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The Brauer–Kuroda formula for higher S-class numbers in dihedral extensions of number fields

by

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1. General setting and statement of the main result. For a number field E and a finite set of places S of E containing the archimedean places, let ζ_E^S be the Dedekind S-zeta function of E. For a complex number s, let $\zeta_E^S(s)^*$ denote the special value of ζ_E^S at s (i.e. the first nontrivial coefficient of the Laurent expansion of ζ_E^S around s). Dedekind, inspired by previous work of Dirichlet, proved the formula

$$\zeta_E^S(0)^* = -\frac{h_E^S}{w_E} R_E^S, \tag{1.1}$$

where h_E^S is the class number of the ring $\mathcal{O}_E^S$ of S-integers of E, w_E is the order of the group of roots of unity of E, and R_E^S is the regulator of $(\mathcal{O}_E^S)^\times$.

There are conjectural analogues of this formula when 0 is replaced by negative integers. More precisely, we recall the definition of motivic cohomology groups in terms of Bloch's higher Chow groups:

$$H^j(\mathcal{O}_E^S, \mathbb{Z}(m)) := CH^m(\mathrm{Spec}(\mathcal{O}_E^S), 2m-j).$$

The Bloch–Kato conjecture, which is now a theorem of Rost, Voevodsky and Weibel (the reader is referred to [We] for appropriate references on this subject), implies that, for any integer $m \geq 2$, $H^1(\mathcal{O}_E^S, \mathbb{Z}(m))$ (resp. $H^2(\mathcal{O}_E^S, \mathbb{Z}(m))$) is a finitely generated $\mathbb{Z}$-module (resp. a finite abelian group). Then the original Lichtenbaum conjecture (see [Li]) can be modified to the following conjecture [Ka, 4.7.4]:

$$\zeta_E^S(1-m)^* = (-1)^{t_{E,m}} \frac{h_{E,m}^S}{w_{E,m}^S} R_{E,m}^S, \quad m \geq 2. \tag{1.2}$$

Here $h_{E,m}^S$ is the order of $H^2(\mathcal{O}_E^S, \mathbb{Z}(m))$, $w_{E,m}^S$ is the order of the tor-

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sion subgroup of $H^1(\mathcal{O}_E^S, \mathbb{Z}(m))$ and $R_{E,m}^S$ is the (motivic) regulator of $H^1(\mathcal{O}_E^S, \mathbb{Z}(m))$ (see Section 3). If S' is a finite set of places of E containing S, then the natural map

$$H^1(\mathcal{O}_E^S, \mathbb{Z}(m)) \to H^1(\mathcal{O}_E^{S'}, \mathbb{Z}(m)),$$

induced by the inclusion $\mathcal{O}_E^S \subseteq \mathcal{O}_E^{S'}$, can be shown to be an isomorphism which commutes with the motivic regulator map (whose definition and properties are recalled in Section 3). Then one can easily see that both $w_{E,m}^S$ and $R_{E,m}^S$ are independent of S and therefore we shall denote them simply by $w_{E,m}$ and $R_{E,m}$, respectively. Finally, $t_{E,m} \in \mathbb{N}$ is given by

$$t_{E,m} = \begin{cases} 1 & \text{if } m \equiv 1 \bmod 4, \\ r_1(E) + r_2(E) & \text{if } m \equiv 2 \bmod 4, \\ r_1(E) & \text{if } m \equiv 3 \bmod 4, \\ r_2(E) & \text{if } m \equiv 0 \bmod 4, \end{cases}$$

where $r_1(E)$ (resp. $r_2(E)$) is the number of real places (resp. complex places) of E.

We now list some known facts about the Lichtenbaum conjecture and motivic cohomology; for any further detail and proper attribution of the results we shall mention, we refer the reader to Kolster's and Kahn's excellent surveys [Ko] and [Ka]. For the rest of the paper, m will denote an integer greater than 1. We have

$$H^1(\mathcal{O}_E^S, \mathbb{Z}(m)) \otimes \mathbb{Q} \cong K_{2m-1}(\mathcal{O}_E^S) \otimes \mathbb{Q}. \tag{1.3}$$

In particular, since the $\mathbb{Z}$-rank of $K_{2m-1}(\mathcal{O}_E^S)$ does not depend on S (see [So]), we get, thanks to Borel's rank formula,

$$\operatorname{rk}_{\mathbb{Z}} H^1(\mathcal{O}_E^S, \mathbb{Z}(m)) = \begin{cases} r_1(E) + r_2(E) & \text{if } m \geq 2 \text{ is odd}, \\ r_2(E) & \text{if } m \geq 2 \text{ is even}. \end{cases} \tag{1.4}$$

Finally, by the validity of the Bloch–Kato conjecture, we also have

$$H^j(\mathcal{O}_E^S, \mathbb{Z}(m)) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong H^j_{\text{ét}}(\mathcal{O}_E^S[1/\ell], \mathbb{Z}_\ell(m)) \qquad \text{for } j = 1, 2, \tag{1.5}$$

where ℓ is any prime. Whenever a Galois action is defined on E, the above isomorphisms are invariant under this action.

Using the main conjecture in Iwasawa theory (which is now a theorem thanks to Wiles's proof), the Lichtenbaum conjecture has been proved for $m \geq 2$ even and E totally real abelian and it is also known to hold up to powers of 2 for $m \geq 2$ even and E totally real. Moreover, it is known to hold up to powers of 2 for E abelian and *any* $m \geq 2$.

Now let p be an *odd* prime. Let $D = D_p$ denote the dihedral group of order $2p$; in particular

$$D = \langle \tau, \sigma \mid \tau^p = \sigma^2 = 1,\, \sigma\tau\sigma = \tau^{-1} \rangle.$$

Let L/k be a Galois extension of number fields such that $\mathrm{Gal}(L/k) \cong D$ (in the rest of this paper we shall identify those groups). Let K (resp. K') be the subfield of L fixed by $\langle\sigma\rangle$ (resp. by $\langle\tau^2\sigma\rangle$); in particular $K' = \tau(K)$. Let F be the subfield of L fixed by $\langle\tau\rangle$ and set $G = \mathrm{Gal}(L/F)$ and $\Delta = \mathrm{Gal}(F/k)$.

Let S be a finite set of places of L which is stable under the action of D and contains the archimedean places (we shall consider only sets of places containing the archimedean ones, so we will not further mention this property). For any subfield E of L containing k, the set of places of E which lie below those of S will be denoted by S_E or simply again by S if no misunderstanding is possible. The existence of the nontrivial D-relation (in the sense of [DD], see Definition 2.1 and Example 2.4 of that paper)

$$\{1\} - 2\langle\sigma\rangle - G + 2D, \tag{1.6}$$

together with Artin formalism of L-functions, gives the following formula:

$$\zeta_L^S(s) = \zeta_F^S(s)\frac{\zeta_K^S(s)^2}{\zeta_k^S(s)^2}. \tag{1.7}$$

Considering the special value at 0 of (1.7) and using (1.1), we get

$$h_L^S = h_F^S\frac{(h_K^S)^2}{(h_k^S)^2}\cdot\frac{w_k^2 w_L}{w_K^2 w_F}\cdot\frac{(R_K^S)^2 R_F^S}{(R_k^S)^2 R_L^S}, \tag{1.8}$$

which is commonly referred to as the (*classical*) *Brauer–Kuroda* formula (for dihedral extensions of order $2p$). It can be shown that the w-factor is actually trivial. More interestingly, the factor involving regulators can be expressed as the index of a subgroup defined in terms of units of subextensions of L/k (see [Ba], [Ja], [HK], [Le], ...). Considering special values at negative integers and using (1.2), we get of course, for any $m \geq 2$, a conjectural analogue of (1.8):

$$h_{L,m}^S = h_{F,m}^S\frac{(h_{K,m}^S)^2}{(h_{k,m}^S)^2}\cdot\frac{(w_{k,m})^2 w_{L,m}}{(w_{K,m})^2 w_{F,m}}\cdot\frac{(R_{K,m})^2 R_{F,m}}{(R_{k,m})^2 R_{L,m}} \tag{1.9}$$

(it is easy to see that indeed the signs appearing in (1.2) cancel each other out in (1.9): use for example Lemma 2.37 of [DD]).

In this paper we prove (1.9) without using the Lichtenbaum conjecture and actually in an algebraic way, i.e. we make no use of L-functions at all. To begin with, thanks to the following lemma, the w-term in the above formula is trivial, as in the classical case. As a matter of notation, if A is a ring and M is an A-module, then $\mathrm{tor}_A(M)$ will denote the torsion submodule of M.

LEMMA 1.1. *Let S be a finite set of places of L which is invariant under the action of* $\mathrm{Gal}(L/k)$. *Then*

$$w_{L,m} = w_{F,m} \quad \textit{and} \quad w_{K,m} = w_{k,m}.$$

Proof. Recall that, for any number field E and any prime ℓ, we have

$$\begin{aligned}\mathrm{tor}_{\mathbb{Z}}(H^1(\mathcal{O}_E^S, \mathbb{Z}(m))) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell &\cong \mathrm{tor}_{\mathbb{Z}_\ell}(H^1_{\text{ét}}(\mathcal{O}_E^S[1/\ell], \mathbb{Z}_\ell(m))) \\ &\cong H^0(E, \mathbb{Q}_\ell/\mathbb{Z}_\ell(m))\end{aligned}$$

($m \geq 2$) and the latter has cardinality $v_\ell(\kappa_E(\gamma_E)^m - 1)$, where v_ℓ is the ℓ-adic valuation on $\mathbb{Z}_\ell$ such that $v_\ell(\ell) = 1$, $\kappa_E : \Gamma_E \to \mathbb{Z}_\ell^\times$ is the cyclotomic character evaluated on $\Gamma_E = \mathrm{Gal}(E(\mu_{\ell^\infty})/E)$ and γ_E is any topological generator of Γ_E. Now, since L/k is not abelian, $L \cap F(\mu_{\ell^\infty}) = F$. This shows that

$$v_\ell(\kappa_L(\gamma_L)^m - 1) = v_\ell(\kappa_F(\gamma_F)^m - 1),$$

since restriction maps Γ_L isomorphically onto Γ_F. A similar argument applies to K and k. ∎

The proof of (1.9) is achieved by summing up the following two results (which are proved respectively in Sections 2 and 3), which may be interesting in their own right.

FORMULA 1.2. *The following equality holds:*

$$h_{L,m}^S = p^{-\alpha_m} h_{F,m}^S \left(\frac{h_{K,m}^S}{h_{k,m}^S}\right)^2 u_m,$$

with $\alpha_m = \mathrm{rk}_{\mathbb{Z}_p} H_{F,m} - \mathrm{rk}_{\mathbb{Z}_p} H_{k,m}$, *and*

$$u_m = \frac{(H_{L,m} : H_{F,m} H_{K,m} H_{K',m})((\overline{H}_{F,m})^\Delta : \overline{H}_{k,m})}{((\overline{H}_{F,m})^D : \overline{H}_{k,m})},$$

where, for a number field E, we have set $H_{E,m} = H^1(\mathcal{O}_E^S, \mathbb{Z}_p(m))$ *and* $\overline{H}_{E,m} = H_{E,m}/\mathrm{tor}_{\mathbb{Z}}(H_{E,m})$.

FORMULA 1.3. *With notation as in the previous statement,*

$$p^{-\alpha_m} u_m = \frac{(R_{K,m})^2 R_{F,m}}{(R_{k,m})^2 R_{L,m}}.$$

Note that indeed u_m and α_m do not depend on S because $H_{E,m}$ does not. It will turn out that, as in the classical case, u_m is a power of p.

We will divide the proof of Formula 1.2 into two parts, studying separately p-parts and ℓ-parts for any prime $\ell \neq p$ (using then (1.5) to glue all parts together). It should be stressed that the proof for ℓ-parts with $\ell \neq p$ is really much easier: using just the fact that étale cohomology defines a cohomological Mackey functor (in the sense of Dress; see for example [Bo]) and that $D = D_p$ is not ℓ-hypoelementary, we get even more precise structural relations. On the other hand, the proof for p-parts uses mainly descent and co-descent results for étale cohomology groups (which are described in [KM], see also [Ko]). This is essentially the only piece of arithmetic information which is needed (the rest of the proof being a technical algebraic computation), while any algebraic proof of the classical version of Formula 1.2 (see [Ja], [HK], [Le], ...) uses class field theory. The proof of Formula 1.2 given here can probably be generalized without too much effort to metabelian groups with commutator subgroup of order a power of p and index coprime to p. However this approach seems not to be the best for the general case of an arbitrary Galois group (see below).

In the last section we prove Formula 1.3. The translation of u_m in terms of motivic regulators is done using methods from representation theory which have been introduced by the Dokchitser brothers (see for example [DD]). In particular, we closely follow the strategy of Bartel (see [Ba]), who used Dokchitsers' ideas to prove a statement analogous to ours in the classical case.

After completing the final draft of this paper, I was informed by both A. Bartel and the referee about the existence of a preprint by D. Burns in which he provides a proof of (1.9) that is independent of the conjecture (1.2). Burns's result is much more general than ours (in particular (1.9) is proved for finite Galois extensions with arbitrary Galois group) and the strategy is completely different. He directly proves (1.9) without proving Formulas 1.2 and 1.3 separately.

Notation and standard results

- Throughout the paper, if A is a commutative ring and M is an A-module, $\mathrm{tor}_A(M)$ denotes the torsion submodule of M. We will also use the notation $\overline{M}$ for $M/\mathrm{tor}_A(M)$ without any specific mention of A, since it will be clear from the context which is the ring we are considering. Finally, for any $a \in A$, we set $M[a] = \{m \in M \mid am = 0\}$.
- Let H be a finite group. We denote by $N_H = \sum_{h\in H} h \in A[H]$ the norm element and by $I_H \subseteq A[H]$ the augmentation ideal of the group ring $A[H]$ (again the reference to the ring is omitted). If B is an $A[H]$-module, we use the notation
$$\begin{aligned} B^H &= \{b \in B \mid hb = b \text{ for all } h \in H\},\\ B_H &= B/I_H B,\\ B[N_H] &= \{b \in B \mid N_H b = 0\}. \end{aligned}$$
If ℓ is a prime, $A = \mathbb{Z}_\ell$ and H is a q-group for some prime $q \neq \ell$, then
$$B^H = N_H B \quad \text{and} \quad B[N_H] = I_H B,$$
since B is uniquely q-divisible (being a $\mathbb{Z}_\ell$-module) and hence H-cohomologically trivial.

2. A formula relating higher class numbers and a higher units index. In this section we prove Formula 1.2. First we study the p-part of the problem, which is the most delicate.

The natural number $m \geq 2$ will be fixed throughout the section. For any number field E such that $k \subseteq E \subseteq L$ and any finite set S of places of L, we set

$$U_{E,m}^S = H^1_{\text{ét}}(\mathcal{O}_E^S[1/p], \mathbb{Z}_p(m)) \quad \text{and} \quad A_{E,m}^S = H^2_{\text{ét}}(\mathcal{O}_E^S[1/p], \mathbb{Z}_p(m)).$$

In fact, $U_{E,m}^S$ does not depend on S (see Lemma 2.12) so we shall denote it simply by $U_{E,m}$. We also fix for this section a finite set T of places of L such that

- T is stable under the action of $D = \mathrm{Gal}(L/k)$,
- T contains the ramified places of L/k,
- T contains the places above p.

Since T and m are fixed for this section, we will also use the notation U_E for $U_{E,m}$ and A_E for $A_{E,m}^T$. Both A_E and U_E are abelian groups; however, because of their analogies with the ideal class group and the unit group of E respectively, we are going to use multiplicative notation for them. In particular, if g_1 and g_2 are elements of a group acting on the left on U_E, we have $u^{g_1 g_2} = (u^{g_2})^{g_1}$ for any $u \in U_E$ (and similarly for A_E).

REMARK 2.1. Note that, if Q is a group of automorphisms of E of order 2 which acts trivially on k, then the restriction induces isomorphisms $U_{E^Q} = U_E^Q$ and $A_{E^Q} = A_E^Q$. In fact, $k \subseteq E^Q \subseteq L$ and E/E^Q is a Galois extension. Therefore, thanks to our hypotheses on T, $\mathcal{O}_{E^Q}^T \subseteq \mathcal{O}_E^T$ is an étale Galois extension. The above assertion then follows by the Hochschild–Serre spectral sequence (see [NSW, Proposition 2.1.2 and Corollary 2.1.4]), together with the fact that $H^t(\mathcal{O}_{E^Q}^T, \mathbb{Z}_p(m))$ is Q-cohomologically trivial for any $t \geq 0$.

The following well-known result gives us the description of the G-descent for U_E and A_E (recall that $G = \mathrm{Gal}(L/F)$).

PROPOSITION 2.2 (Kolster–Movahhedi). *The natural map $U_F \to U_L^G$ is an isomorphism and the natural map $A_F \to A_L^G$ fits into the following exact sequence:*

$$0 \to H^1(G, U_L) \to A_F \to A_L^G \to H^2(G, U_L) \to 0.$$

Proof. See [KM, Theorem 1.2], whose hypotheses are satisfied thanks to the properties we required of T. ∎

In the light of Remark 2.1 and Proposition 2.2, we will often identify U_K, $U_{K'}$, U_k and U_F with their images in U_L. In the same way, we will identify A_K and $A_{K'}$ with their images in A_L. We now record the following easy lemma which will be used repeatedly for the rest of this section.

Lemma 2.3. *Let M be a uniquely 2-divisible D-module. Then the Tate isomorphisms $\widehat{H}^j(G,M) \cong \widehat{H}^{j+2}(G,M)$ are Δ-antiequivariant. In particular, for any $j \in \mathbb{Z}$, if $\widehat{H}^j(G,M)$ is finite, we have*

$$|\widehat{H}^j(D,M)| = |\widehat{H}^{j+2}(G,M)|/|\widehat{H}^{j+2}(D,M)|.$$

Proof. A standard application of the Hochschild–Serre spectral sequence shows that the restriction induces isomorphisms

$$\widehat{H}^j(D,M) \cong \widehat{H}^j(G,M)^{\Delta}.$$

Thus we only need to show that Tate's isomorphism $\widehat{H}^j(G,M) \cong \widehat{H}^{j+2}(G,M)$ is Δ-antiequivariant. Recall that the Tate isomorphism is given by the cup product with a fixed generator χ of $H^2(G,\mathbb{Z})$:

$$\widehat{H}^i(G,M) \to \widehat{H}^{i+2}(G,M), \qquad x \mapsto x \cup \chi.$$

The action of $\delta \in \Delta$ on $\widehat{H}^i(G,M)$ is δ_* in the notation of [NSW, Chapter I, §5], and this action is -1 on $H^2(G,\mathbb{Z})$ as can immediately be seen through the isomorphism $H^2(G,\mathbb{Z}) \cong H^1(G,\mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G,\mathbb{Q}/\mathbb{Z})$ (which comes from the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ and the fact that $\mathbb{Q}$ is G-cohomologically trivial, being uniquely p-divisible). Then, by Proposition 1.5.3 of [NSW], $\delta_*(x \cup \chi) = \delta_* x \cup \delta_* \chi = -(\delta_* x) \cup \chi$, which gives the result. ■

The next lemma gives a description of the subgroup $A_K A_{K'} \subseteq A_L$ but there is an analogous version for $U_K U_{K'} \subseteq U_L$ (just replace A by U in the statement).

Lemma 2.4. *The subgroup $A_K A_{K'} \subseteq A_L$ is a D-module and*

$$A_K A_{K'} = \prod_{j=0}^{p-2} A_K^{\tau^j} = \prod_{j=0}^{p-2} A_{\tau^j(K)}.$$

Moreover $I_G A_L \subseteq A_K A_{K'}$.

Proof. For the first assertion, see [HK, Lemma 1]. For the last one, see [Le, Lemma 3.4]. ■

We now start the proof of the p-part of Formula 1.2.

Lemma 2.5. *Define $\iota : A_K \oplus A_{K'} \to A_L$ as $\iota(a,a') = aa'$. Then there is an exact sequence*

$$0 \to H^0(D,A_L) \to A_K \oplus A_{K'} \xrightarrow{\iota} A_L \to H_0(G,A_L)/H_0(G,A_L)^{\Delta} \to 0.$$

Proof. It is easy to see that the map $\mathrm{Ker}\,\iota \to H^0(D,A_L)$ given by $(a,a') \mapsto a$ is indeed an isomorphism. As for the cokernel of ι, note that

$$A_{K'} I_G A_L / I_G A_L = (A_L/I_G A_L)^{\langle \tau^2 \sigma \rangle} = (A_L/I_G A_L)^{\langle \sigma \rangle} = A_K I_G A_L / I_G A_L.$$

Therefore

$$H_0(G,A_L)^{\Delta} = (A_L/I_G A_L)^{\Delta} = A_K A_{K'} I_G A_L / I_G A_L = A_K A_{K'}/I_G A_L,$$

since $I_G A_L \subseteq A_K A_{K'}$ by Lemma 2.4. ■

Lemma 2.6. *The following equality holds:*

$$|H^0(D,A_L)| = \frac{|H^2(D,U_L)|\cdot|A_k|}{|H^1(D,U_L)|}.$$

Proof. Take Δ-invariants of the exact sequence in the statement of Proposition 2.2: the sequence stays exact. We also see that $(A_L^G)^\Delta = A_L^D$ and $A_F^\Delta \cong A_k$ (by Remark 2.1). Furthermore, using the Hochschild–Serre spectral sequence, we find $H^j(G,U_L)^\Delta = H^j(D,U_L)$ for $j\geq 0$, because $H^j(G,U_L)$ is Δ-cohomologically trivial (being uniquely 2-divisible). ■

The next lemma describes the G-codescent for A_L.

Lemma 2.7. *The corestriction map induces isomorphisms* $H_0(G,A_L) \cong A_F$ *and* $H_0(G,A_L)^\Delta \cong A_k$.

Proof. For the first isomorphism use [KM, Proposition 1.3]. Since corestriction commutes with conjugation, the second isomorphism follows from the first and Remark 2.1. ■

In what follows, we shall rewrite the orders of $H^1(D,U_L)$ and $H^2(D,U_L)$ in terms of certain units indexes. We first quote a simple lemma which has already been used by Lemmermeyer (see [Le, Section 5]).

Lemma 2.8. *Let $f: B\to B'$ be a homomorphism of abelian groups and let C be a subgroup of B. Then there is an exact sequence*

$$0\to (C+\operatorname{Ker} f)/C \to B/C \xrightarrow{f} f(B)/f(C)\to 0.$$

In particular, if $f(C)$ is of finite index in $f(B)$ and $\operatorname{Ker} f\cap C$ *is of finite index in* $\operatorname{Ker} f$, *then C is of finite index in B and*

$$(B:C) = (f(B):f(C))\cdot(\operatorname{Ker} f:\operatorname{Ker} f\cap C).$$

Proof. Clear. ■

Remark 2.9. Applying the preceding lemma with $B=B'=U_L$, $f=N_G$ and $C=U_KU_{K'}U_F$, we get

$$\begin{aligned}(U_L &: U_KU_{K'}U_F)\\ &= (N_GU_L : N_G(U_KU_{K'}U_F))\cdot(U_L[N_G]:U_L[N_G]\cap U_KU_{K'}U_F).\end{aligned}$$

Note that $U_KU_{K'}U_F$ is indeed of finite index in U_L because $N_G(U_KU_{K'}U_F)$ is of finite index in N_GU_L (since both are of finite index in $U_F=U_L^G$), and that $U_L[N_G]\cap U_KU_{K'}U_F$, which contains I_GU_L by Lemma 2.4, is of finite index in $U_L[N_G]$.

Recall (see Section 1) that $M[p]$ is the kernel of multiplication by p on the $\mathbb{Z}_p$-module M.

Lemma 2.10. *We have*

$$|H^1(D,U_L)| = |U_L[N_G]/(U_L[N_G]\cap U_KU_{K'}U_F)|\cdot|I_GU_L\cdot U_F[p]/I_GU_L\cdot U_k[p]|$$

and

$$|H^2(D,U_L)| = \frac{|U_F/U_F^p|}{|U_k/U_k^p|\cdot|N_GU_L/N_G(U_KU_{K'}U_F)|}.$$

Proof. Let us prove the first assertion. The norm map

$$U_L \xrightarrow{1+\sigma} N_{\langle\sigma\rangle}U_L = U_L^{\langle\sigma\rangle} = U_K$$

gives a map $U_L[N_D] \to U_L[N_G] \cap U_KU_{K'}U_F$. Consider the induced map

$$\overline{N} : U_L[N_D] \to (U_L[N_G]\cap U_KU_{K'}U_F)/I_GU_L\cdot U_F[p].$$

Note that indeed $U_F[p] \subseteq U_L[N_G]$ (since N_G is raising to the pth power on U_F) and $I_GU_L \subseteq U_KU_{K'}$ (see Lemma 2.4). We also have $I_GU_L \subseteq \operatorname{Ker}\overline{N}$, since, for any $u \in U_L$,

$$u^{(1+\sigma)(1-\tau)} = u^{1-\tau}\cdot u^{(1-\tau^{-1})\sigma} \in I_GU_L. \tag{2.1}$$

Then $\overline{N}$ induces a map

$$U_L[N_D]/I_GU_L \to (U_L[N_G]\cap U_KU_{K'}U_F)/I_GU_L\cdot U_F[p] \tag{2.2}$$

and we claim that this map fits into an exact sequence

$$\begin{aligned}0 \to \operatorname{Ker}\overline{N}/I_GU_L &\to U_L[N_D]/I_GU_L\\ &\to (U_L[N_G]\cap U_KU_{K'}U_F)/I_GU_L\cdot U_F[p] \to 0.\end{aligned}$$

The only nontrivial thing to prove is the surjectivity of the map in (2.2). Take $u \in U_K$, $u' \in U_{K'}$ and $v \in U_F$ such that $N_G(uu'v) = 1$. We can find $t, t' \in U_L$ such that $t^{1+\sigma} = u$ and $(t')^{1+\tau^2\sigma} = u'$. Then

$$1 = N_G(uu'v) = N_D(tt')N_G(v) = N_D(tt')v^p. \tag{2.3}$$

In particular $v^p \in U_L^D = U_k$. Note that

$$U_k \cap U_F^p = U_k^p \tag{2.4}$$

(the surjective map $U_F \xrightarrow{p} U_F^p$ stays surjective after taking Δ-invariants, since $U_F[p]$ is Δ-cohomologically trivial being uniquely 2-divisible). Hence there exists $w \in U_k$ such that $v^p = w^p$, which implies $v = wv_0$ for some $v_0 \in U_F[p]$. Therefore

$$\overline{N}(tt'w^{1/2}) = uu'v \bmod I_GU_L\cdot U_F[p]$$

and the fact that $tt'w^{1/2} \in U_L[N_D]$ is exactly (2.3). This proves that the above short sequence is exact.

Now consider the map

$$\operatorname{Ker}\overline{N}/I_GU_L \xrightarrow{1+\sigma} (I_GU_L\cdot U_F[p]/I_GU_L)^{\Delta}. \tag{2.5}$$

It is clearly surjective since $(I_GU_L\cdot U_F[p]/I_GU_L)^{\Delta} = N_{\Delta}(I_GU_L\cdot U_F[p]/I_GU_L)$, and not only $U_F[p] \subseteq U_L[N_D]$, but actually $U_F[p] \subseteq \operatorname{Ker}\overline{N}$. As for the kernel, note that $I_DU_L \subseteq \operatorname{Ker}\overline{N}$ (since $I_GU_L \subseteq \operatorname{Ker}\overline{N}$ and $(1+\sigma)(1-\sigma) = 0$) and we claim that the kernel of the map in (2.5) is I_DU_L/I_GU_L. Define $Y =$

$\{u \in U_L \mid u^{1+\sigma} \in I_G U_L\}$. Note that $I_G U_L \subseteq Y$ (by (2.1)) and $Y/I_G U_L$ equals the kernel of (2.5). Now, if $u \in Y$, then there exists $v \in U_L$ such that

$$u^{1+\sigma} = v^{1-\tau}$$

and, in particular,

$$u^2 = v^{1-\tau} u^{1-\sigma} \in I_D U_L.$$

Hence $u \in I_D U_L$ and, since clearly $I_D U_L \subseteq Y$, we have in fact equality. Thus we have shown that there is an exact sequence

$$0 \to I_D U_L / I_G U_L \to \operatorname{Ker} \overline{N} / I_G U_L \xrightarrow{1+\sigma} (I_G U_L \cdot U_F[p]/I_G U_L)^{\Delta} \to 0.$$

Moreover

$$\begin{aligned}(I_G U_L \cdot U_F[p]/I_G U_L)^{\Delta} &\cong (U_F[p]/I_G U_L \cap U_F[p])^{\Delta} = U_k[p]/I_G U_L \cap U_k[p] \\ &\cong I_G U_L \cdot U_k[p]/I_G U_L.\end{aligned}$$

Then

$$\begin{aligned}&(U_L[N_G] : U_L[N_G] \cap U_K U_{K'} U_F) \\ &\quad = \frac{(U_L[N_G] : I_G U_L \cdot U_F[p])}{(U_L[N_G] \cap U_K U_{K'} U_F : I_G U_L \cdot U_F[p])} \\ &\quad = \frac{(U_L[N_G] : I_G U_L)}{(I_G U_L \cdot U_F[p] : I_G U_L) \cdot (U_L[N_D] : \operatorname{Ker} \overline{N})} \\ &\quad = \frac{|\widehat{H}^{-1}(G, U_L)| (\operatorname{Ker} \overline{N} : I_D U_L)}{(I_G U_L \cdot U_F[p] : I_G U_L) \cdot (U_L[N_D] : I_D U_L)} \\ &\quad = \frac{|\widehat{H}^{-1}(G, U_L)|}{|\widehat{H}^{-1}(D, U_L)|} \cdot \frac{(I_G U_L \cdot U_k[p] : I_G U_L)}{(I_G U_L \cdot U_F[p] : I_G U_L)} = \frac{|H^1(D, U_L)|}{(I_G U_L \cdot U_F[p] : I_G U_L \cdot U_k[p])}\end{aligned}$$

by Lemma 2.3. Hence the first assertion of Lemma 2.10 is proved.

We now prove the second assertion (this part is actually the same as the last part of the proof of [Le, Theorem 2.4]). Note that $N_G U_K = N_D U_L = N_G U_{K'}$; in particular $N_G U_K \subseteq U_L^D = U_k$. Therefore

$$\begin{aligned}(N_G U_L : N_G(U_K U_{K'} U_F)) = (N_G U_L : U_F^p N_G U_K) &= \frac{(U_F : U_F^p N_G U_K)}{(U_F : N_G U_L)} \\ &= \frac{(U_F : U_F^p)}{(U_F^p N_G U_K : U_F^p) \cdot (U_F : N_G U_L)} \\ &= \frac{(U_F : U_F^p) \cdot (U_F^p U_k : U_F^p N_G U_K)}{(U_F^p U_k : U_F^p) \cdot (U_F : N_G U_L)}.\end{aligned}$$

Now

$$(U_F^pU_k : U_F^pN_GU_K) = \frac{(U_F^pU_k : U_F^p)}{(U_F^pN_GU_K : U_F^p)} = \frac{(U_k : U_F^p \cap U_k)}{(N_GU_K : U_F^p \cap N_GU_K)}$$
$$= \frac{(U_k : U_k^p)}{(N_GU_K : U_k^p)} = (U_k : N_GU_K),$$

because of (2.4) and

$$U_k^p = N_GU_k \subseteq U_F^p \cap N_GU_K = U_F^p \cap N_DU_L \subseteq U_F^p \cap U_k = U_k^p.$$

Therefore, using once more (2.4),

$$(N_GU_L : N_G(U_KU_{K'}U_F)) = \frac{(U_F : U_F^p)(U_k : N_GU_K)}{(U_k : U_k^p)(U_F : N_GU_L)}$$
$$= \frac{|\widehat{H}^0(D,U_L)|}{|\widehat{H}^0(G,U_L)|}\,\frac{(U_F : U_F^p)}{(U_k : U_k^p)} = \frac{(U_F : U_F^p)}{(U_k : U_k^p)|H^2(D,U_L)|},$$

thanks to Lemma 2.3 and $\widehat{H}^0(G,U_L) \cong \widehat{H}^2(G,U_L)$. ∎

Recall (see Section 1) that $\overline{M}$ is our notation for the torsion-free quotient of the $\mathbb{Z}_p$-module M.

Lemma 2.11. *We have*

$$I_GU_L \cap U_F[p] = I_GU_L \cap U_F \quad \textit{and} \quad I_GU_L \cap U_k[p] = I_GU_L \cap U_k.$$

Furthermore, there is an isomorphism

$$I_GU_L \cap U_F \cong \overline{U}_L^G/\overline{U}_F,$$

which is Δ-antiequivariant. In particular, it induces an isomorphism

$$I_GU_L \cap U_F/I_GU_L \cap U_k \cong \overline{U}_L^D/\overline{U}_F^\Delta = \overline{U}_L^D/\overline{U}_k.$$

Proof. The first assertion follows from the fact that G acts trivially on U_F and U_k and therefore

$$I_GU_L\cap U_F \subseteq U_L[N_G]\cap U_F = U_F[p] \quad \text{and} \quad I_GU_L\cap U_k \subseteq U_L[N_G]\cap U_k = U_k[p].$$

Now consider the map

$$\phi : I_GU_L \cap U_F \to \overline{U}_L^G/\overline{U}_F$$

defined by $\phi(u^{1-\tau}) = \overline{u} \bmod \overline{U}_F$ for any $u \in U_L$ such that $u^{1-\tau} \in U_F$ ($\overline{u}$ is the class of u in $\overline{U}_L$). First of all, the definition of ϕ does not depend on the choice of u: namely, if $v^{1-\tau} = u^{1-\tau}$, then $vu^{-1} \in U_L^G = U_F$. Of course, the image of ϕ is contained in $(\overline{U}_L/\overline{U}_F)^G = \overline{U}_L^G/\overline{U}_F$ (this last equality comes from $H^1(G,\overline{U}_F) = \mathrm{Hom}(G,\overline{U}_F) = 0$ since $\overline{U}_F$ is a free $\mathbb{Z}_p$-module with trivial G-action). Moreover ϕ is clearly a homomorphism. To see that it is injective, suppose that $\phi(u^{1-\tau}) = \overline{1} \bmod \overline{U}_F$. This means that there exist $\zeta \in \mathrm{tor}(U_L)$ and $v \in U_F$ such that $u = v\zeta$. But since $\mathrm{tor}(U_L) = \mathrm{tor}(U_F)$

by Lemma 1.1, this implies $u^{1-\tau}=1$. Hence ϕ is injective. To prove the surjectivity of ϕ, choose an element $\overline{u}\in\overline{U}_L^G$. This means $u^\tau=u\xi$ for some $\xi\in\operatorname{tor}(U_L)$. Then we have $u^{1-\tau}\in\operatorname{tor}(U_L)=\operatorname{tor}(U_F)\subseteq U_F$ and $\phi(u^{1-\tau})=\overline{u}\bmod\overline{U}_F$.

The map ϕ is Δ-antiequivariant; in other words, if δ generates Δ, we have

$$\phi((u^{1-\tau})^\delta)=\phi(u^{1-\tau})^{-\delta} \tag{2.6}$$

for any $u^{1-\tau}\in I_GU_L\cap U_F$. In fact,

$$(u^{1-\tau})^\delta=u^{\sigma(1-\tau)}=u^{(1-\tau^{-1})\sigma}=u^{(1-\tau)(\sum_{i=0}^{p-2}\tau^i)\sigma}.$$

Therefore

$$\phi((u^{1-\tau})^\delta)=\overline{u^{(\sum_{i=0}^{p-2}\tau^i)\sigma}}\bmod\overline{U}_F.$$

Hence, in order to verify (2.6), we have to check that

$$v:=u^{(\sum_{i=0}^{p-2}\tau^i)\sigma+\sigma}\in U_F=U_L^G.$$

Now

$$\begin{aligned}v^{1-\tau^{-1}}&=u^{(1-\tau^{-1})(\sum_{i=0}^{p-2}\tau^i)\sigma+(1-\tau^{-1})\sigma}=(u^{1-\tau})^{\sigma(\sum_{i=0}^{p-2}\tau^{-i})+\sigma}\\&=(u^{1-\tau})^{(p-1)\sigma+\sigma}=(u^{1-\tau})^{p\sigma}=1,\end{aligned}$$

since $u^{1-\tau}\in U_F\cap I_GU_L$ (which means that τ acts trivially on it) and it has order p by the first assertion of the lemma. This proves that ϕ is Δ-antiequivariant. To get the last claim of the lemma note that

$$(I_GU_L\cap U_F)^\Delta=I_GU_L\cap U_F\cap U_k=I_GU_L\cap U_k$$

and, since uniquely 2-divisible modules are Δ-cohomologically trivial,

$$(\overline{U}_L^G/\overline{U}_F)^\Delta=\overline{U}_L^D/\overline{U}_F^\Delta\quad\text{and}\quad\overline{U}_F^\Delta=\overline{U}_k.$$

Therefore ϕ induces an isomorphism

$$I_GU_L\cap U_F/I_GU_L\cap U_k\cong\overline{U}_L^D/\overline{U}_k.\ \blacksquare$$

The following two lemmas will allow us to get results for finite sets of places which are more general than our fixed T.

LEMMA 2.12 (Soulé). *Let S be any subset of T which is stable under the action of D and let S' be the union of S and the set of places of L above p. Then, for any subfield E of L containing k,*

$$H^1_{\text{ét}}(\mathcal{O}_E^S[1/p],\mathbb{Z}_p(m))\cong H^1_{\text{ét}}(\mathcal{O}_E^T,\mathbb{Z}_p(m))\cong H^1_{\text{ét}}(E,\mathbb{Z}_p(m))$$

and there is an exact sequence

$$\begin{aligned}0\to H^2_{\text{ét}}(\mathcal{O}_E^S[1/p],\mathbb{Z}_p(m))&\to H^2_{\text{ét}}(\mathcal{O}_E^T,\mathbb{Z}_p(m))\\&\to\bigoplus_{w\in(T\smallsetminus S')_E}H^1_{\text{ét}}(\mathcal{O}_E/\mathfrak{l}_w,\mathbb{Z}_p(m-1))\to 0\end{aligned}$$

where $\mathcal{O}_E/\mathfrak{l}_w$ is the residue field of E at w.

Proof. See [So, Proposition 1]. ■

LEMMA 2.13. *With notation as in Lemma* 2.12, *for any* $m' \geq 1$, *the function*

$$\varphi(H) = \prod_{w \in (T \setminus S')_{L^H}} |H^1_{\text{ét}}(\mathcal{O}_{L^H}/\mathfrak{l}_w, \mathbb{Z}_p(m'))|, \tag{2.7}$$

which is defined on the set of subgroups of D *and takes values in* $\mathbb{N}$, *is trivial on* D-*relations* (*in the sense of* [DD, Section 2.3]).

Proof. The function defined in (2.7) is actually a product over v in $(T \setminus S')_k$ of functions φ_v defined by

$$\varphi_v(H) = \prod_{w \mid v \text{ in } L^H} |H^1_{\text{ét}}(\mathcal{O}_{L^H}/\mathfrak{l}_w, \mathbb{Z}_p(m'))|.$$

For any v as above, let f_v be the absolute inertia index of v and ℓ_v the rational prime below v. Consider the function $\psi_v : \mathbb{N} \to \mathbb{N}$ defined by

$$\psi_v(f) = |H^1_{\text{ét}}(\mathbb{F}_{\ell_v^{f_v+f}}, \mathbb{Z}_p(m'))|.$$

Then, if D_v denotes a decomposition group of v in L/k, we have

$$\varphi_v(H) = \prod_{x \in H \backslash D / D_v} \psi_v\Big(\frac{[D_v : I_v]}{[H \cap D_v^x : H \cap I_v^x]}\Big).$$

In particular φ_v is trivial on D-relations by [DD, Theorem 2.36(f)], and therefore the same holds for φ. ■

The next proposition can be seen as the p-part of Formula 1.2.

PROPOSITION 2.14. *Let* S *be a finite set of places of* L *which is stable under the action of* D. *Then*

$$|A^S_{L,m}| = p^{-\alpha_m} |A^S_{F,m}| \, \frac{|A^S_{K,m}|^2}{|A^S_{k,m}|^2} \, \frac{(U_{L,m} : U_{K,m} U_{K',m} U_{F,m})}{((\overline{U}_{L,m})^D : \overline{U}_{k,m})},$$

where $\alpha_m = \operatorname{rk}_{\mathbb{Z}_p} U_{F,m} - \operatorname{rk}_{\mathbb{Z}_p} U_{k,m} = \operatorname{rk}_{\mathbb{Z}} H_{F,m} - \operatorname{rk}_{\mathbb{Z}} H_{k,m}$ *is as in Formula* 1.2.

Proof. First we prove the proposition in the case where $S = T$. Thanks to Lemma 2.11, we have

$$\begin{aligned}(I_G U_L \cdot U_F[p] : I_G U_L \cdot U_k[p]) &= \frac{(I_G U_L \cdot U_F[p] : I_G U_L)}{(I_G U_L \cdot U_k[p] : I_G U_L)} \\ &= \frac{(U_F[p] : I_G U_L \cap U_F[p])}{(U_k[p] : I_G U_L \cap U_k[p])} = \frac{(U_F[p] : I_G U_L \cap U_F)}{(U_k[p] : I_G U_L \cap U_k)} \\ &= \frac{(U_F[p] : U_k[p])}{(I_G U_L \cap U_F : I_G U_L \cap U_k)} = \frac{(U_F[p] : U_k[p])}{(\overline{U}_L^D : \overline{U}_k)}.\end{aligned}$$

Therefore

$$\frac{(U_F:U_F^p)}{(U_k:U_k^p)}=p^{\alpha_m}(U_F[p]:U_k[p]),$$

where $\alpha_m=\mathrm{rk}_{\mathbb{Z}_p}U_{F,m}-\mathrm{rk}_{\mathbb{Z}_p}U_{k,m}=\mathrm{rk}_{\mathbb{Z}}H^1_{F,m}-\mathrm{rk}_{\mathbb{Z}}H^1_{k,m}$ (the last equality comes from (1.5)) is as in Formula 1.2. Now consider the exact sequence of Lemma 2.5: we get, using all the preceding lemmas and Remark 2.9,

$$\begin{aligned}|A_L|&=\frac{|A_K|^2|H_0(G,A_L)|\,|H^1(D,U_L)|}{|A_k|\,|H_0(G,A_L)^{\Delta}|\,|H^2(D,U_L)|}\\&=|A_F|\,\frac{|A_K|^2}{|A_k|^2}\,\frac{(U_L:U_FU_KU_{K'})(I_GU_L\cdot U_F[p]:I_GU_L\cdot U_k[p])(U_k:U_k^p)}{(U_F:U_F^p)}\\&=p^{-\alpha_m}|A_F|\,\frac{|A_K|^2}{|A_k|^2}\,\frac{(U_L:U_FU_KU_{K'})}{(\overline{U}_L^D:\overline{U}_k)}.\end{aligned}$$

To get the statement for any S, note that, by Lemma 2.13 and the second assertion of Lemma 2.12, the function

$$H\mapsto\frac{|A^T_{L^H,m}|}{|A^S_{L^H,m}|}$$

is trivial on the relation (1.6). ■

We now deal with the general proof of Formula 1.2. We are going to use the language and some results of the theory of cohomological Mackey functors; instead of recalling definitions we prefer to directly refer the reader to [Bo, Section 1]. The next result is essentially a consequence of the fact that $D=D_p$ is not ℓ*-hypoelementary* (a group is ℓ-hypoelementary if it has a normal ℓ-subgroup with cyclic quotient) if ℓ is any prime different from p.

PROPOSITION 2.15. *Let ℓ be a rational prime different from p. Let S be a finite set of places of L which is stable under the action of $D=\mathrm{Gal}(L/k)$. Then there is an isomorphism of abelian groups*

$$\begin{aligned}&H^2_{\text{ét}}(\mathcal{O}_L^S[1/\ell],\mathbb{Z}_\ell(m))\oplus H^2_{\text{ét}}(\mathcal{O}_k^S[1/\ell],\mathbb{Z}_\ell(m))^{\oplus 2}\\&\qquad\cong H^2_{\text{ét}}(\mathcal{O}_F^S[1/\ell],\mathbb{Z}_\ell(m))\oplus H^2_{\text{ét}}(\mathcal{O}_K^S[1/\ell],\mathbb{Z}_\ell(m))^{\oplus 2}.\end{aligned}$$

Proof. Note that the function which assigns to any subgroup H of D the abelian group $H^2_{\text{ét}}(\mathcal{O}^S_{L^H}[1/\ell],\mathbb{Z}_\ell(m))$ is a cohomological Mackey functor on D. Since $\ell\neq p$, $D=D_p$ is not ℓ-hypoelementary, which allows us to apply Theorem 1.8 of [Bo] to conclude. ■

Together with Formula 1.3 and a generalization of a result of Brauer (see [Ba, Theorem 5.1]), the fact that $D=D_p$ is not ℓ-hypoelementary if $\ell\neq p$ can also be used to give a proof of the next lemma. Here we give another proof which shows that one can prove Formula 1.2 without using Formula 1.3.

LEMMA 2.16. *Let S be a finite set of places of L which is stable under the action of $D = \mathrm{Gal}(L/k)$. Then the number u_m is a power of p. More precisely the following equality holds:*

$$\begin{aligned}(2.8)\qquad &\frac{(U_{L,m} : U_{K,m}U_{K',m}U_{F,m})}{((\overline{U}_{L,m})^D : \overline{U}_{k,m})}\\ &\qquad = \frac{(H_{L,m} : H_{F,m}H_{K,m}H_{K',m})((\overline{H}_{F,m})^{\Delta} : \overline{H}_{k,m})}{((\overline{H}_{L,m})^D : \overline{H}_{k,m})} = u_m,\end{aligned}$$

where $H_{E,m} = H^1(\mathcal{O}_E^S, \mathbb{Z}(m))$ for any subfield E of L containing k.

Proof. Note that, thanks to (1.5), in (2.8) the greatest power of p dividing the right-hand side equals the left-hand side. Then what remains to show is that, for any fixed prime $\ell \neq p$,

$$(2.9)\qquad \frac{(V_L : V_K V_{K'} V_F)((\overline{V}_F)^D : \overline{V}_k)}{((\overline{V}_L)^D : \overline{V}_k)} = 1,$$

where

$$V_E = H^1_{\text{ét}}(\mathcal{O}_E^S[1/\ell], \mathbb{Z}_\ell(m)).$$

But this can be proved using Lemma 2.8. The details are as follows: first of all

$$\begin{aligned}(2.10)\qquad &(V_L : V_K V_{K'} V_F)\\ &\quad = (N_G V_L : N_G(V_K V_{K'} V_F)) \cdot (V_L[N_G] : V_L[N_G] \cap V_K V_{K'} V_F) = 1,\end{aligned}$$

since

- $N_G(V_L) = V_L^G = V_F$ (because V_L is G-cohomologically trivial and an appropriate version of Remark 2.1 applies);
- $N_G(V_F) = V_F^p = V_F$ (because G acts trivially on V_F and V_F is uniquely p-divisible);
- $V_L[N_G] = I_G V_L \subseteq V_K V_{K'}$ (because V_L is G-cohomologically trivial and an appropriate version of Lemma 2.4 holds).

Moreover $\overline{V}_L^G = \overline{V}_F$ (because $\mathrm{tor}_{\mathbb{Z}_\ell}(V_L)$ is G-cohomologically trivial and $V_L^G = V_F$), showing that (2.9) holds. ∎

The proof of Formula 1.2 is then achieved, because (1.5) allows us to glue together Propositions 2.14 and 2.15 and Lemma 2.16. Note that, if $H_{F,m}$ has no p-torsion, then $u_m = (H_{L,m} : H_{K,m}H_{K',m}H_{F,m})$ (this follows easily from Lemma 2.11).

3. Computations with regulators. In this section we prove Formula 1.3, translating the higher units index

$$u_m = \frac{(H_{L,m} : H_{F,m}H_{K,m}H_{K',m})((\overline{H}_{F,m})^{\Delta} : \overline{H}_{k,m})}{((\overline{H}_{L,m})^{D} : \overline{H}_{k,m})}$$

of Formula 1.2 in terms of motivic regulators, whose definition and basic properties we now briefly review (we refer the reader to [Ne, §1]). Recall from Section 1 that, if E is any number field, we have set $H_{E,m} = H^1(\mathcal{O}_E^S, \mathbb{Z}(m))$.

Let $m \geq 2$ be a natural number. If $X(E) = \mathrm{Hom}(E, \mathbb{C})$ is the set of complex embeddings of E, the mth regulator map we shall consider is a homomorphism

$$\rho_{E,m} : H_{E,m} \to \bigoplus_{\beta \in X(E)} (2\pi i)^{m-1}\mathbb{R}.$$

This map is obtained by composing the natural map $H_{E,m} \to H_{E,m} \otimes \mathbb{Q}$ with the Beilinson regulator map $K_{2m-1}(\mathcal{O}_E) \otimes \mathbb{Q} \to \bigoplus_{\beta \in X(E)} (2\pi i)^{m-1}\mathbb{R}$ via the isomorphism (1.3). Moreover, the image of ρ is contained in the subgroup of $\bigoplus_{\beta \in X(E)} (2\pi i)^{m-1}\mathbb{R}$ which is fixed by complex conjugation:

$$\rho_{E,m} : H_{E,m} \to \Big(\bigoplus_{\beta \in X(E)} (2\pi i)^{m-1}\mathbb{R} \Big)^{+}.$$

The kernel of ρ is exactly the torsion subgroup of $H_{E,m}$ and, thanks to Borel's theorem (and the fact that Beilinson's regulator map is twice Borel's regulator map, see [BG]), we know that ρ induces an isomorphism

$$H_{E,m} \otimes \mathbb{R} \cong \Big(\bigoplus_{\beta \in X(E)} (2\pi i)^{m-1}\mathbb{R} \Big)^{+}.$$

Then the mth regulator of E, denoted $R_{E,m}$, is the covolume of the lattice $\rho_{E,m}(H_{E,m})$ as a subset of the real vector space $(\bigoplus_{\beta \in X(E)} (2\pi i)^{m-1}\mathbb{R})^{+}$. Finally, thanks to the functorial properties of the Beilinson regulator, if a Galois action is defined on E, then ρ is invariant under this action.

REMARK 3.1. If E/E' is a finite extension, then the natural map $H_{E',m} \to H_{E,m}$ is injective and, if E/E' is also Galois, then $H_{E',m} \to H_{E,m}^{\mathrm{Gal}(E/E')}$ is an isomorphism. This can be seen as follows: using (1.5) and Lemma 2.12, it remains to check that the restriction

$$H^1(E', \mathbb{Z}_\ell(m)) \to H^1(E, \mathbb{Z}_\ell(m))$$

is injective for any prime ℓ. It is easy to see that it is enough to prove the injectivity of the above map when E/E' is Galois. In that case the

Hochschild–Serre spectral sequence gives an exact sequence

$$0 \to H^1(\mathrm{Gal}(E/E'), H^0(E, \mathbb{Z}_\ell(m))) \to H^1(E', \mathbb{Z}_\ell(m)) \\ \to H^0(\mathrm{Gal}(E/E'), H^1(E, \mathbb{Z}_\ell(m))) \to H^2(\mathrm{Gal}(E/E'), H^0(E, \mathbb{Z}_\ell(m)))$$

which allows us to conclude because $H^0(E, \mathbb{Z}_\ell(m)) = 0$ (since $m \neq 0$). In what follows, $H_{E',m}$ will be identified with its image in $H_{E,m}$.

To translate u_m in terms of regulators, we are going to use the technique of regulator costants, introduced by the Dokchitser brothers. We first define a scalar product on $H_{E,m}$ in the following way: denote by $\langle -, - \rangle_{E,\infty}$ the standard scalar product on $\mathbb{C}^{|X(E)|}$. Then, for $u, v \in H_{E,m}$, we set

$$\langle u, v \rangle_{E,m} = \langle \rho_{E,m}(u), \rho_{E,m}(v) \rangle_{E,\infty}.$$

It is immediate to see that $\langle -, - \rangle_{E,m}$ is a $\mathbb{Z}$-bilinear map on $H_{E,m} \times H_{E,m}$ which takes values in $\mathbb{R}$. Now, if E/E' is a finite extension and $u, v \in H_{E',m}$, we have

$$\langle u, v \rangle_{E,m} = [E : E'] \langle u, v \rangle_{E',m}. \tag{3.1}$$

To see this, consider the commutative diagram

$$\begin{array}{ccc} H_{E,m} & \xrightarrow{\rho_{E,m}} & (\bigoplus_{\beta \in X(E)} (2\pi i)^{m-1} \mathbb{R})^+ \\ \uparrow & & \uparrow \\ H_{E',m} & \xrightarrow{\rho_{E',m}} & (\bigoplus_{\beta' \in X(E')} (2\pi i)^{m-1} \mathbb{R})^+ \end{array}$$

where the left vertical arrow is the natural map induced by the inclusion $E' \subseteq E$ and the right vertical map is the diagonal embedding, i.e.

$$\{x_{\beta'}\}_{\beta' \in X(E')} \mapsto \{\tilde{x}_\beta\}_{\beta \in X(E)} \quad \text{where } \tilde{x}_\beta = x_{\beta|_{E'}}.$$

Now we have

$$\langle \{x_{\beta'}\}, \{y_{\beta'}\} \rangle_{E',\infty} = \sum_{\beta' \in X(E')} x_{\beta'} y_{\beta'},$$

$$\langle \{\tilde{x}_\beta\}, \{\tilde{y}_\beta\} \rangle_{E,\infty} = \sum_{\beta' \in X(E')} \sum_{\beta | \beta'} \tilde{x}_\beta \tilde{y}_\beta = [E : E'] \sum_{\beta' \in X(E')} x_{\beta'} y_{\beta'},$$

and therefore (3.1) is true.

Note also that, if E/E' is Galois, then $\langle -, - \rangle_{E,m}$ is invariant with respect to the Galois action. Finally, the regulator map being trivial on $\mathrm{tor}_{\mathbb{Z}}(H_{E,m})$, $\langle -, - \rangle_{E,m}$ defines a $\mathbb{Z}$-bilinear map on $\overline{H}_{E,m}$.

Next we recall the definition of the regulator (or Dokchitser) constant in our particular case (see [DD, Definition 2.13 and Remark 2.27] or [Ba, Definition 2.5]).

Definition 3.2. Let M be a $\mathbb{Z}[D]$-module which is $\mathbb{Z}$-free of finite rank. Let $\langle \cdot, \cdot \rangle$ be a D-invariant nondegenerate $\mathbb{Z}$-bilinear pairing on M with values in $\mathbb{R}$. Then the *regulator constant* of M is

$$\mathcal{C}(M) = \frac{\det(\langle -,-\rangle)\det(|D|^{-1}\langle -,-\rangle_{|_{M^D}})^2}{\det(|G|^{-1}\langle -,-\rangle_{|_{M^G}})\det(|\langle\sigma\rangle|^{-1}\langle -,-\rangle_{|_{M^{\langle\sigma\rangle}}})^2} \in \mathbb{Q}^\times.$$

We will be interested in the case where $M = \overline{H}_{L,m}$ and we will use $\langle -,-\rangle_{L,m}$ to compute its regulator constant. We have already observed that $\langle -,-\rangle_{L,m}$ is Galois invariant, and Proposition 3.3 below shows that it is nondegenerate, $R_{L,m}$ being a positive real number.

For any subfield E of L containing k, set

$$\lambda_{E,m} = |\mathrm{coker}(\overline{H}_{E,m} \to \overline{H}_{L,m}^{\mathrm{Gal}(L/E)})|.$$

It is immediate to see that

(3.2) $\quad \lambda_{E,m} = |\ker(H^1(\mathrm{Gal}(L/E), \mathrm{tor}_{\mathbb{Z}}(H_{E,m})) \to H^1(\mathrm{Gal}(L/E), H_{E,m}))|$

and sometimes this description will be useful: for example it shows instantly that $\lambda_{E,m}$ is well-defined (i.e. the order of the above kernel is indeed finite).

The following result is similar to [Ba, Lemma 2.12 and Proposition 2.15]; we sketch the proof since there are slight differences from the proof given in that paper. Recall that $r_1(E)$ (resp. $r_2(E)$) is the number of real places (resp. complex places) of the number field E.

LEMMA 3.3. *We have*

$$\det(\langle\cdot,\cdot\rangle_{E,m}) = ((-1)^{m-1}2)^{r_2(E)}R_{E,m}^2$$

and

$$\mathcal{C}(\overline{H}_{L,m}) = \left(\frac{R_{L,m}R_{k,m}^2}{R_{F,m}R_{K,m}^2}\,\frac{\lambda_{F,m}\lambda_{K,m}^2}{\lambda_{L,m}\lambda_{k,m}^2}\right)^2.$$

Proof. The determinant of

$$\langle -,-\rangle_\infty : \Big(\bigoplus_{\beta\in X(E)}(2\pi i)^{m-1}\mathbb{R}\Big)^+ \times \Big(\bigoplus_{\beta\in X(E)}(2\pi i)^{m-1}\mathbb{R}\Big)^+ \to \mathbb{R}$$

is easily seen to be $((-1)^{m-1}2)^{r_2(E)}$. Then the first assertion follows by well-known properties of scalar products and the definition of $R_{E,m}$.

The proof of the second claim follows from the fact that, for any subgroup J of D,

$$\lambda_{L^J,m}^2\det(|J|^{-1}\langle -,-\rangle_{L,m}|_{\overline{H}_{L,m}^J}) = \det(\langle -,-\rangle_{L^J,m})$$

(this follows easily from (3.2) and (3.1), see also the proof of Proposition 2.15 in [Ba]) and the fact that the function

$$J \mapsto ((-1)^{m-1}2)^{r_2(L^J)},$$

defined on the set of subgroups of D and taking values in $\mathbb{N}$, is trivial on relations (see [DD, Example 2.37]). ∎

The preceding lemma shows that the regulator constant of $\overline{H}_{L,m}$ is related to regulators. The next one shows that it is also related to a higher unit index. First we need some notation: we denote by 1_D the trivial $\mathbb{Q}[D]$-module (and in general, if D' is a subgroup of D, by $1_{D'}$ the trivial $\mathbb{Q}[D']$-module), by ε the $\mathbb{Q}[D]$-module given by the sign, and by ω an irreducible $(p-1)$-dimensional $\mathbb{Q}[D]$-module (which is unique up to isomorphism). As is well-known, $K_0(\mathbb{Q}[D])$ is a torsion-free $\mathbb{Z}$-module of rank 3 which is spanned by 1_D, ε and ω. We denote by $\langle -,-\rangle$ the symmetric scalar product on $K_0(\mathbb{Q}[D])$ for which the basis $\{1_D,\varepsilon,\omega\}$ is orthonormal.

Lemma 3.4 (Bartel). *With the notation above, we have*

$$\mathcal{C}(\overline{H}_{L,m})^{-1/2} = p^{\langle H_{L,m}\otimes\mathbb{Q},1_D-\varepsilon-\omega\rangle/2}(\overline{H}_{L,m} : \overline{H}_{L,m}^{\langle\sigma\rangle}\overline{H}_{L,m}^{\langle\omega^2\sigma\rangle}\overline{H}_{L,m}^{G}).$$

Proof. See [Ba, Lemma 4.8]. ∎

Lemma 3.5. *The following equality holds:*

$$\frac{(R_{K,m})^2R_{F,m}}{(R_{k,m})^2R_{L,m}} = p^{-\alpha_m}(\overline{H}_{L,m} : \overline{H}_{L,m}^{\langle\sigma\rangle}\overline{H}_{L,m}^{\langle\tau^2\sigma\rangle}\overline{H}_{L,m}^{G})\frac{\lambda_{F,m}\lambda_{K,m}^2}{\lambda_{k,m}^2},$$

where $\alpha_m = \mathrm{rk}_{\mathbb{Z}}\, H_{F,m} - \mathrm{rk}_{\mathbb{Z}}\, H_{k,m}$ *is the same as in Formula* 1.2.

Proof. Thanks to Lemmas 3.3 and 3.4, we only have to show that

$$\langle H_{L,m}\otimes\mathbb{Q}, 1_D-\varepsilon-\omega\rangle = -2\alpha_m.$$

The lemma is then an immediate consequence of the following remarks. First, note that

$$\mathrm{Ind}_{\{1\}}^D 1_{\{1\}} = 1_D+\varepsilon+2\omega \quad\text{and}\quad \mathrm{Ind}_G^D 1_G = 1_D+\varepsilon$$

as $\mathbb{Q}[D]$-modules. Then, using Frobenius reciprocity (and Remark 3.1), we get

$$\begin{aligned}
\mathrm{rk}_{\mathbb{Z}}\, H_{k,m} &= \mathrm{rk}_{\mathbb{Z}}\, H_{L,m}^D = \langle H_{L,m}\otimes\mathbb{Q}, 1_D\rangle,\\
\mathrm{rk}_{\mathbb{Z}}\, H_{F,m} &= \mathrm{rk}_{\mathbb{Z}}\, H_{L,m}^G = \langle H_{L,m}\otimes\mathbb{Q}, \mathrm{Ind}_G^D 1_G\rangle,\\
\mathrm{rk}_{\mathbb{Z}}\, H_{L,m} &= \mathrm{rk}_{\mathbb{Z}}\, H_{L,m}^{\{1\}} = \langle H_{L,m}\otimes\mathbb{Q}, \mathrm{Ind}_{\{1\}}^D 1_{\{1\}}\rangle.
\end{aligned}$$

Finally, using (1.4), we get

$$\mathrm{rk}_{\mathbb{Z}}\, H_{L,m} = p\cdot\mathrm{rk}_{\mathbb{Z}}\, H_{F,m},$$

since every infinite prime of F splits completely in L/F. ∎

In order to get the proof of Formula 1.3, we need to compare the right-hand side of the formula of Lemma 3.5 with u_m. For this we need some results about $\lambda_{E,m}$.

Lemma 3.6. *Let Q be any of the subgroups of order* 2 *in D and let M be a finitely generated $\mathbb{Z}_2$-module on which D acts. For any $j\geq 1$, the restriction* *induces isomorphisms*

$$H^j(D,M) \cong H^j(Q,M).$$

Proof. First note that $H^j(D,M)$ is an abelian 2-group since M is a finitely generated $\mathbb{Z}_2$-module. Therefore the restriction map

$$H^j(D,M) \to H^j(Q,M) \tag{3.3}$$

is injective since Q is a 2-Sylow subgroup of D (see [CE, Theorem 10.1 in Chapter XII]). Furthermore, by the description of the image of the restriction given in [CE, Theorem 10.1 in Chapter XII], we see that the map in (3.3) is in fact bijective, since Q has trivial intersection with any of its conjugates different from Q itself (in the language of [CE], any element of $H^j(Q,M)$ is *stable*). ∎

Lemma 3.7. *The following equality holds:*

$$(\overline{H}_{L,m} : \overline{H}_{L,m}^{\langle\sigma\rangle}\overline{H}_{L,m}^{\langle\tau^2\sigma\rangle}\overline{H}_{L,m}^{G})\frac{\lambda_{F,m}\lambda_{K,m}^2}{\lambda_{k,m}^2} = (H_{L,m} : H_{K,m}H_{K',m}H_{F,m})\frac{\lambda_{K,m}}{\lambda_{k,m}}.$$

Proof. Using the notation introduced in Section 2, we have, thanks to (2.10) and Lemma 1.1,

$$\begin{aligned}(H_{L,m} : H_{K,m}H_{K',m}H_{F,m}) &= (U_{L,m} : U_{K,m}U_{K',m}U_{F,m})\\ &= (\overline{U}_{L,m} : \overline{U}_{K,m}\overline{U}_{K',m}\overline{U}_{F,m}).\end{aligned}$$

With similar arguments, it is easy to prove that we also have

$$(\overline{H}_{L,m} : \overline{H}_{L,m}^{\langle\sigma\rangle}\overline{H}_{L,m}^{\langle\tau^2\sigma\rangle}\overline{H}_{L,m}^{G}) = (\overline{U}_{L,m} : \overline{U}_{K,m}\overline{U}_{K',m}\overline{U}_{L,m}^{G}).$$

Furthermore, $\lambda_{K,m}$ is a power of 2, and if, for a prime ℓ, $(\lambda_{k,m})_\ell$ denotes the exact power of ℓ dividing $\lambda_{k,m}$, then we have

$$(\lambda_{k,m})_2 = \lambda_{K,m}$$

(use (3.2) and Lemma 3.6 applied to $M = \mathrm{tor}_{\mathbb{Z}}(H_{L,m}) \otimes \mathbb{Z}_2$ and $M = H_{L,m} \otimes \mathbb{Z}_2$). Thus it remains to check that

$$(\overline{U}_{K,m}\overline{U}_{K',m}\overline{U}_{L,m}^{G} : \overline{U}_{K,m}\overline{U}_{K',m}\overline{U}_{F,m}) = \frac{\lambda_{F,m}}{(\lambda_{k,m})_p}. \tag{3.4}$$

Note that

$$(\overline{U}_{K,m}\overline{U}_{K',m}\overline{U}_{L,m}^{G} : \overline{U}_{K,m}\overline{U}_{K',m}\overline{U}_{F,m}) = (\overline{U}_{L,m}^{G} : (\overline{U}_{K,m}\overline{U}_{K',m}\overline{U}_{F,m})^G).$$

In the rest of the proof we drop the subscript m. Note that $\lambda_F = 1$ or p (since $\mathrm{tor}_{\mathbb{Z}_p}(U_F)$ is cyclic; see the proof of Lemma 1.1) and $(\lambda_k)_p \le \lambda_F$ (this is easily shown using for example (3.2)). Since (3.4) is trivially true if $\overline{U}_L^G = \overline{U}_F$, we shall suppose that $\overline{U}_L^G \ne \overline{U}_F$, i.e. $\lambda_F = p$.

Suppose first that $I_GU_L \cap U_k \ne I_GU_L \cap U_F$ (which is equivalent to $\lambda_F = (\lambda_k)_p = p$ by Lemma 2.11). Let $u \in U_L$ be such that the class of $[u] \in \overline{U}_L$ is nontrivial in $\overline{U}_L^G/\overline{U}_F$; then $u^{\tau-1}=\zeta$ generates $I_GU_L\cap U_F$, which is cyclic of order p (note that we also have $u^{\tau^{-1}-1}=\zeta^{-1}$). Then $u^{1+\sigma}\in U_K\subseteq U_KU_{K'}U_F$ and

$$(u^{1+\sigma})^{\tau-1}=u^{(\tau-1)(1+\sigma)}=u^{\tau-1}u^{(\tau-1)\sigma}=u^{\tau-1}u^{\sigma(\tau^{-1}-1)}=\zeta^{1-\sigma}=\zeta^2$$

because σ acts as -1 on $I_GU_L\cap U_F$ (the latter being different from $I_GU_L\cap U_k$ by assumption). This shows at once that $[u]^{1+\sigma}\in(\overline{U}_K\overline{U}_{K'}\overline{U}_F)^G$ but $[u]^{1+\sigma}\notin\overline{U}_F$ (since σ acts trivially on $\overline{U}_L^G/\overline{U}_F$ by Lemma 2.11 and $[u]\notin\overline{U}_F$) and therefore $\overline{U}_L^G=(\overline{U}_K\overline{U}_{K'}\overline{U}_F)^G$ (since $\overline{U}_L^G/\overline{U}_F$ has order p).

Now suppose that $I_GU_L\cap U_k=I_GU_L\cap U_F$; then we have to show that $(\overline{U}_K\overline{U}_{K'}\overline{U}_F)^G=\overline{U}_F$. First of all we observe that

$$U_K^p\cap U_k=U_k^p. \tag{3.5}$$

In fact, of course $U_K^p\cap U_k\supseteq U_k^p$. Conversely, if $u\in U_K$ and $u^p\in U_k$, then $u^{p(\tau-1)}=1$. In particular $u^{\tau-1}\in I_GU_L\cap U_L[p]=I_GU_L\cap U_F=I_GU_L\cap U_k$ (use Lemmas 1.1 and 2.11), which implies that

$$u^{\sigma(\tau-1)}=u^{\tau-1}.$$

But we also have

$$u^{\sigma(\tau-1)}=u^{(\tau^{-1}-1)\sigma}=u^{\tau^{-1}-1}$$

because $u\in U_K$. Hence we get $u^{\tau^2}=u$. Therefore $u\in U_F\cap U_K=U_k$ and hence $U_K^p\cap U_k\subseteq U_k^p$, which proves (3.5). Now any element in $(\overline{U}_K\overline{U}_{K'}\overline{U}_F)^G$ can be written as $[uv^\tau w]$ with $u,v\in U_K$ and $w\in U_F$ satisfying

$$(uv^\tau w)^{\tau-1}\in\operatorname{tor}(U_L).$$

Actually $(uv^\tau w)^{\tau-1}=(uv^\tau)^{\tau-1}=\xi\in U_k[p]$ (from $I_GU_L\cap U_k=I_GU_L\cap U_F$ and Lemma 1.1). This implies that

$$\frac{u^{p\tau}}{v^{p\tau}}=\frac{u^p}{v^{p\tau^2}}\in U_{K'}.$$

In particular

$$\left(\frac{u^p}{v^{p\tau^2}}\right)^{\tau^2\sigma}=\frac{u^p}{v^{p\tau^2}}.$$

A quick calculation now gives $(uv)^p=(uv)^{p\tau^2}$. This means that $(uv)^p\in U_F\cap U_K=U_k$. Then $(uv)^p\in U_K^p\cap U_k=U_k^p$ thanks to (3.5). So $u=v^{-1}z$ with $z\in U_k$ (recall that $U_k[p]=U_K[p]$ by Lemma 1.1) and therefore

$$(uv^\tau w)^{\tau-1}=(uv^\tau)^{\tau-1}=\left(\frac{v^\tau}{v}\right)^{\tau-1}=\xi\in U_k[p].$$

This implies

$$\frac{v^{p\tau}}{v^p}\in U_F\cap I_GU_L=U_F[p]\cap I_GU_L,$$

which means that $v^\tau/v \in \mathrm{tor}_{\mathbb{Z}_p}(U_L) \subseteq U_F$. Then $\xi = 1$ and $[uv^\tau w] \in \overline{U}_F$. This concludes the proof. ∎

Collecting all the results we have proved so far, we get the proof of Formula 1.3 and therefore that of the higher Brauer–Kuroda relation (1.9).

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