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On the *abc* conjecture in algebraic number fields

by

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To Professor W. M. Schmidt on his 75th birthday

In our paper we obtain new, effective results (cf. Theorems 1 and 2 in Section 3) towards the truth of the uniform *abc* conjecture in number fields. Our theorems improve upon the earlier estimates established in this direction. Our Theorems 1 and 2 are deduced from some recent explicit results of Yu and the author [24] (cf. Theorem A in Section 2) concerning S-unit equations. Our proofs (cf. Section 4) depend ultimately on the best known estimates for linear forms in logarithms of algebraic numbers.

1. The *abc* conjecture in $\mathbb{Z}$.

For positive integers a, b and c, we define the radical $N(a,b,c)$ of a, b and c by

$$N(a,b,c)=\prod_{\substack{p\mid abc\\ p\text{ a prime}}} p.$$

Thus $N(a,b,c)$ is the greatest square-free factor of abc.

The *abc* conjecture. *For any $\varepsilon>0$, there is a constant C_ε, which depends only on ε, such that for every triple of positive integers a, b, c satisfying*

$$a+b=c \quad \text{and} \quad \gcd(a,b)=1 \tag{1.1}$$

we have

$$c<C_\varepsilon(N(a,b,c))^{1+\varepsilon}. \tag{1.2}$$

This conjecture was first formulated by Oesterlé [36] and Masser [31] in 1985. The conjecture has a very extensive literature. It was pointed out that the exponent $1+\varepsilon$ is best possible in (1.2). Further, some refinements and

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more explicit versions were formulated; see [1], [2] and [17]. The conjecture has many profound consequences; cf. [1], [6], [9], [13], [16], [19], [28], [29], [34], [35], [37], [43], [47], [48] and the references given there.

Although the conjecture seems completely out of reach, there are some effective results towards its truth. By means of the theory of logarithmic forms Stewart and Tijdeman [43] and Stewart and Yu [44], [45] obtained upper bounds for c as a function of $N(a,b,c)$. The best known upper bounds are due to Stewart and Yu [45].

Let $P(n)$ denote the greatest prime factor of a positive integer n with the convention that $P(1)=1$. Further, denote by $\log_i$ the ith iterate of the logarithmic function with $\log_1=\log$. In [45] Stewart and Yu proved that if a, b, c are positive integers which satisfy (1.1) then

$$\log c < p' N^{c_1 \log_3 N^*/\log_2 N^*} \tag{1.3}$$

and

$$\log c < c_2 N^{1/3} (\log N)^3, \tag{1.4}$$

where $p'=\min(P(a),P(b),P(c))$, $N=N(a,b,c)$, $N^*=\max(N,16)$ and c_1, c_2 are effectively computable positive absolute constants. Furthermore, Chim Kwok Chi [12], following the proof of (1.3), has proved (1.3) with $c_1=710$.

2. S-unit equations in number fields. Let K be an algebraic number field of degree d with class number h and regulator R. Let M_K denote the set of places on K, S_∞ the set of infinite places, and S a finite subset of M_K which contains S_∞. Let s be the cardinality of S, $\mathfrak{p}_1,\dots,\mathfrak{p}_t$ the prime ideals corresponding to the finite places in S, $N(\mathfrak{p}_i)$ the norm of $\mathfrak{p}_i$, $i=1,\dots,t$,

$$P=\max_i N(\mathfrak{p}_i),$$

and R_S the S-regulator of K (see e.g. [11]). Then we have (cf. [11, (18)])

$$R_S = i_S R \prod_{i=1}^{t} \log N(\mathfrak{p}_i), \tag{2.1}$$

where i_S is a positive divisor of h. As usual, O_S and O_S^* will denote the ring of S-integers and the group of S-units of K, respectively. If in particular $S=S_\infty$, O_S is just the ring of integers O_K and O_S^* the unit group O_K^* in K.

For any $\gamma\in\overline{\mathbb{Q}}^*$, we denote by $h(\gamma)$ the absolute logarithmic height of γ. For brevity, we write $\log^* a$ for $\max(\log a,1)$ if $a>0$.

Let $\alpha,\beta\in K^*$ with $H:=\max(h(\alpha),h(\beta),1)$, and consider the S-unit equation

$$\alpha x+\beta y=1 \quad \text{in } x,y\in O_S^*. \tag{2.2}$$

Equations of this type have a great number of applications; cf. [14], [22], [23], [24], [41] and the references therein.

Equation (2.2) has only finitely many solutions. The first explicit upper bounds for the heights of the solutions of (2.2) were given by the author [20], [21] by means of the theory of logarithmic forms. Later several authors, including Kotov and Trelina [27], Schmidt [40], Sprindžuk [42], Bugeaud and Győry [11], Haristoy [25] and Győry and Yu [24] derived effective bounds for the solutions by using logarithmic form estimates.

Bugeaud and Győry [11] derived the bound

$$\max(h(x), h(y)) \leq C_1 P R_S (\log^* R_S)^2 H \tag{2.3}$$

for the solutions of (2.2) with $C_1 = c_3(c_4 ds)^{c_5 s}$, where c_3, c_4 and c_5 are explicitly given positive absolute constants. Győry and Yu [24] improved this to

$$\max(h(x), h(y)) \leq C_2 P R_S (\log^* R_S) H, \tag{2.4}$$

where C_2 is of the same form as C_1 but with smaller absolute constants c_3, c_4 and c_5.

Bombieri [3] and Bombieri and Cohen [4], [5] have developed another effective method in Diophantine approximation, based on an extended version of the Thue–Siegel principle, the Dyson lemma and some geometry of numbers. Bugeaud [10], following their approach and combining it with estimates for linear forms in two and three logarithms, derived the following bound for the solutions of (2.2):

$$\max(h(x), h(y)) \leq C_3 P (\log^* P) R_S \max(C_4 P (\log^* P) R_S, H), \tag{2.5}$$

where C_3 and C_4 are of the same form as C_1 and C_2, but the absolute constants in C_3 and C_4 are larger than the corresponding ones in C_2.

In some applications, the parameters depending on S play a crucial role. The bounds occurring in (2.3)–(2.5) contain the factor s^s, and this is the dominating factor in terms of S whenever $t > \log P$.

The following theorem provides a bound for the solutions which does not contain any factor of the form s^s or t^t. This fact will enable us to improve upon the earlier effective results obtained in the direction of the *abc* conjecture.

Let r denote the unit rank of O_K, and let

$$c_6 = \begin{cases} 0 & \text{if } r = 0, \\ 1/d & \text{if } r = 1, \\ 29 e r! r \sqrt{r-1} \log d & \text{if } r \geq 2. \end{cases}$$

Further, let

$$\mathcal{R} = \max(h, c_6 d R).$$

Theorem A (Győry and Yu). *If $t > 0$, then all solutions x, y of* (2.2) *satisfy*

$$\max(h(x), h(y)) \le c_7 h R(\log^* R)\mathcal{R}^{t+1}(\log^* \mathcal{R})(P/\log^* P) R_S H, \tag{2.6}$$

where

$$c_7 = (2^{13.32} d)^t (r+1)^{4r+13.5} 2^{10r+64} d^{r+5} (\log^*(2d))^6.$$

If in particular $t > 0$ and $r = 0$, the bound in (2.6) *can be replaced by*

$$c_8 h^{t+2} (\log^* h)(P/\log^* P)\Big\{\prod_{i=1}^{t} \log N(\mathfrak{p}_i)\Big\} H \tag{2.7}$$

with

$$c_8 = 2^{10t+22} t^{3.5} d^{t+2} (\log^*(2d))^3.$$

Further, if $t = 0$, one can replace the bound in (2.6) *by*

$$c_9 R(\log^* R) H, \tag{2.8}$$

where

$$c_9 = (r+1)^{2r+9} 2^{3.2(r+2)} \log(2r+2)(d \log^*(2d))^3.$$

In Section 4 we shall deduce Theorem A from Theorems 1 and 2 of Győry and Yu [24].

In the special case $K = \mathbb{Q}$, Theorem A implies the following. Let A, B, C and a, b, c be non-zero rational integers such that

$$Aa + Bb + Cc = 0 \tag{2.9}$$

and $\max(|A|, |B|, |C|) = H$, $|abc| > 1$, where both A, B, C and a, b, c are relatively prime.

Corollary. *We have*

$$\log \max(|a|, |b|, |c|) \le 2^{10t+22} t^4 (P/\log^* P)\Big(\prod_{p \mid abc} \log p\Big) \log^* H, \tag{2.10}$$

where $P = P(abc)$ and t denotes the number of distinct prime factors of abc.

3. The *abc* conjecture in number fields. Keeping the notation of Section 2, let again K be an algebraic number field of degree d with ring of integers O_K and unit rank r. Let Δ_K be the absolute value of the discriminant of K, and let M_K denote the set of places on K. For $v \in M_K$, we choose an absolute value $|\ |_v$ in the following way: if v is infinite and corresponds to $\sigma : K \to \mathbb{C}$, then we put $|\alpha|_v = |\sigma(\alpha)|^{d_v}$ for $\alpha \in K$, where $d_v = 1$ or 2 according as $\sigma(K)$ is contained in $\mathbb{R}$ or not; if v is a finite place corresponding to a prime ideal $\mathfrak{p}$ of O_K, then we put $|\alpha|_v = N(\mathfrak{p})^{-\operatorname{ord}_{\mathfrak{p}} \alpha}$ for $\alpha \in K \setminus \{0\}$ and $|0|_v = 0$. Here, for $\alpha \neq 0$, $\operatorname{ord}_{\mathfrak{p}} \alpha$ denotes the exponent of $\mathfrak{p}$ in the prime ideal factorization of the principal fractional ideal (α).

We define the *height* of $(a,b,c) \in (K^*)^3$ as

$$H_K(a,b,c) = \prod_{v \in M_K} \max(|a|_v, |b|_v, |c|_v). \tag{3.1}$$

Further, the *radical* of $(a,b,c) \in (K^*)^3$ is defined as

$$N_K(a,b,c) = \prod_v N(\mathfrak{p})^{\mathrm{ord}_{\mathfrak{p}} p}, \tag{3.2}$$

where p is a rational prime with $p\mathbb{Z} = \mathfrak{p} \cap \mathbb{Z}$, and the product is over all finite v such that $|a|_v$, $|b|_v$, $|c|_v$ are not all equal. Finally, we denote by $P_K(a,b,c)$ the greatest factor $N(\mathfrak{p})$ in (3.2).

There have been several proposals for generalizing the *abc* conjecture to algebraic number fields. In 1987, Vojta [47] proposed a very general conjecture and, as a consequence, suggested the first number field version of (1.2). Later, Vojta's version was refined by Elkies [13], Broberg [7], Granville and Stark [18], Browkin [8] and Masser [32]. The following version is due to Masser [32].

Uniform *abc* conjecture in number fields. *Let K be an algebraic number field of degree d, and Δ_K the absolute value of its discriminant. Then for every $\varepsilon > 0$ there exists C_ε, depending only on ε, such that*

$$H_K(a,b,c) < C_\varepsilon^d (\Delta_K N_K(a,b,c))^{1+\varepsilon} \tag{3.3}$$

for all $a,b,c \in K^*$ which satisfy $a+b+c=0$.

We note that (3.3) is best possible in terms of ε. Further, the upper bound is uniform in the sense that it has good behaviour under field extensions. In particular, for $K = \mathbb{Q}$ this general conjecture reduces to the classical *abc* conjecture formulated in Section 1.

The effective results concerning S-unit equations can be used to obtain weaker but unconditional and effective bounds for $H_K(a,b,c)$. Let

$$a+b+c=0, \qquad \text{where } a,b,c \in K^*. \tag{3.4}$$

Further, let S be the smallest subset of M_K containing S_∞, such that $v \in S$ for every finite place v for which $|a|_v$, $|b|_v$, $|c|_v$ are not all equal. Then

$$x = -a/c, \qquad y = -b/c$$

is a solution of the S-unit equation

$$x + y = 1 \qquad \text{in } x, y \in O_S^*. \tag{3.5}$$

Now having a bound for $h(x)$ and $h(y)$, we can derive a bound for $H_K(a,b,c)$.

Surroca [46], using the bound (2.3) due to Bugeaud and Győry [11], derived from (3.4) the estimate

$$\log H_K(a,b,c) < ((c_{10} d \Delta_K)^{c_{11}} N_K(a,b,c)^{c_{12}})^d, \tag{3.6}$$

where c_{10}, c_{11} and c_{12} denote effectively computable positive absolute constants.

The inequality (3.6) could be derived with smaller absolute constants from the explicit version of (2.4), established by Győry and Yu [24]. From our Theorem A we can, however, deduce better bounds for $H_K(a,b,c)$ in terms of $N_K(a,b,c)$.

For later applications, we prove Theorem 1 below in a completely explicit form, with as good constants c_{13}, c_{14} as possible.

THEOREM 1. *If $a,b,c \in K^*$ satisfy* (3.4), *then*

$$\log H_K(a,b,c) < c_{13}\Delta_K^{3/2}(\log^* \Delta_K)^{3d-1}(P/\log^* P)N^{d(c_{14}\log^* \Delta_K + 19.2\log_3 N^*)/\log_2 N^*} \tag{3.7}$$

where

$$P = P_K(a,b,c), \qquad N = N_K(a,b,c), \qquad N^* = \max(N,16),$$

$$c_{13} = \begin{cases} 2^{23} & \text{if } d=1, \\ 2^{27} & \text{if } d=2 \text{ and } r=0, \\ 2^{98}d^8(\log^* d)^6 & \text{if } r=1, \\ (r+1)^{5r+14}2^{10r+74}d^{r+8}(\log 2d)^8 & \text{if } r \geq 2, \end{cases}$$

and

$$c_{14} = \begin{cases} 12.4 & \text{if } d=1, \\ 14.7 & \text{if } d=2 \text{ and } r=0, \\ 7.4d & \text{if } r=1, \\ 2.9d\log d & \text{if } r \geq 2. \end{cases}$$

If in particular $S = S_\infty$ (i.e. if $a/c, b/c \in O_K^*$), then

$$\log H_K(a,b,c) < c_{15}\Delta_K^{1/2}(\log^* \Delta_K)^d, \tag{3.8}$$

where

$$c_{15} = (r+1)^{2r+9}2^{4(r+2)}(d\log^*(2d))^4.$$

Our Theorem 1 can be compared with Corollary 2 of Győry and Yu [24], where it is additionally assumed that a, b, c are S-units for some finite subset S of M_K. Further, instead of $N_K(a,b,c)$, the product, N_0, of the distinct prime factors of $N(\mathfrak{p}_1\cdots\mathfrak{p}_t)$ is considered in [24], where $\mathfrak{p}_1,\dots,\mathfrak{p}_t$ are the prime ideals corresponding to the finite places in S. However, N_0 can be arbitrarily large relative to $N_K(a,b,c)$. Finally, apart from a common proportional factor of a, b, c, in [24] $\max(h(a),h(b),h(c))$ and not $H_K(a,b,c)$ is estimated from above. On the other hand, we note that a weaker version of our Theorem 1 can be deduced from Corollary 2 of [24].

The following theorem is a consequence of Theorem 1. It provides a considerable improvement of (3.6) in terms of $N_K(a,b,c)$.

THEOREM 2. *Let $a, b, c \in K^*$ with $a + b + c = 0$, and $N = N_K(a, b, c)$. Then for every $\varepsilon > 0$, we have*

$$\log H_K(a, b, c) < c_{16} N^{1+\varepsilon}, \tag{3.9}$$

where $c_{16} = c_{16}(d, \Delta_K, \varepsilon)$ is an effectively computable positive constant which depends only on d, Δ_K and ε. Further, if

$$N > \max(\exp\exp(\max(\Delta_K, e)), \Delta_K^{2/\varepsilon}),$$

then

$$\log H_K(a, b, c) < c_{17}(\Delta_K N)^{1+\varepsilon} \tag{3.10}$$

with an effectively computable positive constant $c_{17} = c_{17}(d, \varepsilon)$ depending only on d and ε.

A comparison of the special case $K = \mathbb{Q}$ of Theorems 1 and 2 with (1.3) and (1.4) is in order. Theorem 2 implies that if a, b and c are coprime positive integers such that $a + b = c$, then, for every $\varepsilon > 0$, we have

$$\log c < c_{16} N^{1+\varepsilon}, \tag{3.11}$$

where N denotes the product of the distinct prime factors of abc, and c_{16} is now an effectively computable number which depends only on ε. This estimate is slightly weaker than (1.4). One of the reasons of this difference is that we obtained (3.11) as a special case of (3.9), while in [45] the authors gave a direct proof for (1.4) and (1.3) and utilized some specific properties of $\mathbb{Z}$, e.g. that $a + b = c$ and $b > a$ imply $2b > c > b$.

In this special situation Theorem 1 gives

$$\log c < 2^{23}(P/\log^* P) N^{653 \log_3 N^*/\log_2 N^*},$$

where $P = P(abc)$ and $N^* = \max(N, 16)$. This is comparable with (1.3).

The *abc* conjecture presented in Section 1 requires an upper bound for c in terms of ε and N only. Baker [1], [2] and Granville [17] formulated such refinements which involve also the number, t, of distinct prime factors of abc. The Corollary in Section 2 implies in this direction that if a, b, c are coprime positive integers with $a + b = c$ then

$$\log c < 2^{10t+22} t^4 (P/\log^* P) \prod_{p \mid abc} \log p, \tag{3.12}$$

whence, using $P \le N$ and $\prod_{p|abc} \log p \le (\log N/t)^t$,

$$\log c < (2^{10t+22}/t^{t-4}) N (\log N)^t \tag{3.13}$$

follows.

In general (3.12) gives a better upper bound for c than (1.3) with $c_1 = 710$. For example, if $a = 11^2$, $b = 3^2 5^2 7^3$, $c = 2^{21} 23$ then $a + b = c$ (de Weger) and the bound in (1.3) is $> 2^{4950}$, while the bound in (3.12) is $< 2^{100}$. In this example $2^{10t+22}t^4 > 2^{93}$ is the dominating factor.

4. Proofs. Before proving our Theorems 1 and 2, we deduce Theorem A from Theorems 1 and 2 of Győry and Yu [24]. We note that the proofs of Theorems 1 and 2 in [24] depend on the logarithmic form estimates due to Matveev [33] and Yu [49].

Proof of Theorem A. First suppose that $t > 0$. We apply Theorem 2 of [24] to our equation (2.2). Let x, y be a solution of (2.2). Then, using the notation of Theorem 2 in [24], there are σ, ϱ_1, ϱ_2, ϱ_3 in O_S^* such that

$$x = \sigma\varrho_1, \qquad y = \sigma\varrho_2, \qquad -1 = \sigma\varrho_3 \tag{4.1}$$

and

$$\max_i h(\varrho_i) \le E/2. \tag{4.2}$$

Here E denotes the bound occurring in (2.6) resp. (2.7) in our Theorem A. In fact, in [24], (4.2) is proved with a slightly smaller bound. However, using (2.1), we can choose here E as an upper bound. It follows now from (4.1) and (4.2) that

$$h(\sigma) \le E/2,$$

whence, by (4.2), we get $\max(h(x), h(y)) \le E$.

For $t = 0$, Theorem 1 of [24] gives (2.8) immediately. ■

Proof of the Corollary. Denote by S the set of places on $\mathbb{Q}$ which consists of the infinite place and the finite ones corresponding to the prime factors of abc. Then (2.9) implies that $x = -a/c$, $y = -b/c$ is a solution of the S-unit equation

$$(A/C)x + (B/C)y = 1.$$

Here $\max(h(A/C), h(B/C)) \le \log^* H$. Further,

$$\log\max(|a|, |b|, |c|) \le \max(h(a/c), h(b/c)).$$

On applying now the bound (2.7) in Theorem A to this solution x, y, (2.10) follows. ■

Proof of Theorem 1. We shall use, in refined form, some ideas from the proof of Corollary 2 of Győry and Yu [24].

Consider the relation

$$a + b + c = 0 \qquad \text{with } a, b, c \in K^*, \tag{3.4}$$

and choose S as in Section 3, i.e. let S be the smallest subset of M_K containing S_∞, such that $v \in S$ for every finite place v for which $|a|_v$, $|b|_v$, $|c|_v$ are not all equal. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the prime ideals corresponding to the finite places of S. Then $x = -a/c$, $y = -b/c$ is a solution of the equation (3.5) $x + y = 1$ in S-units x, y. Then, by our Theorem A, we have

$$\max(h(x), h(y)) \le E_0,$$

where E_0 denotes the bound with the choice $H = 1$, which occurs in (2.6), (2.7) or (2.8), according as $t > 0$, $r \ge 1$ or $t > 0$, $r = 0$ or $t = 0$, respectively.

Using the product formula, we infer that

$$\begin{aligned} H_K(a,b,c) &= \prod_{v \in M_K} \max(|a/c|_v, |b/c|_v, 1) \\ &\le \prod_{v \in M_K} \max(|a/c|_v, 1) \prod_{v \in M_K} \max(|b/c|_v, 1). \end{aligned}$$

Hence it follows that

$$\begin{aligned} \log H_K(a,b,c) &\le \sum_{v \in M_K} \log \max(|a/c|_v, 1) + \sum_{v \in M_K} \log \max(|b/c|_v, 1) \\ &= d(h(x) + h(y)) \le 2dE_0. \end{aligned} \tag{4.3}$$

We shall now give an upper bound for E_0 in terms of the parameters occurring in (3.7) and (3.8), respectively. First consider the case $t = 0$. The case $d = 1$ being trivial, we assume that $d > 1$. We use the fact that

$$hR \le \Delta_K^{1/2} (\log^* \Delta_K)^{d-1}, \tag{4.4}$$

which follows from a result of [30]. For $K = \mathbb{Q}$ and $K = \mathbb{Q}(\sqrt{-3})$ this is obviously true, because then $h = R = 1$. In the remaining cases (4.4) follows from (2) of [30] and from

$$\omega_K \le 20d \log_2 d \quad \text{if } d \ge 3, \tag{4.5}$$

where ω_K denotes the number of roots of unity in K. Since $\Phi(\omega_K)$ divides d, where Φ denotes Euler's function, (4.5) immediately follows from Theorem 15 of [39].

We have $(\log X)^B \le (B/2\varepsilon)^B X^\varepsilon$ if $X > 0$, $B > 0$ and $\varepsilon > 0$. Using this we infer that

$$(\log^* \Delta_K)^{d-1} \le (d-1)^{d-1} \Delta_K^{1/2}. \tag{4.6}$$

Hence in view of (4.4) and $h \ge 1$ we get

$$R \le \Delta_K^{1/2} (\log^* \Delta_K)^{d-1} \le d^d \Delta_K.$$

Thus we obtain

$$\log^* R \le d \log d + \log^* \Delta_K \le 2d(\log^* d) \log^* \Delta_K. \tag{4.7}$$

Here we utilized the trivial observation that

$$A + B = \left(\frac{1}{A} + \frac{1}{B}\right) AB \quad \text{for } A, B > 0 \tag{4.8}$$

(which will also be needed later) and that $\Delta_K \geq 3$ if $d > 1$. Now (4.4) and (4.7) give

$$R(\log^* R) \leq 2d(\log^* d)\Delta_K^{1/2}(\log^* \Delta_K)^d. \tag{4.9}$$

If we take into consideration that

$$2^{3.2(r+2)+1}\log(2r+2) < 2^{4(r+2)},$$

(2.8) and (4.9) imply (3.8).

Next consider the case $t > 0$. First assume that $r \geq 1$. It follows from (2.1) that

$$R_S \leq hR\prod_{i=1}^{t}\log N(\mathfrak{p}_i). \tag{4.10}$$

We have $R_K \geq 0.2052$; (cf. [15]). Hence we infer from (4.4) that

$$\mathcal{R} \leq c_{18}\Delta_K^{1/2}(\log^* \Delta_K)^{d-1}, \tag{4.11}$$

where $c_{18} = \max(c_6 d, 4.88)$. Now (4.4) and (4.11) imply that

$$h^2R^2(\log^* R)\mathcal{R}(\log^* \mathcal{R}) \leq 4d^2 c_{18}(\log c_{18})\Delta_K^{3/2}(\log^* \Delta_K)^{3d-1}. \tag{4.12}$$

Here we have used the fact that

$$\log c_{18} + 0.5 + (d-1)/e \leq 1.32d\log c_{18}.$$

We shall now estimate from above the product $\prod_{i=1}^{t}\log N(\mathfrak{p}_i)$. Denote by N_0 the product of the distinct prime factors of $N = N_K(a,b,c)$, and by t_0 the number of these primes. By assumption $t > 0$, hence $t_0 > 0$. Further, it follows that $t \leq dt_0$ and

$$N_0 \leq N \leq N_0^d. \tag{4.13}$$

Let

$$N_0^* = \max(N_0, 16).$$

Then obviously $N_0^* \leq N^*$. It follows from explicit estimates of [39] and [38] that

$$t_0 < 1.5\log N_0/\log_2 N_0^*,$$

whence

$$t < 1.5d\log N/\log_2 N^*. \tag{4.14}$$

We have $\prod_{i=1}^{t} N(\mathfrak{p}_i) \leq N^*$. Hence it follows that

$$\prod_{i=1}^{t}\log N(\mathfrak{p}_i) \leq \left(\frac{\log N}{t}\right)^t. \tag{4.15}$$

It is easy to check that

$$\log\left(\frac{d\log N}{t}\right)\left(\frac{d\log N}{t}\right)^{-1} \leq \frac{1}{e}.$$

If now $\log_2 N^* < 1.5e$, then

$$\frac{\log_2 N^*}{\log_3 N^*} \le \max\left(\frac{\log_2 16}{\log_3 16}, \frac{1.5e}{\log(1.5e)}\right) \le 19.16/e,$$

whence

$$\log\left(\frac{d\log N}{t}\right)\left(\frac{d\log N}{t}\right)^{-1} \le 19.16\,\frac{\log_3 N^*}{\log_2 N^*}. \tag{4.16}$$

Otherwise, if $\log_2 N^* \ge 1.5e$, then using (4.14) we get

$$\frac{d\log N}{t} > \frac{\log_2 N^*}{1.5} \ge e,$$

which implies that

$$\begin{aligned}\log\left(\frac{d\log N}{t}\right)\left(\frac{d\log N}{t}\right)^{-1} &\le \log\left(\frac{\log_2 N^*}{1.5}\right)\left(\frac{\log_2 N^*}{1.5}\right)^{-1}\\ &\le 1.5\,\frac{\log_3 N^*}{\log_2 N^*}.\end{aligned}$$

This proves that (4.16) holds in both cases. But we infer from (4.16) that

$$t\log\left(\frac{d\log N}{t}\right) \le 19.16d\,\frac{\log N\log_3 N^*}{\log_2 N^*}.$$

Together with (4.15) this gives

$$\prod_{i=1}^{t}\log N(\mathfrak{p}_i) \le \frac{1}{d^t}\,N^{19.16d\log_3 N^*/\log_2 N^*}. \tag{4.17}$$

We recall that c_7 denotes the constant occurring in Theorem A. We write $c_7 = c_7' \cdot c_7''$, where

$$c_7' = (2^{13.32}d)^t, \qquad c_7'' = (r+1)^{4r+13.5}2^{10r+64}d^{r+5}(\log 2d)^6. \tag{4.18}$$

We now distinguish two cases. First let $r \ge 2$. Then $d \ge 3$ and $\Delta_K \ge 23$ (see e.g. [26]). It is easy to see that $\log c_{18} \le 1.3d\log d$ and hence

$$8d^3c_7''c_{18}\log c_{18} \le (r+1)^{5r+14}2^{10r+74}d^{r+9}(\log 2d)^8. \tag{4.19}$$

Further, it follows from (4.11), (4.18) and (4.14) that

$$\begin{aligned}(c_7'\mathcal{R})^t &\le (2^{13.32}dc_{18}\Delta_K^{1/2}(\log\Delta_K)^{d-1})^{1.5d\log N/\log_2 N^*}\\ &\le N^{1.5d\log(2^{13.32}dc_{18}\Delta_K^{1/2}(\log\Delta_K)^{d-1})/\log_2 N^*}.\end{aligned} \tag{4.20}$$

Using (4.6), (4.8), $d \ge 3$ and $\Delta_K \ge 23$, we infer that

$$\begin{aligned}\log(2^{13.32}dc_{18}\Delta_K^{1/2}(\log\Delta_K)^{d-1}) &\le 5.11d\log d + \log\Delta_K\\ &\le 1.9332(d\log d)\log\Delta_K.\end{aligned}$$

Hence

$$(c_7'\mathcal{R})^t \le N^{2.9d^2(\log d)\log \Delta_K/\log_2 N^*}. \tag{4.21}$$

Thus, (2.6), (4.3), (4.12), (4.17), (4.19) and (4.21) imply (3.7) for $r \ge 2$ with

$$c_{13} = (r+1)^{5r+14}2^{10r+74}d^{r+8}(\log 2d)^8 \quad \text{and} \quad c_{14} = 2.9d\log d.$$

Next consider the case $r = 1$. Then $2 \le d \le 4$, $\Delta_K \ge 5$ (cf. [26]), $c_6 = 1/d$, $c_{18} = 4.88$ and $c_7'' = 2^{91.5}d^6(\log 2d)^6$. Hence

$$8d^3c_7''c_{18}\log c_{18} < 2^{98}d^9(\log 2d)^6.$$

Further, by means of (4.6), (4.8), $d \le 4$ and $\Delta_K \ge 5$ we infer that

$$\log(2^{13.32}dc_{18}\Delta_K^{1/2}(\log \Delta_K)^{d-1}) \le 7.11d + \log \Delta_K \le 4.92d\log \Delta_K.$$

Thus

$$(c_7'\mathcal{R})^t \le N^{7.4d^2\log \Delta_K/\log_2 N^*}.$$

This implies as in the case $r \ge 2$ that (3.7) holds with $c_{13} = 2^{98}d^8(\log d)^6$ and $c_{14} = 7.4d$.

Finally, assume that $r = 0$ when $d = 1$ or 2. Then, by Theorem A,

$$2dE_0 = c_8h^{t+2}(\log^* h)(P/\log^* P)\prod_{i=1}^{t}\log N(\mathfrak{p}_i).$$

Further, (4.4), (4.6), (4.8) and (4.17) imply (3.7) with $c_{13} = 2^{27}$, $c_{14} = 14.7$ if $d = 2$ and $r = 0$, and $c_{13} = 2^{23}$, $c_{14} = 12.4$ if $d = 1$. ∎

We now deduce Theorem 2 from Theorem 1.

Proof of Theorem 2. We first prove (3.10). By assumption

$$N > \max(\exp\exp(\max(\Delta_K, e)), \Delta_K^{2/\varepsilon}). \tag{4.22}$$

This implies that

$$\log^* \Delta_K \le \log_3 N^*.$$

Hence

$$d(c_{14}\log^* \Delta_K + 19.2\log_3 N^*) < c_{19}\log_3 N^* \tag{4.23}$$

with $c_{19} = d(c_{14}+19.2)$ which depends only on d and can be given explicitly. Using the fact that $r+1 \le d$, $P \le N$,

$$(\log^* \Delta_K)^{3d-1} \le (3d)^{3d-1}\Delta_K^{1/2},$$

and, by (4.22), $\Delta_K < N^{\varepsilon/2}$, we infer from (3.7) and (4.23) that

$$\log H_K(a,b,c) < c_{20}\Delta_K N^{1+\varepsilon/2+c_{19}\log_3 N^*/\log_2 N^*} \tag{4.24}$$

with some effectively computable c_{20} depending only on d. If now

$$c_{19}\log_3 N^*/\log_2 N^* < \varepsilon/2,$$

then (4.24) implies (3.10). Otherwise, it follows that $N \leq N_0(d, \varepsilon)$ with some effectively computable N_0 which depends only on d and ε, and (4.24) gives again (3.10) with another effectively computable c_{17}.

Finally, if (4.22) holds, (3.9) follows at once from (3.10). Otherwise, if (4.22) does not hold, (3.9) is an immediate consequence of (3.7). ∎

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