid \nu \in T \ (= T_1)\} \\ &= \{u + v\alpha + \alpha^2 \mid u, v \in \mathbb{Z} \text{ such that } (u + v\alpha + \alpha^2)/(f'(\alpha)) \gg 0\}. \end{aligned}$$

By using Maple, we can determine S and compute the values of $N_{K/\mathbb{Q}}$ of its elements. For an example, let $m = 10$; then $m^2 + 3m + 9 = 139$ and we have the following table:

(u, v), where $u + v\alpha + \alpha^2 \in S$	$N_{K/\mathbb{Q}}(u + v\alpha + \alpha^2)$
$(-12, 10),\ (0, -1),\ (-1, 11)$	-1
$(-1, 10),\ (-1, 0),\ (-11, 10)$	-23
$(-10, -10),\ (-1, 9),\ (-2, 1)$	-59
$(-9, 10),\ (-3, 2),\ (-1, 8)$	-103
$(-10, 9),\ (-2, 10),\ (-1, 1)$	-131
$(-8, 10),\ (-4, 3),\ (-1, 7)$	-149
$(-7, 10),\ (-5, 4),\ (-1, 6)$	-191
$(-9, 8),\ (-3, 10),\ (-1, 2)$	-199
$(-6, 10),\ (-6, 5),\ (-1, 5)$	-223
$(-8, 7),\ (-4, 10),\ (-1, 3)$	-233
$(-7, 6),\ (-5, 10),\ (-1, 4)$	-239
$(-9, 9),\ (-2, 9),\ (-2, 2)$	-251
$(-8, 9),\ (-2, 8),\ (-3, 3)$	-353
$(-8, 8),\ (-3, 9),\ (-2, 3)$	-383
$(-7, 9),\ (-4, 4),\ (-2, 7)$	-431
$(-7, 7),\ (-4, 9),\ (-2, 4)$	-461
$(-6, 9),\ (-5, 5),\ (-2, 6)$	-479
$(-6, 6),\ (-5, 9),\ (-2, 5)$	-491
$(-7, 8),\ (-3, 4),\ (-3, 8)$	-523
$(-6, 8),\ (-4, 5),\ (-3, 7)$	-613
$(-6, 7),\ (-4, 8),\ (-3, 5)$	-619
$(-5, 6),\ (-5, 8),\ (-3, 6)$	-647
$(-5, 7),\ (-4, 6),\ (-4, 7)$	-701

From the above table, we know that $N_{K/\mathbb{Q}}(u + v\alpha + \alpha^2)$ is -1 or a prime for any $u + v\alpha + \alpha^2 \in S$. Thus from Corollary 3.2, the class number of K_{10} is 1.

REMARK. In [4], Lettl showed that there are only 7 simplest cubic fields with class number 1 and their m's are -1, 1, 2, 4, 7, 8, 10. By using Maple, we determined S and checked that $N_{K/\mathbb{Q}}(u+v\alpha+\alpha^2)$ is ± 1 or a prime for any $u+v\alpha+\alpha^2 \in S$ for each $m=-1,1,2,4,7,8,10$. Thus we proved that for the case of the simplest cubic field K_m, the condition in Corollary 3.2 is also a necessary condition for the class number of K_m to be 1.

On the other hand, using the similar method to [1], Kim and Hwang [2] obtained the same result.

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