Perfect powers in \( q \)-binomial coefficients

by

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1. Products of consecutive terms of Lucas sequences and powers.

Let \( \alpha \) and \( \beta \) be complex nonzero numbers with \( \alpha/\beta \) not a root of 1 such that \( r := \alpha + \beta \) and \( s := -\alpha\beta \) are nonzero coprime integers. Then the sequence \( \{u_n\}_{n \geq 0} \) of general term

\[
(1.1) \quad u_n := \frac{\alpha^n - \beta^n}{\alpha - \beta}
\]

for all \( n \geq 0 \) is called the Lucas sequence of roots \( \alpha \) and \( \beta \). It consists of integers. It was shown in [LS] that the equation

\[
(1.2) \quad u_{n+1} \cdots u_{n+k} = y^t
\]

has only finitely many positive integer solutions \((n, k, y, t)\) with \( t \geq 2 \) and prime, which are furthermore effectively computable. Moreover, the method of proof of this result from [LS] makes it possible to actually find all such solutions immediately once we know all solutions of the equation

\[
(1.3) \quad u_n = y^t \quad \text{in integers } n \geq 1, \ y \geq 1, \ \text{and } t \geq 2 \text{ prime.}
\]

Indeed, let \( n_1, \ldots, n_r \) be all indices \( n \) participating in solutions to equation (1.3). Let \( P(m) \) be the largest prime factor of a nonzero integer \( m \) with the convention that \( P(\pm 1) = 1 \). For any prime number \( p \), let \( z(p) \) be the order of appearance of \( p \) in \( \{u_n\}_{n \geq 1} \), that is, the smallest positive integer \( k \) such that \( p | u_k \). It is known that \( z(p) \) exists for all primes \( p \) coprime to \( s \), while primes \( p \) dividing \( s \) never appear in the factorization of any \( u_n \) for \( n \geq 1 \).

Put

\[
(1.4) \quad P_1 := \max\{3, P(n_1), \ldots, P(n_r)\}.
\]

A particular case of the main result in [LS] is the following.

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Main Theorem 1.1. All solutions of the equation
\begin{equation}
  u_{n+1} \cdots u_{n+k} = by^t
\end{equation}
with \( n \geq k \geq 2 \), and \( t \geq 2 \) prime, and \( z(p) \leq k \) for all primes \( p \) dividing \( b \) have the property that \( P((n+1)(n+2)\cdots(n+k)) \leq P_1 \).

Proof. Put \( p := P((n+1)(n+2)\cdots(n+k)) \). Since \( n \geq k \), a well-known theorem of Sylvester asserts that \( p > k \). Assume also that \( p > P_1 \) and we shall see that this leads to a contradiction. Rewrite equation (1.5) as
\begin{equation}
  u_p^a \left( \frac{u_{n+i_0}}{u_p^a} \right) \prod_{i \neq i_0, 1 \leq i \leq k} u_{n+i} = by^t.
\end{equation}
The argument following (2.1) on page 301 in [LS] shows that \( u_p^a \) is coprime to the remaining factors on the left-hand side of (1.6) above. Let us go quickly through this argument. Since \( \gcd(u_a, u_b) = \gcd(a, b) \) for all positive integers \( a \) and \( b \), it follows that \( \gcd(u_p^a, u_{n+i}) = \gcd(p^a, n+i) = u_1 = 1 \) for all \( i \neq i_0 \) in \( \{1, \ldots, k\} \). Furthermore, it is well-known that if \( q \) is a prime factor dividing both \( u_p^a \) and \( u_{n+i_0}/u_p^a \), then \( q \mid (n+i_0)/p^a = m \). Thus, \( q \mid m \). If \( q \) divides the discriminant \( \Delta := r^2 + 4s \) of \( \{u_n\}_{n \geq 1} \), then \( q \mid u_q \), and since also \( q \mid u_p^a \), it follows that \( q = p \), but then \( q \) cannot divide \( m \). Thus, \( q \) does not divide \( \Delta \), so \( q \equiv \pm 1 \pmod{z(q)} \). Since also \( q \mid u_p^a \), it follows that \( p \mid z(q) \). Hence, \( q \equiv \pm 1 \pmod{p} \), and since \( p \geq 5 \), we see that \( q \geq 2p-1 > p \), so again \( q \) cannot divide \( m \). This shows that indeed \( u_p^a \) is coprime to the remaining factors on the left-hand side of (1.6). Observe that since \( p > k \), the number \( u_p^a \) is also clearly coprime to \( b \) since \( a \) is divisible only by primes \( q \) having \( z(q) \leq k \). Hence, \( u_p^a = y_1^t \) for some divisor \( y_1 \) of \( y \), therefore \( p^a \in \{n_1, \ldots, n_r\} \), which contradicts the fact that \( p > P_1 \).

In [LS], it was shown that equation (1.2) has no solutions in the particular case when the Lucas sequence is the sequence \( \{F_n\}_{n \geq 1} \) of Fibonacci numbers. The interesting feature of that proof is that all solutions of equation (1.3) for this sequence, i.e., all the perfect powers of exponent \( >1 \) in the Fibonacci sequence, were not yet known at the time [LS] was written, thus the proof from [LS] does not make use of this information. Since now we know thanks to work of Bugeaud, Mignotte and Siksek [BMS] that the set of solutions \( n \) to equation (1.3) for the case of the Fibonacci numbers is \( \{1, 2, 6, 12\} \), we deduce that Theorem 1.1 has the following immediate corollary.

Corollary 1.2. Equation (1.5) has no solutions with \( n \geq k \geq 2 \) for the case of the Fibonacci sequence.
was shown that the Fibonomial coefficient is never a perfect power. However, Theorem \[1.1\] tells us that if \( k \geq 2 \), then \( P((n+1)\cdots(n+k)) \leq 3 \). Thus, putting \( x := n+k \) and \( y := n+k-1 \), we find that \( x-y = 1 \) and \( P(xy) \leq 3 \). Hence, \( \{x,y\} = \{3^2,2^2\} \), where \( 3^2 - 2^2 = \pm 1 \). The largest solution of the above Diophantine equation is \( 3^2 - 2^2 = 1 \). Thus, \( x \leq 9 \), and now a computation by hand convinces us that there is no solution to equation (1.5) with \( n \geq k \geq 2 \). □

Recall that given a Lucas sequence \( \{u_n\}_{n \geq 1} \), the \textit{u-binomial coefficient} is defined as

\[
\binom{m}{k}_u := \frac{u_{m-k+1} \cdots u_m}{u_1 \cdots u_k}.
\]

Since \( \binom{m}{k}_u = \binom{m}{m-k}_u \), and we are interested only in these quantities as integers, we shall assume that \( m \geq 2k \), therefore \( n := m-k \geq k \). In [MT], it was shown that the Fibonomial coefficient is never a perfect power. However, writing this as

\[ F_{n+1} \cdots F_{n+k-1} = by^t, \quad \text{where} \ b := F_1 \cdots F_k, \]

the result from [MT] is easily seen to be an immediate consequence of Corollary \[1.2\]. Moreover, Corollary \[1.2\] can be used to deal with other Lucas sequences also. In what follows, we give two examples.

Take first the Pell sequence \( \{P_n\}_{n \geq 1} \), which is the Lucas sequence with roots \( \alpha := 1+\sqrt{2} \) and \( \beta := 1-\sqrt{2} \). The only solutions of equation (1.3) have \( n \in \{1,7\} \) (see [C] and [P]). Hence, Theorem \[1.1\] tells us that all solutions of equation (1.5) for this sequence have \( P((n+1)\cdots(n+k)) \leq 7 \). Putting \( x := n+k \), \( y := n+k-1 \), we get \( x-y = 1 \) and \( P(xy) \leq 7 \). All solutions of this particular Diophantine equation appear in [A] (see also Chapter 6 of de Weger’s Ph.D. dissertation [W]). These give us certain possibilities, the largest one being \( x = 4375 \). If \( P(x-2) > 7 \), then \( k = 2 \). This means that \( P_{n+1}P_{n+2}/P_2 \) is a perfect power. Since \( P_2 = 2 \), and \( P_n \) is even if and only if \( n \) is, it follows that if we put \( m \in \{n,n+1\} \) such that \( m \) is odd, then \( P_m \) is a perfect power, therefore \( m \leq 7 \). Thus, either \( x \leq 8 \), or \( P(x-2) \leq 7 \). The positive integers \( x \) with \( P(x(x-1)(x-2)) \leq 7 \) are \( x = 50,16 \), and \( x \in [3,10] \), and one checks by hand that these do not lead to any convenient solution either. Hence, we record the following corollary.

\textbf{COROLLARY 1.3.} \textit{The only solution of the equation} \( \binom{m}{k}_P = y^t \) \textit{with} \( m \geq 2k \geq 2 \) \textit{and} \( t \geq 2 \) \textit{is} \( (m,k) = (7,1) \).

More generally, let \( D \geq 3 \), and let \( \alpha := v_1 + \sqrt{D}u_1 \) and \( \beta := v_1 - \sqrt{D}u_1 \), where \( (v_1,u_1) \) is the minimal solution in positive integers of the Pell equation \( v^2 - Du^2 = 1 \). Let \( \{u_n\}_{n \geq 1} \) be the Lucas sequence of roots \( \alpha \) and \( \beta \). The numbers \( u_n \) are precisely all the possible positive integer solutions \( u \) in the
Pell equation $v^2 - Du^2 = 1$. All perfect powers in the sequence $\{u_n\}_{n \geq 1}$ for all $2 \leq D \leq 100$ appear in Theorem 7.1 in [B]. A careful inspection of this list of solutions from [B] reveals that all solutions of equation (1.3) for these sequences have $n \in \{1, 2\}$, therefore again if $n \geq k \geq 2$ is a solution of (1.5), then with $x := n + k$ and $y := n + k - 1$, we have $P(xy) \leq 3$. Thus, again $x \leq 9$, and now a quick check convinces us that there is no convenient solution with $k \geq 2$ to these Diophantine equations. We record this result as follows.

**Corollary 1.4.** Let $\{u_n\}_{n \geq 1}$ be the Lucas sequence consisting of the components $u$ of the positive integer solutions $(v, u)$ to the Pell equation $v^2 - Du^2 = 1$. Then, for $2 \leq D \leq 100$, all solutions of the equation $\left[\begin{array}{c} m \\ k \end{array}\right]_u = y^t$ with $m \geq 2k \geq 2$ and $t \geq 2$ prime have $k = 1$.

## 2. Perfect powers in $q$-binomial coefficients.

From now on until the rest of the paper, we work with the Lucas sequence $\{u_n\}_{n \geq 1}$ of roots $\alpha := q$ and $\beta := 1$, where $q \geq 2$ is an integer. In this case, the $u$-binomial coefficient is called the $q$-binomial coefficient. In [LS], it was proved that the Diophantine equation (1.2) has no solution with $k \geq 2$ for this sequence (treating also $q \geq 2$ as an unknown integer). Again, the proof of this result from [LS] circumvents knowledge of the solutions to equation (1.3), which in this case is

\[(2.1) \, \frac{q^n - 1}{q - 1} = y^t \quad \text{in integers } q \geq 2, \, n \geq 3, \, y \geq 2, \, \text{and } t \geq 2 \text{ prime},\]

because the complete list of solutions to equation (2.1) is not yet known. The known solutions are

\[(2.2) \, \frac{3^5 - 1}{3 - 1} = 11^2, \quad \frac{7^4 - 1}{7 - 1} = 20^2, \quad \frac{18^3 - 1}{18 - 1} = 7^3.\]

We next study the $q$-binomial coefficients which are powers. Our result is the following.

**Main Theorem 2.1.** All the solutions of the equation

\[(2.3) \, \left[\begin{array}{c} m \\ k \end{array}\right]_q = y^t \quad \text{in integers } q \geq 2, \, m \geq 2k \geq 1, \, y \geq 2, \, \text{and } t \geq 2 \text{ prime} \]

have $k = 1$.

This complements a result of K. Győry [G1], [G2] who found all binomial coefficients which are perfect powers. The proof uses a result of Pethő [P] concerning perfect powers in the Pell sequence.

We record a couple of known facts that turn out to be useful.

**Lemma 2.2.** Assume that $(q, n, y, t)$ is a solution of equation (2.1) with $n \geq 3, \, \text{and } t \geq 2 \text{ prime}. \, \text{Then}: \]
Perfect powers in $q$-binomial coefficients

(i) $q$ is not square.
(ii) If $(q,n,y,t)$ is not in the list (2.2), then both $n$ and $t$ are odd, and if $p$ is the smallest prime factor of $n$, then either $p \geq 29$, or $p = t \in \{17, 19, 23\}$.

The proofs of the statements summarized in Lemma 2.2 can be found in [BHM], [BMRS], and [M].

We shall also need the following extension of a result of Faulkner [F].

Lemma 2.3. Assume that $m \geq 2.5k \geq 5$. Then $P\left(\binom{m}{k}\right) > 2k + 1$ except if the pair $(m,k)$ belongs to the set

(2.4) \{ (5,2), (6,2), (9,2), (8,3), (9,3), (10,2), (10,3), (10,4), (15,6),
          (16,2), (16,3), (16,6), (25,2), (28,11), (50,3), (81,2) \}.

Proof. We follow Faulkner [F]. Let $p_k$ stand for the smallest prime factor $\geq 2k$. In [F], it is shown that $P\left(\binom{m}{k}\right) \geq p_k$ for all $m \geq 2k \geq 4$, except when $(m,k) = (9,2), (10,3)$. In his proof, Faulkner used the inequality

(2.5) $\theta(x) < 1.01624x$, valid for all $x > 0$,

where $\theta(x) := \sum_{p \leq x} \log p$, which he had taken from [RS1]. We shall use instead the sharper inequality

(2.6) $\theta(x) < 1.001102x$, valid for all $x > 0$,

from [RS2], which appeared nine years after Faulkner’s paper. So, let us follow the proof of the theorem in [F] by replacing everywhere inequality (2.5) by (2.6), and using also the inequality

(2.7) $\pi(x) < 1.25506x/\log x$, valid for all $x > 1$.

Instead of pointing out what one should change where, we simply go through the entire proof. First, a computation in the range $5 \leq m \leq 1100$ leads to the solutions shown in (2.4). So from now on, $m \geq 1101$.

Assume that $\binom{m}{k}$ has no prime factor $p > 2k + 1$. Then

(2.8) $\left(\frac{m}{k}\right)^k \leq \binom{m}{k} = \prod_{p^a \mid \binom{m}{k}} p^a \leq m^{\pi(2k+1)}$.

The last inequality above is based on the fact due to Erdős that if $p^a \mid \binom{m}{k}$, then $p^a \leq m$. Using inequality (2.7), we get

$$\frac{m}{k} < m^{\pi(2k+1)/k} < m^{1.25506(2+1/k)/\log(2k)}.$$  

The right-hand side above is $< m^{1/2}$ for $k \geq 79$. However, since $\pi(2k+1)/k < 1/2$ holds also for $k = 77, 78$ and 79, we conclude that

$$\frac{m}{k} < m^{1/2} \quad \text{for } k \geq 76,$$
which is the last display on page 107 in [F]. Thus, the assumption is false for $76 \leq k \leq m^{1/2}$. Assume next that $k > m^{1/2}$. Returning to inequality (2.8), and using (2.6) and (2.7), we have

$$\left(\frac{m}{k}\right)^k \leq \left(\frac{m}{k}\right) \leq \prod_{p \leq 2k} p \prod_{p \leq \sqrt{m}} p^a \leq \exp(1.001102(2k + 1))m^{2\cdot1.25506\sqrt{m}/\log m}.$$ 

Hence,

$$\left(\frac{m}{k}\right)^k < \exp(1.001102(2k + 1) + 2 \cdot 1.25506\sqrt{m}).$$

Replacing $\sqrt{m}$ by $k$ and taking $k$th roots in (2.10) above, we find that $m < \exp(4.55)k$ for $k \geq 76$. However, when $k > \exp(-4.55)m$, we may replace $\sqrt{m}$ by $(4.55)k/\sqrt{m}$ in (2.10) above and take $k$th roots to obtain

$$m < \exp\left(1.001102\left(2 + 1/k\right) + 2 \cdot 1.25506\frac{\exp(4.55)}{\sqrt{m}}\right)k < \exp(3.49)k,$n

provided that $m > 3 \cdot \exp(9.1)$, i.e., for $m > 26900$. The initial assumption is then false if $m^{1/2} < k \leq \exp(-4.55)m$, or $\exp(-4.55)m < k \leq \exp(-3.49)m$, with $m > 26900$.

A simple induction argument shows that

$$\binom{9k}{k} > \frac{21.3^k}{3^k} \quad \text{for } k = 1, 2, 3, \ldots.$$ 

Thus, for $9k \leq m < 3^{3.49}k$, it follows from inequality (2.9) that

$$\frac{21.3^k}{3^k} \leq \binom{m}{k} < \exp(1.001102(2k + 1) + 2 \cdot 1.25506\sqrt{m}).$$

Taking $k$th roots, we obtain

$$(2.11) \quad 21.3 < \exp\left(1.001102\left(2 + 1/k\right) + 2 \cdot 1.25506\frac{\sqrt{m}}{k} + \frac{\log(3k)}{k}\right).$$

Since

$$\frac{\sqrt{m}}{k} < \frac{\exp(3.49)}{\sqrt{m}} < \frac{\exp(3.49)}{\sqrt{26900}} \quad \text{and} \quad \frac{\log(3k)}{k} \leq \frac{\log 228}{76},$$

for $26900 < m < \exp(3.49)k$ and $k \geq 76$, a simple calculation shows that (2.11) is false. Our assumption is therefore false for $9k \leq m < \exp(3.49)k$ with $m > 26900$ and $k \geq 76$.

The case $k \geq 76$ and $9k \leq m \leq 26000$ follows as in [F], since the maximum gap, or difference, between consecutive primes $< 26900$ is $< 76$. Hence, in this range, $\binom{m}{k}$ has a prime factor $> 8k > 2k + 1$. 

Thus, it remains to deal with the case when $k < 76$.

For $3k \leq m < 9k$, the interval $((8m/9, m)$ is contained in $(m - k, m)$. Next, as argued in [F], the interval $(8m/9, m)$ contains a prime $p$ for all $m \geq 54$, which is our case. To see that this prime $p$ satisfies $p > 2k + 1$, it suffices to observe that $p > 8m/9 > (8/9) \cdot (2.5k) = 20k/9$, so indeed $p > 2k + 1$ whenever $(20/9)k > 2k + 1$, which is true for $k \geq 5$. However, for $k \leq 4$ and $m \geq 1101$, we have $(8/9) \cdot 1101 \geq 978 > 9 \geq 2k + 1$, so the desired conclusion holds in this case also.

It remains to treat three situations, namely:

(i) $k \in \{2, 3, 4, 5, 6, 7\}$;
(ii) $8 \leq k < 76$ and $m > 3k$;
(iii) $8 \leq k$ and $2.5k \leq m \leq 3k$.

Let us deal with situation (i). If the desired conclusion does not hold, then $P \left( \left( \begin{array}{c} m \\ k \end{array} \right) \right) \leq 13$. Assume first that $k \in \{5, 6, 7\}$. Then at most one of the numbers in the interval $[m - k + 1, m]$ is a multiple of 13, at most one is a multiple of 11, and at most one is a multiple of 7. Since there are $k \geq 5$ numbers in this interval, it follows that there exist two, say $x > y$, such that $x - y \leq 7$ and $P(xy) \leq 5$. All solutions to this Diophantine equation appear in [W]. The largest is $6 = 486 - 480 = 2 \cdot 3^5 - 2^5 \cdot 3 \cdot 5$, leading to $n \leq 480 + 7 = 487 < 1100$, which is a range already covered. Suppose now that $k \in \{2, 3, 4\}$. Then $P \left( \left( \begin{array}{c} m \\ k \end{array} \right) \right) \leq 7$. Hence, with $x := m$, $y := m - 1$, we deduce that $x - y = 1$ and $P(xy) \leq 7$. The results from [A] or [W] give us a certain list of possibilities. There are only two of them with $m = x \geq 1101$, namely $x = 2401, 4305$. In both cases, $x - 2 > 2k + 1$ is prime, therefore $k = 2$, but both 2401 and 4305 are multiples of $7 > 2 \cdot 2 + 1$. So, case (i) does not lead to new exceptional pairs $(m, k)$ failing the desired property.

Let us now deal with situation (ii). Let us note here that the interval $[m - k + 1, m]$ is contained in $[2k + 2, m]$, so it is enough to show that the interval $[m - k + 1, m]$ contains primes. We follow the arguments on page 108 in [F]. A result of D. H. Lehmer shows that the product of seven consecutive integers $\geq 36$ contains a multiple of a prime $\geq 43$. Thus, the desired conclusion holds for $8 \leq k \leq 20$. Hence, we may assume that $k \geq 21$. Now if the desired conclusion is violated for some $m$ and $k$, then

$$\frac{m^k}{k!} \left( 1 - \frac{k(k-1)}{2m} \right) \leq \frac{m^k}{k!} \left( 1 - \frac{1}{m} \right) \cdots \left( 1 - \frac{k-1}{m} \right) = \binom{m}{k} \leq m^{\pi(2k+1)}.$$  

Assume that $m > (k^2 - 1)/2$. Then the factor in parentheses on the left-hand side above is $> 1/(k + 1)$, so the above inequality implies

$$m^{k-\pi(2k+1)} \leq (k + 1)!,$$

or, in other words,

$$m < (k + 1)!^{1/(k-\pi(2k+1))}.$$
The upper bound above is $< 1100$ for all $k$ in the range $k \in [21, 75]$. Hence, $1101 \leq m \leq (k^2 - 1)/2$ giving $k \geq 47$. The same argument shows that the desired conclusion holds for $m > 3000 > (75^2 - 1)/2$. Hence, it remains to cover the range $1101 < m \leq 3000$ and $47 \leq k \leq 75$. This follows as in [F] because the maximum gap between consecutive primes for $m < 3000$ is $< 46$.

Finally, let us deal with situation (iii). Observe that $2.5k > 2k + 1$ for all $k \geq 8$. Assume next that the interval $(2.5k, 3k)$ does not contain any prime number. Using the fact that \[ \pi(x) > \frac{x}{\log x - 1.5} \quad \text{and} \quad \pi(x) < \frac{x}{\log x - 0.5} \quad \text{for all } x > 67 \] (see Theorem 2 in [RS1]), we get
\[ \pi(3k) - \pi(2.5k) > \frac{3k}{\log(3k) - 0.5} - \frac{2.5k}{\log(2.5k) - 1.5}. \]
The function appearing on the right-hand side above is positive for $k \geq 663$ and a short calculation reveals that $\pi(3k) - \pi(2.5k) > 0$ for all $k \geq 8$, which finishes the argument for (iii) and hence the proof of the lemma.

Proof of Theorem 2.1. Assume that there is a solution $(m, k, y, q, t)$ of equation (2.3) with $m \geq 2k \geq 4$, and $t$ prime. We handle various cases.

Case 1: $k = 2$. Let $n \in \{m - 1, m\}$ be such that $n$ is even and let \[ m = \{n, n + \delta\}, \quad \text{where} \quad \delta \in \{\pm 1\}. \] Then
\[ \left[ \begin{array}{c} m \\ 2 \end{array} \right] = \left( \frac{q^n + \delta - 1}{q - 1} \right) \left( \frac{q^2 (n/2)^2 - 1}{q^2 - 1} \right) = y^l, \]
and the two factors in the middle are coprime. Thus, $(x^{n/2} - 1)/(x - 1) = y^l_1$ with the perfect square $x := q^2$ and with some divisor $y_1$ of $y$. If $n/2 \geq 3$, this is impossible by (i) of Lemma 2.2 while if $n/2 = 2$, then $q^2 + 1 = x + 1 = y^l_1$, which is not possible with $q \geq 2$ by known results on the Catalan equation. From now on, $k \geq 3$.

Case 2: $t = 2$. Put $n := m - k$ and observe that $n \geq k$. Put $p := P((n + 1) \cdots (n + k))$ and observe that $p > k$ by Sylvester’s theorem. Since $k \geq 3$, we have $p \geq 5$. Let, as in the proof of Theorem 1.1, $i_0$ stand for the unique index in $\{1, \ldots, k\}$ such that $p | n + i_0$, and write $n + i_0 := p^a l$ for some integers $a \geq 1$ and $l$ coprime to $p$. The argument from the proof of Theorem 1.1 shows that $u_p^v = y^\gamma_1$ for some divisor $y_1$ of $y$. Lemma 2.2 shows that $(q, p^a) = (3,5)$. Thus, $P(m(m - 1)(m - 2)) = 5$, and the only possibility is $m = 6$, for which $p = 5$ and $q = 3$. However, $[6]_3^3 = 33880$ is not a perfect square. From now on, $t \geq 3$. 


Case 3: \( k = 3 \). This is a warm-up for the more general case that follows, but it is worth doing it separately since it has its particularities. Let \( \mathcal{M} \) be the set of even indices in the interval \([m - 2, m]\). Then \( \mathcal{M} = \{m - 2, m\} \) or \( \{m - 1\} \), according to whether \( m \) is even or odd. Let \( p := P(\prod_{m \in \mathcal{M}} m) \), and assume that \( p \geq 5 \). Let \( m_0 \) be the unique index in \( \mathcal{M} \) such that \( p | m_0 \). Write \( m_0 = 2p^a l \), where \( a \) and \( l \) are positive integers with \( l \) coprime to \( p \). Rewrite equation (2.3) as

\[
\frac{q^{2p^a} - 1}{q - 1} \frac{q^{m_0} - 1}{q^{2p^a} - 1} \prod_{n \in \{m-2,m-1,m\}, n \neq m_0} \frac{q^n - 1}{q - 1} = \left( \frac{q^3 - 1}{q - 1} \right)^y. \tag{2.12}
\]

We now argue that the first factor on the left-hand side in (2.12) above is coprime to all other factors on the left and also to \( u_3 = (q^3-1)/(q-1) \), which is on the right-hand side. Indeed, if \( r \) is a prime dividing both \( (q^{2p^a} - 1)/(q^2 - 1) \) and \( (q^n - 1)/(q - 1) \) for some \( n \neq m_0 \) in \([m - 2, m]\), then \( r | (q^{2n} - 1)/(q^2 - 1) \). Hence, \( r \) divides both \( w_{p^a} \) and \( w_n \), where \( \{w_j\}_{j \geq 1} \) is the Lucas sequence of roots \( \alpha := q^2 \) and \( \beta := 1 \). But this is impossible since \( p^a \) and \( n \) are coprime. The same argument shows that \( (q^{2p^a} - 1)/(q^2 - 1) \) is coprime to \( (q^3 - 1)/(q - 1) \), which is a divisor of \( (q^6 - 1)/(q^2 - 1) \), because we are assuming that \( p > 3 \). Finally, assume that \( r \) divides both \( (q^{2p^a} - 1)/(q^2 - 1) \) and \( (q^{m_0} - 1)/(q^{2p^a} - 1) \). Then \( r \) must divide \( m_0/2 \). However, since \( r \) divides either \( (q^{p^a} - 1)/(q - 1) \) or \( (q^{p^a} + 1)/(q + 1) \), it follows that either \( r = p \) (and this happens if and only if \( q \equiv \pm 1 \pmod{p} \)), or \( r \equiv 1 \pmod{p} \). In both cases, \( r \geq p > P(m_0/2) \), so \( r \) cannot divide \( m_0/2 \). Now we deduce that \( (q^{2p^a} - 1)/(q^2 - 1) = y_1^t \) for some divisor \( y_1 \) of \( y \), which is impossible by (i) of Lemma 2.2.

Thus, \( p \leq 3 \). Suppose next that \( m \) is even. Then \( m = 2m_0 \), where \( P(m_0(m_0 - 1)) \leq 3 \). Since \( m_0 \geq 3 \), it follows that the only possibilities are \( m_0 \in \{3, 4, 9\} \), so \( m \in \{6, 8, 18\} \). When \( m = 6 \), we get

\[
\binom{6}{3}_q = \left( \frac{q^5 - 1}{q - 1} \right) \left( \frac{(q^4 - 1)(q^6 - 1)}{(q^2 - 1)(q^3 - 1)} \right) = y^t,
\]

and in the middle above the first factor is coprime to the cofactor. Hence, we get \( (q^5 - 1)/(q - 1) = y_1^t \) for some divisor \( y_1 \) of \( t \), which is impossible for \( t \geq 3 \) by (ii) of Lemma 2.2. Similarly, when \( m = 8 \), we get an equation of the form \( (q^7 - 1)/(q - 1) = y_1^t \), which is impossible by (ii) of Lemma 2.2. Finally, when \( m = 18 \), we get

\[
\binom{18}{3}_3 = (q^8 + 1) \left( \frac{q^8 - 1}{q^2 - 1} \right) \left( \frac{(q^{17} - 1)(q^{18} - 1)}{(q - 1)(q^3 - 1)} \right) = y^t.
\]

In the middle, either \( q^8 + 1 \) is coprime to the remaining cofactor, or the only common prime factor of \( q^8 + 1 \) with the remaining cofactor is 2, and if it
is 2, then \( q \) is odd, so \( 2 \| q^8 + 1 \). Hence, \( q^8 + 1 = \delta y_1^t \) for some divisor \( y_1 \) of \( y \) and some \( \delta \in \{1, 2\} \), which is impossible.

Suppose next that \( n \) is odd and \( P(n-1) \leq 3 \). Then \( m-1 = 2^a3^b \geq 6 \) for some \( a \geq 1 \) and \( b \geq 0 \). Furthermore, \( m-1 \) is coprime to both \( m-2 \) and \( m \). Hence, \( (q^{m-1} - 1)/(q-1) = y_1^t \) for some divisor \( y_1 \) of \( t \), but this is impossible by (ii) of Lemma 2.2.

**Case 4:** \( m \geq 2.5k \). Here, we write \( m = 2m_0 + m_1, k = 2k_0 + k_1 \), with integers \( m_0, k_0 \geq 2 \) and \( m_1, k_1 \in \{0, 1\} \). Observe that since \( m \geq 2.5k \), we get \( m/2 \geq 2.5k/2 \), so that

\[
m_0 = \lfloor m/2 \rfloor \geq \lfloor 2.5k/2 \rfloor = \lfloor 2.5k_0 + 1.25k_1 \rfloor = \begin{cases} 2.5k_0 & \text{if } k_0 = 0, \\ 2.5k_0 + 1 & \text{if } k_0 = 1. \end{cases}
\]

By Lemma 2.3, we get \( P((m_0 - k_0 + 1) \cdots m_0) > 2k_0 + 1 \geq k \) except when \((m_0, k_0) \) belongs to the list (2.4). Assume that we are in the good case, and we treat the exceptions later. Observe that

\[
\{2m_0 - 2k_0 + 2, 2m_0 - 2k_0 + 4, \ldots, 2m_0\} \subseteq \{m - k + 1, m - k + 2, \ldots, m\}.
\]

Write \( \mathcal{N} \) for the set of even integers in \([m-k+1, m]\) and put \( p := P(\prod_{n \in \mathcal{N} \setminus \{m\}} m) \). From the above display it follows that \( p > k \). In particular, there exists a unique number \( n_0 \in \mathcal{N} \) such that \( p \mid n_0 \). Write \( n_0 := 2p^{a_l}l \), with some integers \( a \geq 1 \) and \( l \) coprime to \( p \). We then rewrite equation (2.3) as

\[
(2.13) \quad \frac{q^{2^{p^{a_l}} - 1}}{q^2 - 1} \Bigg( \frac{q^{2^{p^{a_l}} - 1}}{q^2 - 1} \Bigg) \prod_{n \in \{m-k+1, \ldots, m\}, n \neq n_0} \frac{q^n - 1}{q - 1} = \left( \prod_{3 \leq i \leq k} \frac{q^i - 1}{q - 1} \right) y_1^t.
\]

Arguing as before, we see that the first factor on the left-hand side in (2.13) above, \( w_p := (q^{2^{p^{a_l}}} - 1)/(q^2 - 1) \), is coprime to \( (q^n - 1)/(q^2 - 1) \) for all \( n \neq n_0 \) in \([m-k+1, \ldots, m]\), just because \( (q^n - 1)/(q - 1) \) is a divisor of \( w_n := (q^{2^{n}} - 1)/(q - 1) \), and \( n \) and \( p \) are coprime. The same argument shows that the first factor on the left-hand side in (2.13) above is coprime to each of the factors \( (q^i - 1)/(q - 1) \) for \( i = 3, \ldots, k \) from the right-hand side, just because \( p > k \). Finally, since \( p \) is the largest prime factor of \( n_0 \) and \( n_0/(2p^{a_l}) \) is coprime to \( p \), it follows that the first factor on the left-hand side above is also coprime to the second factor on the same side. Hence, \( (q^{2^{p^{a_l}}} - 1)/(q^2 - 1) = y_1^t \) for some divisor \( y_1 \) of \( y \), which is impossible by (ii) of Lemma 2.2.

It remains to deal with the exceptions. Observe that if \( k_1 = 0 \), then \( k \) is even, and we only want the largest prime factor of \((m_0 - k_0 + 1) \cdots m_0 \) to exceed \( 2k_0 = k \), and by Faulkner’s result this is so except when \((m_0, k_0) = (9, 2), (10, 3) \). When \( k_1 = 1 \), we get in fact \( m_0 \geq 2.5k_0 + 1 \). A quick look in
the list (2.4) shows that the only pairs \((m, k)\) to consider when \(k_1 = 1\) are
\[(6, 2), (9, 2), (9, 3), (10, 2), (10, 3), (16, 2),
(16, 3), (16, 6), (25, 2), (50, 3), (81, 2).
\]
Hence, we just need to deal with the following set of 26 exceptional pairs \((m, k)\):
\[
\{(18, 4), (19, 4), (20, 6), (21, 6), (12, 5), (13, 5), (18, 5), (19, 5), (18, 7)
(19, 7), (20, 5), (21, 5), (20, 7), (21, 7), (32, 5), (33, 5), (32, 7), (33, 7),
(32, 13), (33, 13), (50, 5), (51, 5), (100, 7), (101, 7), (162, 5), (163, 5)\}.
\]
In all the above cases except when \((m, k) = (32, 13), (33, 13)\), we have \(k \leq 8\),
so the interval \([m - k + 1, m]\) contains at most one multiple of 8, whereas in
the two exceptional cases above the interval \([m - k + 1, m]\) contains at most
one multiple of 16. In all the above cases, this multiple of 8 (or 16) is one of
8, 16 or 32, except for the following pairs \((m, k)\):
\[(2.15) \quad (13, 5), (21, 5), (50, 5), (51, 5), (100, 7), (101, 7), (162, 5), (163, 5).
\]
Thus, for such equations except when \((m, k)\) belongs to the list (2.15), we have
\[
\left[\frac{m}{k}\right]_q = ((q^\delta)^4 + 1)C = y^t, \quad \text{where } \delta \in \{1, 2, 4\},
\]
where the greatest common divisor of \(q^{4\delta} + 1\) and the cofactor \(C\) either is 1,
or is 2, but if it is 2, then \(q\) is odd and \(2 \parallel q^{4\delta} + 1\). Hence, we get the relation
\(x^4 + 1 = \delta_1 y_1^t\) with \(x := q^{\delta}\), \(\delta_1 \in \{1, 2\}\) and some divisor \(y_1\) of \(y\), but this is
impossible.

For \((m, k) = (13, 5)\), the only multiple of 11 in \([m - k + 1, m]\) is 11, so
by arguments similar to the above ones we conclude that equation (2.3) for
this pair implies that \((q^{11} - 1)/(q - 1) = y_1^t\) for some divisor \(y_1\) of \(y\), and
this is impossible by (ii) of Lemma 2.2. For \((m, k) = (21, 5)\), we see that
17 and 19 are two primes in \([m - k + 1, m]\). Hence, we get the equations
\((q^{17} - 1)/(q - 1) = y_1^t\) and \((q^{19} - 1)/(q - 1) = y_2^t\) for divisors \(y_1\) and \(y_2\) of \(y\). Now
(ii) of Lemma 2.2 implies that \(t\) must be equal to both 17 and 19, which is
impossible. Finally, for the last six cases \(m \in \{50, 51, 100, 101, 162, 163\}\), we
work with the numbers 49, 98, 161, respectively. Namely, when \(m \in \{50, 51\}\)
then 49 is the only multiple of 7 in the interval \([m - k + 1, m]\). Hence,
equation (2.3) yields \((q^{49} - 1)/(q - 1) = y_1^t\) for some divisor \(y_1\) of \(y\), and this
is impossible by (ii) of Lemma 2.2. When \(m \in \{100, 101\}\), we write equation
(2.3) as
\[
\left(\frac{q^{49} - 1}{q^7 - 1}\right)\left(q^{49} + 1\right)\prod_{\substack{n \in [m - k + 1, m] \\ n \neq 98}} \left(\frac{q^n - 1}{q - 1}\right) = y^t \prod_{i=1}^{6} \left(\frac{q^i - 1}{q - 1}\right).
\]
Since no number \( n \neq 98 \) in \([m - k + 1, m]\) is a multiple of 7, it follows that the first factor on the left-hand side above is coprime to all numbers of the form \((q^n - 1)/(q - 1)\) for \( n \neq 98 \in [m - k + 1, m]\), and for the same reason it is also coprime to \((q^i - 1)/(q - 1)\) for \( i = 1, \ldots, 6\). Finally, \((q^{49} - 1)/(q^7 - 1)\) is also coprime to \(q^{49} + 1\), since clearly their only common factor could be 2, but this is not the case since \((q^{49} - 1)/(q^7 - 1)\) is odd. Hence, we find that \((q^{49} - 1)/(q^7 - 1) = y_1^t\) holds for some divisor \(y_1\) of \(y\). This gives a solution to the equation \((q^7 - 1)/(q - 1) = y_1^t\) with \(q_1 := q^7\), which does not exist by (ii) of Lemma 2.2.

Finally, when \( m \in \{162, 163\} \) then note that 161 = 7·23 is a multiple of 7 and is coprime to all \([m - k + 1, m]\) and to all \( i \in [1, k]\). Hence, by the previous arguments, we get a solution to the equation \((q^{161} - 1)/(q - 1) = y_1^t\) for some divisor \(y_1\) of \(y\), and this is impossible by (ii) of Lemma 2.2.

CASE 5: \( m < 2.5k \). Let \( a \) be the largest positive integer such that \( k/2 < 2^a \leq k \). Observe that since \( k \geq 4 \), we have \( a \geq 2 \). The interval \([m - k + 1, m]\) contains one or two multiples of \(2^a\), but not three of them. If one of them is \(5 \cdot 2^a\), we get \( m \geq 5 \cdot 2^a > 2.5k\), which is a contradiction. Since also \( m - k + 1 > k \), the only possible multiples of \(2^a\) in \([m - k + 1, m]\) are \(\{2^{a+1}, 3 \cdot 2^a, 2^{a+2}\}\).

Suppose first that \(2^{a_1} \in [m - k + 1, m]\) for some \(a_1 \in \{a + 1, a + 2\}\). Observe that \(a_1\) is unique because not both \(2^{a+1}\) and \(2^{a+2}\) can belong to \([m - k + 1, m]\). Then equation (2.3) implies that

\[
(q^{2^{a_1} - 1} + 1)\left(\prod_{n \in [m - k + 1, m], n \neq 2^{a_1}} \left(\frac{q^n - 1}{q - 1}\right)\right) = y_1^t \prod_{i=1}^{k} \left(\frac{q^i - 1}{q - 1}\right).
\]

As before, it follows that the only common prime that the first factor on the left can share either with the remaining factors on the left or with the factors \((q^i - 1)/(q - 1)\) for \( i = 1, \ldots, k \) appearing on the right above is 2, and this happens when \( q \) is odd, but in that case \( 2 \not| q^{2^{a_1} - 1} + 1 \). Indeed, this follows because aside from 2, all other primes dividing \(q^{2^{a_1} - 1} + 1\) have order of appearance \(z(p) = 2^{a_1}\), but there is no \( n \neq 2^{a_1} \) in \([m - k + 1, m]\) which is a multiple of \(2^{a_1}\), and similarly there is no \( i \in [1, k]\) which is a multiple of \(2^{a_1}\) either. Hence, we get a solution to the equation \(x^2 + 1 = \delta y_1^t\) with \(\delta \in \{1, 2\}\), \(y_1\) some divisor of \(y\) and \(x := q^{2^{a_1} - 2}\), which is impossible because this last equation has no positive integer solutions with \(x > 1\).

Finally, assume that \(3 \cdot 2^a \in [m - k + 1, m]\). We then have

\[
\left(\frac{q^{3 \cdot 2^a} - 1}{q - 1}\right) \prod_{n \in [m - k + 1, m], n \neq 3 \cdot 2^a} \left(\frac{q^n - 1}{q - 1}\right) = \prod_{i=2}^{k} \left(\frac{q^i - 1}{q - 1}\right) y_1^t.
\]
Consider the divisor \( q^{2a} - q^{2a-1} + 1 = \Phi(2a) \) of \( u_{3,2a} \). Here, \( \Phi_n(X) \in \mathbb{Z}[X] \) denotes the cyclotomic polynomial whose roots are the primitive roots of unity of order \( n \). Note that the divisor considered is odd and it is also coprime to 3 because \( q^{2a-1} \equiv 0, 1 \pmod{3} \), so \( q^{2a} - q^{2a-1} + 1 \equiv 1 \pmod{3} \). Then every prime factor \( p \) of \( q^{2a} - q^{2a-1} + 1 \) has \( z(p) = 3 \cdot 2a \), and since \( 3 \cdot 2a \) divides neither any \( n \neq 3 \cdot 2a \in [m-k+1, m] \) nor any \( i \in [1, k] \), it follows that \( q^{2a} - q^{2a-1} + 1 = y_1^t \) for some divisor \( y_1 \) of \( y \). This leads to \( (x^3 - 1)/(x - 1) = y_1^t \) with \( x := q^{2a-1} - 1 \). The only solution is \( (x, y_1, t) = (18, 7, 3) \), leading to \( q^{2a-1} = 19 \), which is false since 19 is not a perfect square. This finishes the proof of Theorem 2.1.

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