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Perfect powers in q-binomial coefficients

by

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1. Products of consecutive terms of Lucas sequences and powers. Let α and β be complex nonzero numbers with α/β not a root of 1 such that $r := \alpha + \beta$ and $s := -\alpha\beta$ are nonzero coprime integers. Then the sequence $\{u_n\}_{n\geq 0}$ of general term

$$u_n := \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{for all } n \geq 0 \tag{1.1}$$

is called the *Lucas sequence* of roots α and β. It consists of integers. It was shown in [LS] that the equation

$$u_{n+1} \cdots u_{n+k} = y^t \tag{1.2}$$

has only finitely many positive integer solutions (n, k, y, t) with $t \geq 2$ and prime, which are furthermore effectively computable. Moreover, the method of proof of this result from [LS] makes it possible to actually find all such solutions immediately once we know all solutions of the equation

$$u_n = y^t \quad \text{in integers } n \geq 1,\ y \geq 1,\ \text{and } t \geq 2 \text{ prime.} \tag{1.3}$$

Indeed, let $n_1, \ldots, n_r$ be all indices n participating in solutions to equation (1.3). Let $P(m)$ be the largest prime factor of a nonzero integer m with the convention that $P(\pm 1) = 1$. For any prime number p, let $z(p)$ be the *order of appearance* of p in $\{u_n\}_{n\geq 1}$, that is, the smallest positive integer k such that $p \mid u_k$. It is known that $z(p)$ exists for all primes p coprime to s, while primes p dividing s never appear in the factorization of any u_n for $n \geq 1$.

Put

$$P_1 := \max\{3, P(n_1), \ldots, P(n_r)\}. \tag{1.4}$$

A particular case of the main result in [LS] is the following.

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Main Theorem 1.1. *All solutions of the equation*

$$u_{n+1}\cdots u_{n+k}=by^t \tag{1.5}$$

with $n\ge k\ge 2$, and $t\ge 2$ prime, and $z(p)\le k$ for all primes p dividing b have the property that $P((n+1)(n+2)\cdots(n+k))\le P_1$.

Proof. Put $p:=P((n+1)(n+2)\cdots(n+k))$. Since $n\ge k$, a well-known theorem of Sylvester asserts that $p>k$. Let $i_0\in\{1,\dots,k\}$ be the unique index such that $p\,|\,n+i_0$, and write $n+i_0=:p^a m$ with some positive integers a and m, where $p\nmid m$. Observe that p is coprime to $n+i$ for all $i\ne i_0$ in $\{1,\dots,k\}$. Assume also that $p>P_1$ and we shall see that this leads to a contradiction. Rewrite equation (1.5) as

$$u_{p^a}\left(\frac{u_{n+i_0}}{u_{p^a}}\right)\prod_{\substack{i\ne i_0\\ 1\le i\le k}} u_{n+i}=by^t. \tag{1.6}$$

The argument following (2.1) on page 301 in [LS] shows that u_{p^a} is coprime to the remaining factors on the left-hand side of (1.6) above. Let us go quickly through this argument. Since $\gcd(u_a,u_b)=u_{\gcd(a,b)}$ for all positive integers a and b, it follows that $\gcd(u_{p^a},u_{n+i})=u_{\gcd(p^a,n+i)}=u_1=1$ for all $i\ne i_0$ in $\{1,\dots,k\}$. Furthermore, it is well-known that if q is a prime factor dividing both u_{p^a} and u_{n+i_0}/u_{p^a}, then $q\,|\,(n+i_0)/p^a=m$. Thus, $q\,|\,m$. If q divides the discriminant $\Delta:=r^2+4s$ of $\{u_n\}_{n\ge 1}$, then $q\,|\,u_q$, and since also $q\,|\,u_{p^a}$, it follows that $q=p$, but then q cannot divide m. Thus, q does not divide Δ, so $q\equiv\pm 1 \pmod{z(q)}$. Since also $q\,|\,u_{p^a}$, it follows that $p\,|\,z(q)$. Hence, $q\equiv\pm1 \pmod p$, and since $p\ge 5$, we see that $q\ge 2p-1>p$, so again q cannot divide m. This shows that indeed u_{p^a} is coprime to the remaining factors on the left-hand side of (1.6). Observe that since $p>k$, the number u_{p^a} is also clearly coprime to b since b is divisible only by primes q having $z(q)\le k$. Hence, $u_{p^a}=y_1^t$ for some divisor y_1 of y, therefore $p^a\in\{n_1,\dots,n_r\}$, which contradicts the fact that $p>P_1$. ∎

In [LS], it was shown that equation (1.2) has no solutions in the particular case when the Lucas sequence is the sequence $\{F_n\}_{n\ge1}$ of Fibonacci numbers. The interesting feature of that proof is that all solutions of equation (1.3) for this sequence, i.e., all the perfect powers of exponent >1 in the Fibonacci sequence, were not yet known at the time [LS] was written, thus the proof from [LS] does not make use of this information. Since now we know thanks to work of Bugeaud, Mignotte and Siksek [BMS] that the set of solutions n to equation (1.3) for the case of the Fibonacci numbers is $\{1,2,6,12\}$, we deduce that Theorem 1.1 has the following immediate corollary.

Corollary 1.2. *Equation* (1.5) *has no solutions with $n\ge k\ge 2$ for the case of the Fibonacci sequence.*

Proof. Indeed, since the set of solutions to equation (1.3) is $\{1, 2, 6, 12\}$, Theorem 1.1 tells us that if $k \geq 2$, then $P((n+1)\cdots(n+k)) \leq 3$. Thus, putting $x := n+k$ and $y := n+k-1$, we find that $x-y=1$ and $P(xy) \leq 3$. Hence, $\{x, y\} = \{3^\gamma, 2^\delta\}$, where $3^\gamma - 2^\delta = \pm 1$. The largest solution of the above Diophantine equation is $3^2 - 2^3 = 1$. Thus, $x \leq 9$, and now a computation by hand convinces us that there is no solution to equation (1.5) with $n \geq k \geq 2$. ∎

Recall that given a Lucas sequence $\{u_n\}_{n\geq 1}$, the *u-binomial coefficient* is defined as

$$\begin{bmatrix} m \\ k \end{bmatrix} := \frac{u_{m-k+1}\cdots u_m}{u_1\cdots u_k}.$$

Since $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_u = \left[\begin{smallmatrix} m \\ m-k \end{smallmatrix}\right]_u$, and we are interested only in these quantities as integers, we shall assume that $m \geq 2k$, therefore $n := m-k \geq k$. In [MT], it was shown that the Fibonomial coefficient is never a perfect power. However, writing this as

$$F_{n+1}\cdots F_{n+k-1} = by^t, \qquad \text{where } b := F_1\cdots F_k,$$

the result from [MT] is easily seen to be an immediate consequence of Corollary 1.2. Moreover, Corollary 1.2 can be used to deal with other Lucas sequences also. In what follows, we give two examples.

Take first the Pell sequence $\{P_n\}_{n\geq 1}$, which is the Lucas sequence with roots $\alpha := 1+\sqrt{2}$ and $\beta := 1-\sqrt{2}$. The only solutions of equation (1.3) have $n \in \{1, 7\}$ (see [C] and [P]). Hence, Theorem 1.1 tells us that all solutions of equation (1.5) for this sequence have $P((n+1)\cdots(n+k)) \leq 7$. Putting $x := n+k,\ y := n+k-1$, we get $x-y=1$ and $P(xy) \leq 7$. All solutions of this particular Diophantine equation appear in [A] (see also Chapter 6 of de Weger's Ph.D. dissertation [W]). These give us certain possibilities, the largest one being $x = 4375$. If $P(x-2) > 7$, then $k = 2$. This means that $P_{n+1}P_{n+2}/P_2$ is a perfect power. Since $P_2 = 2$, and P_n is even if and only if n is, it follows that if we put $m \in \{n, n+1\}$ such that m is odd, then P_m is a perfect power, therefore $m \leq 7$. Thus, either $x \leq 8$, or $P(x-2) \leq 7$. The positive integers x with $P(x(x-1)(x-2)) \leq 7$ are $x = 50, 16$, and $x \in [3, 10]$, and one checks by hand that these do not lead to any convenient solution either. Hence, we record the following corollary.

Corollary 1.3. *The only solution of the equation* $\left[\begin{smallmatrix} m \\ k \end{smallmatrix}\right]_P = y^t$ *with* $m \geq 2k \geq 2$ *and* $t \geq 2$ *is* $(m, k) = (7, 1)$.

More generally, let $D \geq 3$, and let $\alpha := v_1 + \sqrt{D}\,u_1$ and $\beta := v_1 - \sqrt{D}\,u_1$, where (v_1, u_1) is the minimal solution in positive integers of the Pell equation $v^2 - Du^2 = 1$. Let $\{u_n\}_{n\geq 1}$ be the Lucas sequence of roots α and β. The numbers u_n are precisely all the possible positive integer solutions u in the

Pell equation $v^2 - Du^2 = 1$. All perfect powers in the sequence $\{u_n\}_{n\geq 1}$ for all $2 \leq D \leq 100$ appear in Theorem 7.1 in [B]. A careful inspection of this list of solutions from [B] reveals that all solutions of equation (1.3) for these sequences have $n \in \{1,2\}$, therefore again if $n \geq k \geq 2$ is a solution of (1.5), then with $x := n+k$ and $y := n+k-1$, we have $P(xy) \leq 3$. Thus, again $x \leq 9$, and now a quick check convinces us that there is no convenient solution with $k \geq 2$ to these Diophantine equations. We record this result as follows.

Corollary 1.4. *Let $\{u_n\}_{n\geq 1}$ be the Lucas sequence consisting of the components u of the positive integer solutions (v,u) to the Pell equation $v^2 - Du^2 = 1$. Then, for $2 \leq D \leq 100$, all solutions of the equation $\begin{bmatrix} m \\ k \end{bmatrix}_u = y^t$ with $m \geq 2k \geq 2$ and $t \geq 2$ prime have $k = 1$.*

2. Perfect powers in q-binomial coefficients. From now on until the rest of the paper, we work with the Lucas sequence $\{u_n\}_{n\geq 1}$ of roots $\alpha := q$ and $\beta := 1$, where $q \geq 2$ is an integer. In this case, the u-binomial coefficient is called the *q-binomial coefficient*. In [LS], it was proved that the Diophantine equation (1.2) has no solution with $k \geq 2$ for this sequence (treating also $q \geq 2$ as an unknown integer). Again, the proof of this result from [LS] circumvents knowledge of the solutions to equation (1.3), which in this case is

$$\frac{q^n - 1}{q - 1} = y^t \quad \text{in integers } q \geq 2,\ n \geq 3,\ y \geq 2, \text{ and } t \geq 2 \text{ prime}, \tag{2.1}$$

because the complete list of solutions to equation (2.1) is not yet known. The known solutions are

$$\frac{3^5 - 1}{3 - 1} = 11^2, \qquad \frac{7^4 - 1}{7 - 1} = 20^2, \qquad \frac{18^3 - 1}{18 - 1} = 7^3. \tag{2.2}$$

We next study the q-binomial coefficients which are powers. Our result is the following.

Main Theorem 2.1. *All the solutions of the equation*

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = y^t \quad \textit{in integers } q \geq 2,\ m \geq 2k \geq 1,\ y \geq 2, \textit{ and } t \geq 2 \textit{ prime} \tag{2.3}$$

have $k = 1$.

This complements a result of K. Győry [G1], [G2] who found all binomial coefficients which are perfect powers. The proof uses a result of Pethő [P] concerning perfect powers in the Pell sequence.

We record a couple of known facts that turn out to be useful.

Lemma 2.2. *Assume that (q,n,y,t) is a solution of equation (2.1) with $n \geq 3$, and $t \geq 2$ prime. Then:*

(i) *q is not square.*
(ii) *If (q,n,y,t) is not in the list (2.2), then both n and t are odd, and if p is the smallest prime factor of n, then either $p \geq 29$, or $p = t \in \{17, 19, 23\}$.*

The proofs of the statements summarized in Lemma 2.2 can be found in [BHM], [BMRS], and [M].

We shall also need the following extension of a result of Faulkner [F].

LEMMA 2.3. *Assume that $m \geq 2.5k \geq 5$. Then $P\big(\binom{m}{k}\big) > 2k+1$ except if the pair (m,k) belongs to the set*

$$\begin{aligned}(2.4)\qquad &\{(5,2),(6,2),(9,2),(8,3),(9,3),(10,2),(10,3),(10,4),(15,6),\\ &(16,2),(16,3),(16,6),(25,2),(28,11),(50,3),(81,2)\}.\end{aligned}$$

Proof. We follow Faulkner [F]. Let p_k stand for the smallest prime factor $\geq 2k$. In [F], it is shown that $P\big(\binom{m}{k}\big) \geq p_k$ for all $m \geq 2k \geq 4$, except when $(m,k) = (9,2),(10,3)$. In his proof, Faulkner used the inequality

$$\theta(x) < 1.01624x, \qquad \text{valid for all } x > 0, \tag{2.5}$$

where $\theta(x) := \sum_{p \leq x} \log p$, which he had taken from [RS1]. We shall use instead the sharper inequality

$$\theta(x) < 1.001102x, \qquad \text{valid for all } x > 0, \tag{2.6}$$

from [RS2], which appeared nine years after Faulkner's paper. So, let us follow the proof of the theorem in [F] by replacing everywhere inequality (2.5) by (2.6), and using also the inequality

$$\pi(x) < 1.25506x/\log x, \qquad \text{valid for all } x > 1. \tag{2.7}$$

Instead of pointing out what one should change where, we simply go through the entire proof. First, a computation in the range $5 \leq m \leq 1100$ leads to the solutions shown in (2.4). So from now on, $m \geq 1101$.

Assume that $\binom{m}{k}$ has no prime factor $p > 2k+1$. Then

$$\left(\frac{m}{k}\right)^k \leq \binom{m}{k} = \prod_{\substack{p^\alpha \| \binom{m}{k}\\ p \leq 2k+1}} p^a \leq m^{\pi(2k+1)}. \tag{2.8}$$

The last inequality above is based on the fact due to Erdős that if $p^a \| \binom{m}{k}$, then $p^a \leq m$. Using inequality (2.7), we get

$$\frac{m}{k} < m^{\pi(2k+1)/k} < m^{1.25506(2+1/k)/\log(2k)}.$$

The right-hand side above is $< m^{1/2}$ for $k \geq 79$. However, since $\pi(2k+1)/k < 1/2$ holds also for $k = 77$, 78 and 79, we conclude that

$$\frac{m}{k} < m^{1/2} \qquad \text{for } k \geq 76,$$

which is the last display on page 107 in [F]. Thus, the assumption is false for $76 \le k \le m^{1/2}$. Assume next that $k > m^{1/2}$. Returning to inequality (2.8), and using (2.6) and (2.7), we have

$$\left(\frac{m}{k}\right)^k \le \binom{m}{k} \le \prod_{p\le 2k+1} p \prod_{\substack{p^a \| \binom{m}{k} \\ p\le\sqrt{m}}} p^a \tag{2.9}$$
$$< \exp(1.001102(2k+1))m^{2\cdot 1.25506\sqrt{m}/\log m}.$$

Hence,

$$\left(\frac{m}{k}\right)^k < \exp(1.001102(2k+1) + 2\cdot 1.25506\sqrt{m}). \tag{2.10}$$

Replacing $\sqrt{m}$ by k and taking kth roots in (2.10) above, we find that $m < \exp(4.55)k$ for $k \ge 76$. However, when $k > \exp(-4.55)m$, we may replace $\sqrt{m}$ by $\exp(4.55)k/\sqrt{m}$ in (2.10) above and take kth roots to obtain

$$m < \exp\left(1.001102(2+1/k) + 2\cdot 1.25506\frac{\exp(4.55)}{\sqrt{m}}\right)k < \exp(3.49)k,$$

provided that $m > 3\cdot\exp(9.1)$, i.e., for $m > 26900$. The initial assumption is then false if $m^{1/2} < k \le \exp(-4.55)m$, or $\exp(-4.55)m < k \le \exp(-3.49)m$, with $m > 26900$.

A simple induction argument shows that

$$\binom{9k}{k} > \frac{21.3^k}{3k} \quad \text{for } k = 1, 2, 3, \ldots.$$

Thus, for $9k \le m < 3^{3.49k}$, it follows from inequality (2.9) that

$$\frac{21.3^k}{3k} \le \binom{m}{k} < \exp(1.001102(2k+1) + 2\cdot 1.25506\sqrt{m}).$$

Taking kth roots, we obtain

$$21.3 < \exp\left(1.001102(2+1/k) + 2\cdot 1.25506\frac{\sqrt{m}}{k} + \frac{\log(3k)}{k}\right). \tag{2.11}$$

Since

$$\frac{\sqrt{m}}{k} < \frac{\exp(3.49)}{\sqrt{m}} < \frac{\exp(3.49)}{\sqrt{26900}} \quad \text{and} \quad \frac{\log(3k)}{k} \le \frac{\log 228}{76},$$

for $26900 < m < \exp(3.49)k$ and $k \ge 76$, a simple calculation shows that (2.11) is false. Our assumption is therefore false for $9k \le m < \exp(3.49)k$ with $m > 26900$ and $k \ge 76$.

The case $k \ge 76$ and $9k \le m \le 26000$ follows as in [F], since the maximum gap, or difference, between consecutive primes < 26900 is < 76. Hence, in this range, $\binom{m}{k}$ has a prime factor $> 8k > 2k+1$.

Thus, it remains to deal with the case when $k < 76$.

For $3k \le m < 9k$, the interval $(8m/9, m)$ is contained in $(m-k, m)$. Next, as argued in [F], the interval $(8m/9, m)$ contains a prime p for all $m \ge 54$, which is our case. To see that this prime p satisfies $p > 2k+1$, it suffices to observe that $p > 8m/9 > (8/9)\cdot(2.5k) = 20k/9$, so indeed $p > 2k+1$ whenever $(20/9)k > 2k+1$, which is true for $k \ge 5$. However, for $k \le 4$ and $m \ge 1101$, we have $(8/9)\cdot 1101 \ge 978 > 9 \ge 2k+1$, so the desired conclusion holds in this case also.

It remains to treat three situations, namely:

(i) $k \in \{2,3,4,5,6,7\}$;
(ii) $8 \le k < 76$ and $m > 3k$;
(iii) $8 \le k$ and $2.5k \le m \le 3k$.

Let us deal with situation (i). If the desired conclusion does not hold, then $P\left(\binom{m}{k}\right) \le 13$. Assume first that $k \in \{5,6,7\}$. Then at most one of the numbers in the interval $[m-k+1, m]$ is a multiple of 13, at most one is a multiple of 11, and at most one is a multiple of 7. Since there are $k \ge 5$ numbers in this interval, it follows that there exist two, say $x > y$, such that $x - y \le 7$ and $P(xy) \le 5$. All solutions to this Diophantine equation appear in [W]. The largest is $6 = 486 - 480 = 2\cdot 3^5 - 2^5\cdot 3\cdot 5$, leading to $n \le 480 + 7 = 487 < 1100$, which is a range already covered. Suppose now that $k \in \{2,3,4\}$. Then $P\left(\binom{m}{k}\right) \le 7$. Hence, with $x := m,\ y := m-1$, we deduce that $x - y = 1$ and $P(xy) \le 7$. The results from [A] or [W] give us a certain list of possibilities. There are only two of them with $m = x \ge 1101$, namely $x = 2401,\ 4305$. In both cases, $x - 2 > 2k+1$ is prime, therefore $k = 2$, but both 2401 and 4305 are multiples of $7 > 2\cdot 2 + 1$. So, case (i) does not lead to new exceptional pairs (m,k) failing the desired property.

Let us now deal with situation (ii). Let us note here that the interval $[m-k+1, m]$ is contained in $[2k+2, m]$, so it is enough to show that the interval $[m-k+1, m]$ contains primes. We follow the arguments on page 108 in [F]. A result of D. H. Lehmer shows that the product of seven consecutive integers ≥ 36 contains a multiple of a prime ≥ 43. Thus, the desired conclusion holds for $8 \le k \le 20$. Hence, we may assume that $k \ge 21$. Now if the desired conclusion is violated for some m and k, then

$$\frac{m^k}{k!}\left(1 - \frac{k(k-1)}{2m}\right) \le \frac{m^k}{k!}\left(1 - \frac{1}{m}\right)\cdots\left(1 - \frac{k-1}{m}\right) = \binom{m}{k} \le m^{\pi(2k+1)}.$$

Assume that $m > (k^2-1)/2$. Then the factor in parentheses on the left-hand side above is $> 1/(k+1)$, so the above inequality implies

$$m^{k-\pi(2k+1)} \le (k+1)!,$$

or, in other words,

$$m < (k+1)!^{1/(k-\pi(2k+1))}.$$

The upper bound above is < 1100 for all k in the range $k \in [21, 75]$. Hence, $1101 \le m \le (k^2-1)/2$ giving $k \ge 47$. The same argument shows that the desired conclusion holds for $m > 3000 > (75^2-1)/2$. Hence, it remains to cover the range $1101 < m \le 3000$ and $47 \le k \le 75$. This follows as in [F] because the maximum gap between consecutive primes for $m < 3000$ is < 46.

Finally, let us deal with situation (iii). Observe that $2.5k > 2k+1$ for all $k \ge 8$. Assume next that the interval $(2.5k, 3k)$ does not contain any prime number. Using the fact that

$$\pi(x) > \frac{x}{\log x - 1.5} \quad \text{and} \quad \pi(x) < \frac{x}{\log x - 0.5} \quad \text{for all } x > 67$$

(see Theorem 2 in [RS1]), we get

$$\pi(3k) - \pi(2.5k) > \frac{3k}{\log(3k) - 0.5} - \frac{2.5k}{\log(2.5k) - 1.5}.$$

The function appearing on the right-hand side above is positive for $k \ge 663$ and a short calculation reveals that $\pi(3k) - \pi(2.5k) > 0$ for all $k \ge 8$, which finishes the argument for (iii) and hence the proof of the lemma. ∎

Proof of Theorem 2.1. Assume that there is a solution (m, k, y, q, t) of equation (2.3) with $m \ge 2k \ge 4$, and t prime. We handle various cases.

Case 1: $k = 2$. Let $n \in \{m-1, m\}$ be such that n is even and let $\{m-1, m\} =: \{n, n+\delta\}$, where $\delta \in \{\pm 1\}$. Then

$$\begin{bmatrix} m \\ 2 \end{bmatrix} = \left(\frac{q^{n+\delta}-1}{q-1}\right)\left(\frac{(q^2)^{n/2}-1}{q^2-1}\right) = y^t,$$

and the two factors in the middle are coprime. Thus, $(x^{n/2}-1)/(x-1) = y_1^t$ with the perfect square $x := q^2$ and with some divisor y_1 of y. If $n/2 \ge 3$, this is impossible by (i) of Lemma 2.2, while if $n/2 = 2$, then $q^2+1 = x+1 = y_1^t$, which is not possible with $q \ge 2$ by known results on the Catalan equation. From now on, $k \ge 3$.

Case 2: $t = 2$. Put $n := m - k$ and observe that $n \ge k$. Put $p := P((n+1)\cdots(n+k))$ and observe that $p > k$ by Sylvester's theorem. Since $k \ge 3$, we have $p \ge 5$. Let, as in the proof of Theorem 1.1, i_0 stand for the unique index in $\{1, \ldots, k\}$ such that $p \mid n + i_0$, and write $n + i_0 := p^a l$ for some integers $a \ge 1$ and l coprime to p. The argument from the proof of Theorem 1.1 shows that $u_{p^a} = y_1^2$ for some divisor y_1 of y. Lemma 2.2 shows that $(q, p^a) = (3, 5)$. Thus, $P(m(m-1)(m-2)) = 5$, and the only possibility is $m = 6$, for which $p = 5$ and $q = 3$. However, $\left[\begin{smallmatrix} 6 \\ 3 \end{smallmatrix}\right]_3 = 33880$ is not a perfect square. From now on, $t \ge 3$.

CASE 3: $k=3$. This is a warm-up for the more general case that follows, but it is worth doing it separately since it has its particularities. Let $\mathcal{M}$ be the set of even indices in the interval $[m-2,m]$. Then $\mathcal{M}=\{m-2,m\}$ or $\{m-1\}$, according to whether m is even or odd. Let $p:=P(\prod_{m\in\mathcal{M}} m)$, and assume that $p\ge 5$. Let m_0 be the unique index in $\mathcal{M}$ such that $p\,|\,m_0$. Write $m_0=:2p^a l$, where a and l are positive integers with l coprime to p. Rewrite equation (2.3) as

$$\frac{q^{2p^a}-1}{q^2-1}\left(\frac{q^{m_0}-1}{q^{2p^a}-1}\right)\prod_{\substack{n\in\{m-2,m-1,m\}\\ n\neq m_0}}\frac{q^n-1}{q-1}=\left(\frac{q^3-1}{q-1}\right)y^t. \tag{2.12}$$

We now argue that the first factor on the left-hand side in (2.12) above is coprime to all other factors on the left and also to $u_3=(q^3-1)/(q-1)$, which is on the right-hand side. Indeed, if r is a prime dividing both $(q^{2p^a}-1)/(q^2-1)$ and $(q^n-1)/(q-1)$ for some $n\neq m_0$ in $[m-2,m]$, then $r\,|\,(q^{2n}-1)/(q^2-1)$. Hence, r divides both w_{p^a} and w_n, where $\{w_j\}_{j\ge1}$ is the Lucas sequence of roots $\alpha:=q^2$ and $\beta:=1$. But this is impossible since p^a and n are coprime. The same argument shows that $(q^{2p^a}-1)/(q^2-1)$ is coprime to $(q^3-1)/(q-1)$, which is a divisor of $(q^6-1)/(q^2-1)$, because we are assuming that $p>3$. Finally, assume that r divides both $(q^{2p^a}-1)/(q^2-1)$ and $(q^{m_0}-1)/(q^{2p^a}-1)$. Then r must divide $m_0/2$. However, since r divides either $(q^{p^a}-1)/(q-1)$ or $(q^{p^a}+1)/(q+1)$, it follows that either $r=p$ (and this happens if and only if $q\equiv\pm1 \pmod p$), or $r\equiv 1 \pmod p$. In both cases, $r\ge p>P(m_0/2)$, so r cannot divide $m_0/2$. Now we deduce that $(q^{2p^a}-1)/(q^2-1)=y_1^t$ for some divisor y_1 of y, which is impossible by (i) of Lemma 2.2.

Thus, $p\le 3$. Suppose next that m is even. Then $m=2m_0$, where $P(m_0(m_0-1))\le 3$. Since $m_0\ge 3$, it follows that the only possibilities are $m_0\in\{3,4,9\}$, so $m\in\{6,8,18\}$. When $m=6$, we get

$$\begin{bmatrix}6\\3\end{bmatrix}_q=\left(\frac{q^5-1}{q-1}\right)\left(\frac{(q^4-1)(q^6-1)}{(q^2-1)(q^3-1)}\right)=y^t,$$

and in the middle above the first factor is coprime to the cofactor. Hence, we get $(q^5-1)/(q-1)=y_1^t$ for some divisor y_1 of t, which is impossible for $t\ge3$ by (ii) of Lemma 2.2. Similarly, when $m=8$, we get an equation of the form $(q^7-1)/(q-1)=y_1^t$, which is impossible by (ii) of Lemma 2.2. Finally, when $m=18$, we get

$$\begin{bmatrix}18\\3\end{bmatrix}_3=(q^8+1)\left(\frac{q^8-1}{q^2-1}\right)\left(\frac{(q^{17}-1)(q^{18}-1)}{(q-1)(q^3-1)}\right)=y^t.$$

In the middle, either q^8+1 is coprime to the remaining cofactor, or the only common prime factor of q^8+1 with the remaining cofactor is 2, and if it is 2, then q is odd, so $2 \,\|\, q^8+1$. Hence, $q^8+1=\delta y_1^t$ for some divisor y_1 of y and some $\delta \in \{1,2\}$, which is impossible.

Suppose next that n is odd and $P(n-1) \le 3$. Then $m-1=2^a3^b \ge 6$ for some $a \ge 1$ and $b \ge 0$. Furthermore, $m-1$ is coprime to both $m-2$ and m. Hence, $(q^{m-1}-1)/(q-1)=y_1^t$ for some divisor y_1 of t, but this is impossible by (ii) of Lemma 2.2.

Case 4: $m \ge 2.5k$. Here, we write $m=2m_0+m_1$, $k=2k_0+k_1$, with integers $m_0,k_0 \ge 2$ and $m_1,k_1 \in \{0,1\}$. Observe that since $m \ge 2.5k$, we get $m/2 \ge 2.5k/2$, so that

$$m_0=\lfloor m/2\rfloor \ge \lfloor 2.5k/2\rfloor = \lfloor 2.5k_0+1.25k_1\rfloor \ge \begin{cases} 2.5k_0 & \text{if } k_0=0,\\ 2.5k_0+1 & \text{if } k_0=1.\end{cases}$$

By Lemma 2.3, we get $P((m_0-k_0+1)\cdots m_0) > 2k_0+1 \ge k$ except when (m_0,k_0) belongs to the list (2.4). Assume that we are in the good case, and we treat the exceptions later. Observe that

$$\{2m_0-2k_0+2, 2m_0-2k_0+4,\ldots,2m_0\} \subseteq \{m-k+1, m-k+2,\ldots,m\}.$$

Write $\mathcal{N}$ for the set of even integers in $[m-k+1,m]$ and put $p:=P(\prod_{n\in\mathcal{N}} m)$. From the above display it follows that $p>k$. In particular, there exists a unique number $n_0 \in \mathcal{N}$ such that $p \mid n_0$. Write $n_0 =: 2p^a l$, with some integers $a \ge 1$ and l coprime to p. We then rewrite equation (2.3) as

$$\frac{q^{2p^a}-1}{q^2-1}\left(\frac{q^{n_0}-1}{q^{2p^a}-1}\right)\prod_{\substack{n\in\{m-k+1,\ldots,m\}\\ n\ne n_0}}\frac{q^n-1}{q-1} = \left(\prod_{3\le i\le k}\frac{q^i-1}{q-1}\right)y^t. \tag{2.13}$$

Arguing as before, we see that the first factor on the left-hand side in (2.13) above, $w_{p^a} := (q^{2p^a}-1)/(q^2-1)$, is coprime to $(q^n-1)/(q^2-1)$ for all $n \ne n_0$ in $[m-k+1,\ldots,m]$, just because $(q^n-1)/(q-1)$ is a divisor of $w_n := (q^{2n}-1)/(q-1)$, and n and p are coprime. The same argument shows that the first factor on the left-hand side in (2.13) above is coprime to each of the factors $(q^i-1)/(q-1)$ for $i=3,\ldots,k$ from the right-hand side, just because $p>k$. Finally, since p is the largest prime factor of n_0 and $n_0/(2p^a)$ is coprime to p, it follows that the first factor on the left-hand side above is also coprime to the second factor on the same side. Hence, $(q^{2p^a}-1)/(q^2-1)=y_1^t$ for some divisor y_1 of y, which is impossible by (ii) of Lemma 2.2.

It remains to deal with the exceptions. Observe that if $k_1=0$, then k is even, and we only want the largest prime factor of $(m_0-k_0+1)\cdots m_0$ to exceed $2k_0=k$, and by Faulkner's result this is so except when $(m_0,k_0)=(9,2),(10,3)$. When $k_1=1$, we get in fact $m_0 \ge 2.5k_0+1$. A quick look in the list (2.4) shows that the only pairs (m,k) to consider when $k_1 = 1$ are

$$\begin{aligned}(6,2),(9,2),(9,3),(10,2),(10,3),(16,2),\\ (16,3),(16,6),(25,2),(50,3),(81,2).\end{aligned} \tag{2.14}$$

Hence, we just need to deal with the following set of 26 exceptional pairs (m,k):

$$\begin{aligned}\{&(18,4),(19,4),(20,6),(21,6),(12,5),(13,5),(18,5),(19,5),(18,7)\\ &(19,7),(20,5),(21,5),(20,7),(21,7),(32,5),(33,5),(32,7),(33,7),\\ &(32,13),(33,13),(50,5),(51,5),(100,7),(101,7),(162,5),(163,5)\}.\end{aligned}$$

In all the above cases except when $(m,k) = (32,13),(33,13)$, we have $k \le 8$, so the interval $[m-k+1,m]$ contains at most one multiple of 8, whereas in the two exceptional cases above the interval $[m-k+1,m]$ contains at most one multiple of 16. In all the above cases, this multiple of 8 (or 16) is one of 8, 16 or 32, except for the following pairs (m,k):

$$(13,5),(21,5),(50,5),(51,5),(100,7),(101,7),(162,5),(163,5). \tag{2.15}$$

Thus, for such equations except when (m,k) belongs to the list (2.15), we have

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = ((q^{\delta})^4+1)C = y^t, \qquad \text{where } \delta \in \{1,2,4\},$$

where the greatest common divisor of $q^{4\delta}+1$ and the cofactor C either is 1, or is 2, but if it is 2, then q is odd and $2 \parallel q^{4\delta}+1$. Hence, we get the relation $x^4+1 = \delta_1 y_1^t$ with $x := q^{\delta}$, $\delta_1 \in \{1,2\}$ and some divisor y_1 of y, but this is impossible.

For $(m,k) = (13,5)$, the only multiple of 11 in $[m-k+1,m]$ is 11, so by arguments similar to the above ones we conclude that equation (2.3) for this pair implies that $(q^{11}-1)/(q-1) = y_1^t$ for some divisor y_1 of y, and this is impossible by (ii) of Lemma 2.2. For $(m,k) = (21,5)$, we see that 17 and 19 are two primes in $[m-k+1,m]$. Hence, we get the equations $(q^{17}-1)/(q-1) = y_1^t$ and $(q^{19}-1)/(q-1) = y_2^t$ for divisors y_1 and y_2 of y. Now (ii) of Lemma 2.2 implies that t must be equal to both 17 and 19, which is impossible. Finally, for the last six cases $m \in \{50,51,100,101,162,163\}$, we work with the numbers 49, 98, 161, respectively. Namely, when $m \in \{50,51\}$ then 49 is the only multiple of 7 in the interval $[m-k+1,m]$. Hence, equation (2.3) yields $(q^{49}-1)/(q-1) = y_1^t$ for some divisor y_1 of y, and this is impossible by (ii) of Lemma 2.2. When $m \in \{100,101\}$, we write equation (2.3) as

$$\left(\frac{q^{49}-1}{q^7-1}\right)(q^{49}+1) \prod_{\substack{n\in[m-k+1,m]\\ n\neq 98}} \left(\frac{q^n-1}{q-1}\right) = y^t \prod_{i=1}^{6}\left(\frac{q^i-1}{q-1}\right).$$

Since no number $n \neq 98$ in $[m-k+1, m]$ is a multiple of 7, it follows that the first factor on the left-hand side above is coprime to all numbers of the form $(q^n - 1)/(q-1)$ for $n \neq 98 \in [m-k+1, m]$, and for the same reason it is also coprime to $(q^i - 1)/(q-1)$ for $i = 1, \ldots, 6$. Finally, $(q^{49} - 1)/(q^7 - 1)$ is also coprime to $q^{49} + 1$, since clearly their only common factor could be 2, but this is not the case since $(q^{49} - 1)/(q^7 - 1)$ is odd. Hence, we find that $(q^{49} - 1)/(q^7 - 1) = y_1^t$ holds for some divisor y_1 of y. This gives a solution to the equation $(q_1^7 - 1)/(q_1 - 1) = y_1^t$ with $q_1 := q^7$, which does not exist by (ii) of Lemma 2.2.

Finally, when $m \in \{162, 163\}$ then note that $161 = 7 \cdot 23$ is a multiple of 7 and is coprime to all $[m-k+1, m]$ and to all $i \in [1, k]$. Hence, by the previous arguments, we get a solution to the equation $(q^{161} - 1)/(q-1) = y_1^t$ for some divisor y_1 of y, and this is impossible by (ii) of Lemma 2.2.

Case 5: $m < 2.5k$. Let a be the largest positive integer such that $k/2 < 2^a \leq k$. Observe that since $k \geq 4$, we have $a \geq 2$. The interval $[m-k+1, m]$ contains one or two multiples of 2^a, but not three of them. If one of them is $5 \cdot 2^a$, we get $m \geq 5 \cdot 2^a > 2.5k$, which is a contradiction. Since also $m - k + 1 > k$, the only possible multiples of 2^a in $[m-k+1, m]$ are $\{2^{a+1}, 3 \cdot 2^a, 2^{a+2}\}$.

Suppose first that $2^{a_1} \in [m-k+1, m]$ for some $a_1 \in \{a+1, a+2\}$. Observe that a_1 is unique because not both 2^{a+1} and 2^{a+2} can belong to $[m-k+1, m]$. Then equation (2.3) implies that

$$(q^{2^{a_1-1}} + 1)\left(\frac{q^{2^{a_1-1}} - 1}{q-1}\right) \prod_{\substack{n \in [m-k+1, m] \\ n \neq 2^{a_1}}} \left(\frac{q^n - 1}{q-1}\right) = y^t \prod_{i=1}^{k} \left(\frac{q^i - 1}{q-1}\right).$$

As before, it follows that the only common prime that the first factor on the left can share either with the remaining factors on the left or with the factors $(q^i - 1)/(q-1)$ for $i = 1, \ldots, k$ appearing on the right above is 2, and this happens when q is odd, but in that case $2 \parallel q^{2^{a_1-1}} + 1$. Indeed, this follows because aside from 2, all other primes dividing $q^{2^{a_1-1}} + 1$ have order of appearance $z(p) = 2^{a_1}$, but there is no $n \neq 2^{a_1}$ in $[m-k+1, m]$ which is a multiple of 2^{a_1}, and similarly there is no $i \in [1, k]$ which is a multiple of 2^{a_1} either. Hence, we get a solution to the equation $x^2 + 1 = \delta y_1^t$ with $\delta \in \{1, 2\}$, y_1 some divisor of y and $x := q^{2^{a_1-2}}$, which is impossible because this last equation has no positive integer solutions with $x > 1$.

Finally, assume that $3 \cdot 2^a \in [m-k+1, m]$. We then have

$$\left(\frac{q^{3 \cdot 2^a} - 1}{q-1}\right) \prod_{\substack{n \in [m-k+1, m] \\ n \neq 3 \cdot 2^a}} \left(\frac{q^n - 1}{q-1}\right) = \prod_{i=2}^{k} \left(\frac{q^i - 1}{q-1}\right) y^t.$$

Consider the divisor $q^{2^a} - q^{2^{a-1}} + 1 = \Phi_6(q^{2^{a-1}}) = \Phi_{3\cdot 2^a}(q)$ of $u_{3\cdot 2^a}$. Here, $\Phi_n(X) \in \mathbb{Z}[X]$ denotes the cyclotomic polynomial whose roots are the primitive roots of unity of order n. Note that the divisor considered is odd and it is also coprime to 3 because $q^{2^{a-1}} \equiv 0, 1 \pmod 3$, so $q^{2^a} - q^{2^{a-1}} + 1 \equiv 1 \pmod 3$. Then every prime factor p of $q^{2^a} - q^{2^{a-1}} + 1$ has $z(p) = 3\cdot 2^a$, and since $3\cdot 2^a$ divides neither any $n \neq 3\cdot 2^a \in [m-k+1, m]$ nor any $i \in [1, k]$, it follows that $q^{2^a} - q^{2^{a-1}} + 1 = y_1^t$ for some divisor y_1 of y. This leads to $(x^3-1)/(x-1) = y_1^t$ with $x := q^{2^{a-1}} - 1$. The only solution is $(x, y_1, t) = (18, 7, 3)$, leading to $q^{2^{a-1}} = 19$, which is false since 19 is not a perfect square. This finishes the proof of Theorem 2.1. ∎

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