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Large gaps between consecutive zeros of the Riemann zeta-function. II

by

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1. Introduction. Subject to the truth of the Riemann Hypothesis (RH), the nontrivial zeros of the Riemann zeta-function can be written as $\rho = 1/2+i\gamma$, where $\gamma \in \mathbb{R}$. Denoting consecutive ordinates of zeros by $0 < \gamma \le \gamma'$, we define the normalized gap

$$\delta(\gamma) := (\gamma' - \gamma)\frac{\log \gamma}{2\pi}.$$

It is well-known that

$$N(T) := \sum_{0<\gamma\le T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

for $T \ge 10$. Hence $\delta(\gamma)$ is 1 on average. It is expected that there are arbitrarily large and arbitrarily small (normalized) gaps between consecutive zeros of the Riemann zeta-function on the critical line, i.e.

$$\lambda := \limsup_{\gamma} \delta(\gamma) = \infty \quad \text{and} \quad \mu := \liminf_{\gamma} \delta(\gamma) = 0.$$

In this article, we focus only on the large gaps, and prove the following theorem.

Theorem 1.1. *Assume RH. Then $\lambda > 2.9$.*

Very little is known about λ unconditionally. Selberg [15] remarked that he could prove $\lambda > 1$. Conditionally, Bredberg [2] showed that $\lambda > 2.766$ under the assumption of RH (see also [13, 12, 7, 11, 6] for work in this direction), and under the Generalized Riemann Hypothesis (GRH) it is known that $\lambda > 3.072$ ([10]; see also [8, 14, 3]). These results either use Hall's approach using Wirtinger's inequality, or exploit the following idea of Mueller [13]:

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Let $H:\mathbb{C}\to\mathbb{C}$ and consider the functions

$$\mathcal{M}_1(H,T)=\int_0^T\left|H\left(\tfrac12+it\right)\right|^2dt$$

and

$$\mathcal{M}_2(H,T;c)=\int_{-c/L}^{c/L}\sum_{0<\gamma\le T}\left|H\left(\tfrac12+i(\gamma+\alpha)\right)\right|^2d\alpha,$$

where $L=\log\frac{T}{2\pi}$. We note that if

$$h(c):=\frac{\mathcal{M}_2(H,T;c)}{\mathcal{M}_1(H,T)}<1$$

as $T\to\infty$, then $\lambda>c/\pi$, and if $h(c)>1$ as $T\to\infty$, then $\mu<c/\pi$.

Mueller [13] applied this idea to $H(s)=\zeta(s)$. Using

$$H(s)=\sum_{n\le T^{1-\varepsilon}}\frac{d_{2.2}(n)}{n^s},$$

where $d_k(n)$ are the coefficients of $\zeta(s)^k$, Conrey, Ghosh and Gonek [7] showed that $\lambda>2.337$. Later [8], assuming GRH and applying

$$H(s)=\zeta(s)\sum_{n\le T^{1/2-\varepsilon}}n^{-s},$$

they obtained $\lambda>2.68$. By considering a more general choice

$$H(s)=\zeta(s)\sum_{n\le T^{1/2-\varepsilon}}\frac{d_r(n)P\left(\frac{\log y/n}{\log y}\right)}{n^s},$$

where $P(x)$ is a polynomial, Ng [14] improved that result to $\lambda>3$ (using $r=2$ and $P(x)=(1-x)^{30}$). In the last two papers, GRH is needed to estimate certain exponential sums resulting from the evaluation of the discrete mean value over the zeros in $\mathcal{M}_2(H,T;c)$. Recently, Bui and Heath-Brown [5] showed how one can use a generalization of the Vaughan identity and the hybrid large sieve inequality to circumvent the assumption of GRH for such exponential sums. In the present paper we use that idea to obtain a weaker version of Ng's result without invoking GRH. It is possible that Feng and Wu's result $\lambda>3.072$ can also be obtained by this method just assuming RH. However, we opt to work on Ng's result for simplicity.

Instead of using the divisor function $d(n)=d_2(n)$, we choose

$$H(s)=\zeta(s)\sum_{n\le y}\frac{h(n)P\left(\frac{\log y/n}{\log y}\right)}{n^s},$$

where $y = T^{\vartheta}$, $P(x)$ is a polynomial and $h(n)$ is a multiplicative function satisfying

$$h(n) = \begin{cases} d(n) & \text{if } n \text{ is square-free,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

In Sections 3 and 4 we shall prove the following two key lemmas.

LEMMA 1.2. *Suppose* $0 < \vartheta < 1/2$. *Then*

$$\mathcal{M}_1(H,T) = \frac{AT(\log y)^9}{6} \int_0^1 (1-x)^3 (\vartheta^{-1} P_1(x)^2 - 2P_1(x)P_2(x))\, dx + O(TL^8),$$

where

$$A = \prod_p \left(1 + \frac{8}{p}\right)\left(1 - \frac{1}{p}\right)^8 \quad \textit{and} \quad P_r(x) = \int_0^x t^r P(x-t)\, dt.$$

LEMMA 1.3. *Suppose* $0 < \vartheta < 1/2$ *and* $P(0) = P'(0) = 0$. *Then*

$$\sum_{0<\gamma\le T} H(\rho + i\alpha) H(1 - \rho - i\alpha) = \frac{ATL(\log y)^9}{6\pi} \int_0^1 (1-x)^3 \operatorname{Re}\Big\{\sum_{j=1}^{\infty} (i\alpha \log y)^j B(j;x)\Big\}\, dx + O_\varepsilon(TL^{9+\varepsilon})$$

uniformly for $\alpha \ll L^{-1}$, *where*

$$\begin{aligned} B(j;u) = &-\frac{2P_1(u)P_{j+2}(u)}{(j+2)!} + \frac{2\vartheta P_2(u)P_{j+2}(u)}{(j+2)!} + \frac{4\vartheta P_1(u)P_{j+3}(u)}{(j+3)!} \\ &- \frac{\vartheta}{(j+2)!} \int_0^u t(\vartheta^{-1} - t)^{j+2} P_1(u) P(u-t)\, dt \\ &+ \frac{\vartheta}{(j+1)!} \int_0^u t(\vartheta^{-1} - t)^{j+1} P_2(u) P(u-t)\, dt \\ &- \frac{\vartheta}{6j!} \int_0^u t(\vartheta^{-1} - t)^{j} P_3(u) P(u-t)\, dt. \end{aligned}$$

Proof of Theorem 1.1. We take $\vartheta = (1/2)^-$. On RH we have

$$\sum_{0<\gamma\le T} \big|H\big(\tfrac12 + i(\gamma + \alpha)\big)\big|^2 = \sum_{0<\gamma\le T} H(\rho + i\alpha) H(1 - \rho - i\alpha).$$

Note that this is the only place we need to assume RH. Lemma 1.3 then implies that

$$\int_{-c/L}^{c/L} \sum_{0<\gamma\leq T} \left|H\big(\tfrac{1}{2}+i(\gamma+\alpha)\big)\right|^2 d\alpha \\ \sim \frac{AT(\log y)^9}{6\pi} \sum_{j=1}^{\infty} \frac{(-1)^j c^{2j+1}}{2^{2j-1}(2j+1)} \int_0^1 (1-x)^3 B(2j;x)\,dx.$$

Hence

$$h(c) = \frac{1}{2\pi}\,\frac{\sum_{j=1}^{\infty} \frac{(-1)^j c^{2j+1}}{2^{2j-1}(2j+1)} \int_0^1 (1-x)^3 B(2j;x)\,dx}{\int_0^1 (1-x)^3 (P_1(x)^2 - P_1(x)P_2(x))\,dx} + o(1)$$

as $T\to\infty$. Consider the polynomial $P(x)=\sum_{j=2}^{M} c_j x^j$. Choosing $M=6$ and running Mathematica's Minimize command, we obtain $\lambda > 2.9$. Precisely, with

$$P(x) = 1000x^2 - 9332x^3 + 30134x^4 - 40475x^5 + 19292x^6,$$

we have

$$h(2.9\pi) = 0.99725\ldots < 1,$$

and this proves the theorem. ∎

Remark 1.4. The above lemmas are unconditional. We note that in the case $r=2$, apart from the arithmetical factor a_3 being replaced by A, Lemma 1.2 is the same as [14, Lemma 2.1] (see also [3, Lemma 2.3]), while Lemma 1.3, under the additional condition $P(0)=P'(0)=0$, recovers Theorem 2 of Ng [14] (and also Lemma 2.6 of Bui [3]) without assuming GRH, though Ng's theorem and Bui's lemma are written in a slightly different and more complicated form. This is as expected because replacing the divisor function $d(n)$ by the arithmetic function $h(n)$ (as defined in (1.1)) in the definition of $H(s)$ only changes the arithmetical factor in the resulting mean value estimates. This substitution, however, makes our subsequent calculations much easier. Our arguments also work if we set $h(n)=d_r(n)$ when n is square-free for some $r\in\mathbb{N}$ without much change, but we choose $r=2$ to simplify various statements and expressions in the paper.

Remark 1.5. In the course of evaluating $\mathcal{M}_2(H,T;c)$, we encounter an exponential sum of type (see Section 4.2)

$$\sum_{n\leq y} \frac{h(n)P\big(\frac{\log y/n}{\log y}\big)}{n} \sum_{m\leq nT/2\pi} a(m)e\Big(-\frac{m}{n}\Big)$$

for some arithmetic function $a(m)$. At this point, assuming GRH, Ng [14] applied Perron's formula to the sum over m, and then moved the line of integration to $\mathrm{Re}(s)=1/2+\varepsilon$. The main term arises from the residue at

$s = 1$ and the error terms in this case are easy to handle. To avoid being subject to GRH, we instead use the ideas of [9] and [5]. That leads to a sum of the type

$$\sum_{n\le y}\frac{\mu(n)h(n)P\big(\frac{\log y/n}{\log y}\big)}{n}.$$

This is essentially a variation of the prime number theorem, and here the polynomial $P(x)$ is required to vanish with order at least 2 at $x = 0$ (see Lemma 2.6). As a result, we cannot make the choice $P(x) = (1-x)^{30}$ as in [14]. Here it is not clear how to choose a "good" polynomial $P(x)$. Our theorem is obtained by numerically optimizing over polynomials $P(x)$ with degree less than 7. It is probable that by considering higher degree polynomials, we can establish Ng's result $\lambda > 3$ under RH only.

Notation. Throughout the remainder of the paper, we write

$$[n]_y := \frac{\log y/n}{\log y}.$$

For $Q, R \in C^\infty([0,1])$ we define

$$Q_r(x) = \int_0^x t^r Q(x-t)\,dt \quad \text{and} \quad R_r(x) = \int_0^x t^r R(x-t)\,dt.$$

We let $\varepsilon > 0$ be an arbitrarily small positive number, which can change from occurrence to occurrence.

2. Various lemmas.

The following two lemmas are [9, Lemmas 2 and 3].

Lemma 2.1. *Suppose that* $A(s) = \sum_{m=1}^\infty a(m)m^{-s}$, *where* $a(m) \ll_\varepsilon m^\varepsilon$, *and* $B(s) = \sum_{n\le y} b(n)n^{-s}$, *where* $b(n) \ll_\varepsilon n^\varepsilon$. *Then*

$$\frac{1}{2\pi i}\int_{a+i}^{a+iT} \chi(1-s)A(s)B(1-s)\,ds = \sum_{n\le y}\frac{b(n)}{n}\sum_{m\le nT/2\pi} a(m)e\Big(-\frac{m}{n}\Big) + O_\varepsilon(yT^{1/2+\varepsilon}),$$

where $a = 1 + L^{-1}$.

Lemma 2.2. *Suppose that* $A_j(s) = \sum_{n=1}^\infty a_j(n)n^{-s}$ *is absolutely convergent for* $\sigma > 1$, $1 \le j \le k$, *and that*

$$A(s) = \sum_{n=1}^\infty \frac{a(n)}{n^s} = \prod_{j=1}^k A_j(s).$$

Then for any $l \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \frac{a(ln)}{n^s} = \sum_{l=l_1\ldots l_k} \prod_{j=1}^{k} \Bigg(\sum_{\substack{n\geq 1\\ (n,\prod_{i<j} l_i)=1}} \frac{a_j(l_j n)}{n^s} \Bigg).$$

We shall need estimates for various divisor-like sums. Throughout the paper, we let

$$F_\tau(n) = \prod_{p|n} (1 + O(p^{-\tau}))$$

for $\tau > 0$, and the implicit constant in the O-term is independent of τ.

LEMMA 2.3. *For any $Q \in C^\infty([0,1])$, there exists an absolute constant $\tau_0 > 0$ such that*

(i) $$\sum_{an\leq y} \frac{h(an)Q([an]_y)}{n}$$
$$= C(\log y)^2 h(a) \prod_{p|a} \left(1 + \frac{2}{p}\right)^{-1} Q_1([a]_y) + O(d(a)F_{\tau_0}(a)L),$$

(ii) $$\sum_{an\leq y} \frac{h(an)Q([an]_y)\log n}{n}$$
$$= C(\log y)^3 h(a) \prod_{p|a} \left(1 + \frac{2}{p}\right)^{-1} Q_2([a]_y) + O(d(a)F_{\tau_0}(a)L^2),$$

where

$$C = \prod_p \left(1 + \frac{2}{p}\right)\left(1 - \frac{1}{p}\right)^2.$$

Proof. By a method of Selberg [15] we have

$$\sum_{n\leq t} \frac{h(an)}{n} = \frac{C(\log t)^2}{2} h(a) \prod_{p|a} \left(1 + \frac{2}{p}\right)^{-1} + O(d(a)F_{\tau_0}(a)L)$$

for any $t \leq T$. The first statement then follows by partial summation.

The second statement is an easy consequence of the first one. ∎

LEMMA 2.4. *For any $Q \in C^\infty([0,1])$, we have*

$$\sum_{n\leq y} \frac{h(n)^2\varphi(n)Q([n]_y)}{n^2} \prod_{p|n} \left(1 + \frac{2}{p}\right)^{-2}$$
$$= \frac{D(\log y)^4}{6} \int_0^1 (1-x)^3 Q(x)\, dx + O(L^3),$$

where

$$D = \prod_p \left[1 + \frac{4(p-1)}{p^2}\left(1+\frac{2}{p}\right)^{-2}\right]\left(1-\frac{1}{p}\right)^4.$$

Proof. The proof is similar to that of the above lemma. ∎

We need a lemma concerning the size of the function $F_{\tau_0}(n)$ on average.

LEMMA 2.5. *Suppose* $-1 \le \sigma \le 0$. *Then*

$$\sum_{n\le y} \frac{d_k(n)F_{\tau_0}(n)}{n}\left(\frac{y}{n}\right)^{\sigma} \ll_k L^{k-1}\min\{|\sigma|^{-1}, L\}.$$

Proof. By [4, Lemma 4.6],

$$\sum_{n\le y} \frac{d_k(n)}{n}\left(\frac{y}{n}\right)^{\sigma} \ll_k L^{k-1}\min\{|\sigma|^{-1}, L\}.$$

We have

$$F_{\tau_0}(n) \le \prod_{p|n}(1+Ap^{-\tau_0}) = \sum_{l|n} l^{-\tau_0}A^{w(l)}$$

for some $A>0$, where $w(n)$ is the number of prime factors of n. Hence

$$\begin{aligned}\sum_{n\le y} \frac{d_k(n)F_{\tau_0}(n)}{n}\left(\frac{y}{n}\right)^{\sigma} &\ll \sum_{l\le y}\frac{d_k(l)A^{w(l)}}{l^{1+\tau_0}}\sum_{n\le y/l}\frac{d_k(n)}{n}\left(\frac{y/l}{n}\right)^{\sigma}\\ &\ll_k L^{k-1}\min\{|\sigma|^{-1}, L\},\end{aligned}$$

since $d_k(l)A^{w(l)} \ll l^{\tau_0/2}$ for sufficiently large l. ∎

LEMMA 2.6. *Let* $F(n) = F(n,0)$, *where*

$$F(n,\alpha) = \prod_{p|n}\left(1-\frac{1}{p^{1+\alpha}}\right).$$

For any $Q \in C^\infty([0,1])$ *satisfying* $Q(0)=Q'(0)=0$, *there exist an absolute constant* $\tau_0>0$ *and some* $\nu \asymp (\log\log y)^{-1}$ *such that*

$$\begin{aligned}(2.1)\quad \mathcal{A}_1(y,Q;a,b,\underline{\alpha}) &:= \sum_{\substack{an\le y\\(n,b)=1}} \frac{\mu(n)h(n)Q([an]_y)}{\varphi(n)n^{\alpha_1}}F(n,\alpha_2)F(n,\alpha_3)\\ &= U_1V_1(b)\left(\frac{Q''([a]_y)}{(\log y)^2} + \frac{2\alpha_1 Q'([a]_y)}{\log y} + \alpha_1^2 Q([a]_y)\right)\\ &\quad + O(F_{\tau_0}(b)L^{-3}) + O_\varepsilon\left(F_{\tau_0}(b)\left(\frac{y}{a}\right)^{-\nu}L^{-2+\varepsilon}\right)\end{aligned}$$

uniformly for $\alpha_j \ll L^{-1}$, $1 \le j \le 3$, *where* $U_1 = U_1(0, \underline{0})$ *and* $V_1(n) = V_1(0, n, \underline{0})$, *with*

$$U_1(s, \underline{\alpha}) = \prod_p \left[1 - \frac{2F(p, \alpha_2)F(p, \alpha_3)}{\varphi(p)p^{s+\alpha_1}}\right]\left(1 - \frac{1}{p^{1+s+\alpha_1}}\right)^{-2},$$

$$V_1(s, n, \underline{\alpha}) = \prod_{p|n} \left[1 - \frac{2F(p, \alpha_2)F(p, \alpha_3)}{\varphi(p)p^{s+\alpha_1}}\right]^{-1}.$$

Remark 2.7. For $Q(x) = x^j$, $j \ge 0$, the argument below shows that the last error term in (2.1) should be replaced by $O_\varepsilon(F_{\tau_0}(b)(y/a)^{-\nu}L^{-j+\varepsilon})$ (see (2.2)). To ensure that the contribution of this, averaging over a, is smaller than that of the main term we need $j \ge 2$, i.e. $Q(0) = Q'(0) = 0$.

Proof of Lemma 2.6. This is essentially a variation of the prime number theorem.

It suffices to consider $Q(x) = \sum_{j\ge 2} a_j x^j$. We have

$$\mathcal{A}_1(y, Q; a, b, \underline{\alpha}) = \sum_{j\ge 2} \frac{a_j j!}{(\log y)^j} \sum_{(n,b)=1} \frac{1}{2\pi i} \int_{(2)} \left(\frac{y}{a}\right)^s \frac{\mu(n)h(n)}{\varphi(n)n^{s+\alpha_1}} F(n, \alpha_2)F(n, \alpha_3) \frac{ds}{s^{j+1}}.$$

The sum over n converges absolutely. Hence

$$\mathcal{A}_1(y, Q; a, b, \underline{\alpha}) = \sum_{j\ge 2} \frac{a_j j!}{(\log y)^j} \frac{1}{2\pi i} \int_{(2)} \left(\frac{y}{a}\right)^s \sum_{(n,b)=1} \frac{\mu(n)h(n)}{\varphi(n)n^{s+\alpha_1}} F(n, \alpha_2)F(n, \alpha_3) \frac{ds}{s^{j+1}}.$$

The sum in the integrand equals

$$\prod_{p\nmid b} \left(1 - \frac{2F(p, \alpha_2)F(p, \alpha_3)}{\varphi(p)p^{s+\alpha_1}}\right) = \frac{U_1(s, \underline{\alpha})V_1(s, b, \underline{\alpha})}{\zeta(1 + s + \alpha_1)^2}.$$

Let $Y = o(T)$ be a large parameter to be chosen later. By Cauchy's theorem, $\mathcal{A}_1(y, Q; a, b, \underline{\alpha})$ is equal to the residue at $s = 0$ plus integrals over the line segments $\mathcal{C}_1 = \{s = it : t \in \mathbb{R}, |t| \ge Y\}$, $\mathcal{C}_2 = \{s = \sigma \pm iY : -c/\log Y \le \sigma \le 0\}$, and $\mathcal{C}_3 = \{s = -c/\log Y + it : |t| \le Y\}$, where c is some fixed positive constant such that $\zeta(1 + s + \alpha_1)$ has no zeros in the region on the right hand side of the contour determined by the $\mathcal{C}_j$'s. Furthermore, we require that for such c we have $1/\zeta(\sigma + it) \ll \log(2 + |t|)$ in this region [16, Theorem 3.11]. Then the integral over $\mathcal{C}_1$ is

$$\ll F_{\tau_0}(b)L^{-j}(\log Y)^2/Y^j \ll_\varepsilon F_{\tau_0}(b)L^{-2}Y^{-2+\varepsilon},$$

since $j \ge 2$. The integral over $\mathcal{C}_2$ is

$$\ll F_{\tau_0}(b)L^{-j}(\log Y)/Y^{j+1} \ll_\varepsilon F_{\tau_0}(b)L^{-2}Y^{-3+\varepsilon}.$$

Finally, the contribution from $\mathcal{C}_3$ is

$$\ll F_{\tau_0}(b)L^{-j}(\log Y)^j\left(\frac{y}{a}\right)^{-c/\log Y} \ll_\varepsilon F_{\tau_0}(b)\left(\frac{y}{a}\right)^{-c/\log Y} L^{-2+\varepsilon}. \tag{2.2}$$

Choosing $Y \asymp L$ gives an error so far of size $O_\varepsilon(F_{\tau_0}(b)(y/a)^{-\nu}L^{-2+\varepsilon}) + O_\varepsilon(F_{\tau_0}(b)L^{-4+\varepsilon})$.

For the residue at $s=0$, we write it as

$$\sum_{j\ge 2}\frac{a_j j!}{(\log y)^j}\,\frac{1}{2\pi i}\oint\left(\frac{y}{a}\right)^s\frac{U_1(s,\underline{\alpha})V_1(s,b,\underline{\alpha})}{\zeta(1+s+\alpha_1)^2}\,\frac{ds}{s^{j+1}},$$

where the contour is a circle of radius $\asymp L^{-1}$ around the origin. This integral is trivially bounded by $O(L^{-2})$, so that taking the first term in the Taylor series of $\zeta(1+s+\alpha_1)$ finishes the proof. ∎

LEMMA 2.8. *For any $Q, R \in C^\infty([0,1])$, there exists an absolute constant $\tau_0 > 0$ such that*

$$\begin{aligned}
&\mathcal{A}_2(y,Q,R;a_1,a_2,\alpha_1)\\
&\quad:= \sum_{\substack{a_1a_2l\le y\\ a_1m\le y}}\frac{h(a_1a_2l)h(a_1m)Q([a_1m]_y)R([a_1a_2l]_y)V_1(a_1a_2lm)}{lm^{1+\alpha_1}}\\
&\quad= U_2(\log y)^4 h(a_1a_2)h(a_1)V_1(a_1a_2)V_2(a_1)V_3(a_2)V_4(a_1a_2)\\
&\qquad\times\int_0^{[a_1]_y} y^{-\alpha_1 t}tQ([a_1]_y-t)R_1([a_1a_2]_y)\,dt + O(d_4(a_1)d(a_2)F_{\tau_0}(a_1a_2)L^3)
\end{aligned}$$

uniformly for $\alpha_1 \ll L^{-1}$, where

$$U_2=\prod_p\left(1+\frac{2V_1(p)}{p}\right)\left[1+\frac{2V_1(p)}{p}\left(1+\frac{2}{p}\right)\left(1+\frac{2V_1(p)}{p}\right)^{-1}\right]\left(1-\frac{1}{p}\right)^4,$$

$$V_2(n)=\prod_{p\mid n}\left(1+\frac{2V_1(p)}{p}\right)^{-1},\qquad V_3(n)=\prod_{p\mid n}\left(1+\frac{2}{p}\right)\left(1+\frac{2V_1(p)}{p}\right)^{-1},$$

$$V_4(n)=\prod_{p\mid n}\left[1+\frac{2V_1(p)}{p}\left(1+\frac{2}{p}\right)\left(1+\frac{2V_1(p)}{p}\right)^{-1}\right]^{-1}.$$

Proof. Use Selberg's method [15] similarly to the proof of Lemma 2.3. One first executes the sum over m, and then the sum over l. ∎

LEMMA 2.9. *For any $Q, R \in C^\infty([0,1])$, we have*

$$\text{(i)}\quad \sum_{l_1l_2\le y}\frac{h(l_1l_2)h(l_1)Q([l_1]_y)R([l_1l_2]_y)}{l_1l_2^{1+\alpha_1}} \times F(l_1,\alpha_2)F(l_1l_2,\alpha_3)V_1(l_1l_2)V_2(l_1)V_3(l_2)V_4(l_1l_2)$$

$$= \frac{W(\log y)^6}{6}\int_0^1\int_0^x(1-x)^3y^{-\alpha_1t_1}t_1Q(x)R(x-t_1)\,dt_1\,dx+O(L^5),$$

$$\text{(ii)}\quad \sum_{pl_1l_2\le y}\frac{\log p}{(p^{1+\alpha_4}-1)p^{\alpha_5}}\,\frac{h(pl_1l_2)h(l_1)Q([l_1]_y)R([pl_1l_2]_y)}{l_1l_2^{1+\alpha_1}} \times F(pl_1,\alpha_2)F(pl_1l_2,\alpha_3)V_1(pl_1l_2)V_2(l_1)V_3(pl_2)V_4(pl_1l_2)$$

$$= \frac{W(\log y)^7}{3}\int_0^1\int_{\substack{t_j\ge0\\ t_1+t_2\le x}}(1-x)^3y^{-\alpha_1t_1-(\alpha_4+\alpha_5)t_2}t_1 \times Q(x)R(x-t_1-t_2)\,dt_1\,dt_2\,dx+O(L^6)$$

uniformly for $\alpha_j \ll L^{-1}$, $1\le j\le 5$, where

$$W=\prod_p\left(1+\frac{2F(p)V_1(p)V_3(p)V_4(p)}{p}+\frac{4F(p)^2V_1(p)V_2(p)V_4(p)}{p}\right)\left(1-\frac{1}{p}\right)^6.$$

Proof. We begin with the first statement. We start with the sum over l_2 on the left hand side, which is

$$\sum_{\substack{l_2\le y/l_1\\ (l_2,l_1)=1}}\frac{h(l_2)R([l_1l_2]_y)}{l_2^{1+\alpha_1}}F(l_2,\alpha_3)V_1(l_2)V_3(l_2)V_4(l_2).$$

As in the proof of Lemma 2.3, this equals

$$\text{(2.3)}\quad \prod_p\left\{W_1(p)^{-1}\left(1-\frac{1}{p}\right)^2\right\}(\log y)^2W_1(l_1) \times\int_0^{[l_1]_y}y^{-\alpha_1t_1}t_1R([l_1]_y-t_1)\,dt_1+O(L),$$

where

$$W_1(n)=\prod_{p|n}\left(1+\frac{2F(p)V_1(p)V_3(p)V_4(p)}{p}\right)^{-1}.$$

Hence the required expression is

$$(2.4)\quad \prod_p \left\{W_1(p)^{-1}\left(1-\frac{1}{p}\right)^2\right\}(\log y)^2 \sum_{l_1\le y}\frac{h(l_1)^2Q([l_1]_y)}{l_1}F(l_1,\alpha_2)F(l_1,\alpha_3)$$
$$\times V_1(l_1)V_2(l_1)V_4(l_1)W_1(l_1)\int_0^{[l_1]_y} y^{-\alpha_1 t_1}t_1R([l_1]_y-t_1)\,dt_1+O(L^5).$$

Using Selberg's method [15] again we have

$$\sum_{l_1\le t}\frac{h(l_1)^2}{l_1}F(l_1,\alpha_2)F(l_1,\alpha_3)V_1(l_1)V_2(l_1)V_4(l_1)W_1(l_1)$$
$$=\prod_p\left\{W_2(p)^{-1}\left(1-\frac{1}{p}\right)^4\right\}\frac{(\log t)^4}{24}+O(L^3)$$

for any $t\le T$, where

$$W_2(n)=\prod_{p\mid n}\left\{1+\frac{4F(p)^2V_1(p)V_2(p)V_4(p)W_1(p)}{p}\right\}^{-1}.$$

Partial summation then implies that (2.4) is equal to

$$\prod_p\left\{W_1(p)^{-1}W_2(p)^{-1}\left(1-\frac{1}{p}\right)^6\right\}\frac{(\log y)^4}{6}$$
$$\times\int_0^1\int_0^x (1-x)^3y^{-\alpha_1t_1}t_1Q(x)R(x-t_1)\,dt_1\,dx+O(L^5).$$

It is easy to check that the arithmetical factor is W, and we obtain the first statement.

For the second statement, we first notice that the contribution of the terms involving p^{-s} with $\operatorname{Re}(s)>1$ is $O(L^6)$. Hence the left hand side of (ii) is

$$2\sum_{l_1l_2\le y}\frac{h(l_1l_2)h(l_1)Q([l_1]_y)}{l_1l_2^{1+\alpha_1}}F(l_1,\alpha_2)F(l_1l_2,\alpha_3)V_1(l_1l_2)V_2(l_1)V_3(l_2)V_4(l_1l_2)$$
$$\times\sum_{\substack{p\le y/l_1l_2\\(p,l_1l_2)=1}}\frac{(\log p)R([pl_1l_2]_y)}{p^{1+\alpha_4+\alpha_5}}+O(L^6).$$

The same argument shows that we can include the terms $p\mid l_1l_2$ in the innermost sum with an admissible error $O(L^6)$, so that the above expression is equal to

$$2\sum_{p\leq y}\frac{\log p}{p^{1+\alpha_4+\alpha_5}}\sum_{l_1l_2\leq y/p}\frac{h(l_1l_2)h(l_1)Q([l_1]_y)R([pl_1l_2]_y)}{l_1l_2^{1+\alpha_1}}$$
$$\times F(l_1,\alpha_2)F(l_1l_2,\alpha_3)V_1(l_1l_2)V_2(l_1)V_3(l_2)V_4(l_1l_2)+O(L^6).$$

We have

$$\sum_{p\leq t}\frac{\log p}{p}=\log t+O(1).$$

The result follows by using (i) and partial summation. ∎

3. Proof of Lemma 1.2. To evaluate $\mathcal{M}_1(H,T)$, we first appeal to Theorem 1 of [1] and obtain

$$\mathcal{M}_1(H,T)=T\sum_{m,n\leq y}\frac{h(m)h(n)P([m]_y)P([n]_y)(m,n)}{mn}$$
$$\times\bigg(\log\frac{T(m,n)^2}{2\pi mn}+2\gamma-1\bigg)+O_B(TL^{-B})+O_\varepsilon(y^2T^\varepsilon)$$

for any $B>0$, where γ is the Euler constant. Using the Möbius inversion formula

$$f((m,n))=\sum_{\substack{l|m\\ l|n}}\sum_{d|l}\mu(d)f\Big(\frac{l}{d}\Big),$$

we can write the above as

$$T\sum_{l\leq y}\sum_{d|l}\frac{\mu(d)}{dl}\sum_{m,n\leq y/l}\frac{h(lm)h(ln)P([lm]_y)P([ln]_y)}{mn}$$
$$\times\bigg(\log\frac{T}{2\pi d^2mn}+2\gamma-1\bigg)+O_B(TL^{-B}).$$

We next replace the term in the bracket by $\log\frac{T}{2\pi mn}$. This produces an error of size

$$\ll T\sum_{l\leq y}\frac{d(l)^2}{l}\bigg(\sum_{n\leq y/l}\frac{d(n)}{n}\bigg)^2\sum_{d|l}\frac{\log d}{d}\ll TL^8.$$

Hence $\mathcal{M}_1(H,T)$ equals

$$T\sum_{l\le y}\frac{\varphi(l)}{l^2}\sum_{m,n\le y/l}\frac{h(lm)h(ln)P([lm]_y)P([ln]_y)}{mn}(L-\log m-\log n)+O(TL^8)$$
$$=TL\sum_{l\le y}\frac{\varphi(l)}{l^2}\Big(\sum_{n\le y/l}\frac{h(ln)P([ln]_y)}{n}\Big)^2$$
$$-2T\sum_{l\le y}\frac{\varphi(l)}{l^2}\sum_{m,n\le y/l}\frac{h(lm)h(ln)P([lm]_y)P([ln]_y)\log n}{mn}+O(TL^8).$$

The result follows by Lemmas 2.3–2.5. Here we use the fact (easy to verify) that $C^2D=A$. ■

4. Proof of Lemma 1.3. We define $H(s)=\zeta(s)G(s)$, i.e.

$$G(s)=\sum_{n\le y}\frac{h(n)P([n]_y)}{n^s}.$$

By Cauchy's theorem we have

$$\sum_{0<\gamma\le T}H(\rho+i\alpha)H(1-\rho-i\alpha)$$
$$=\frac{1}{2\pi i}\int_{\mathcal C}\frac{\zeta'}{\zeta}(s)\zeta(s+i\alpha)\zeta(1-s-i\alpha)G(s+i\alpha)G(1-s-i\alpha)\,ds,$$

where $\mathcal C$ is the positively oriented rectangle with vertices at $1-a+i$, $a+i$, $a+iT$ and $1-a+iT$. Here $a=1+L^{-1}$ and T is chosen so that the distance from T to the nearest γ is $\gg L^{-1}$. It is standard that the contribution from the horizontal segments of the contour is $O_\varepsilon(yT^{1/2+\varepsilon})$.

We denote the contribution from the right edge by $\mathcal N_1$, where

$$\mathcal N_1=\frac{1}{2\pi i}\int_{a+i}^{a+iT}\chi(1-s-i\alpha)\frac{\zeta'}{\zeta}(s)\zeta(s+i\alpha)^2G(s+i\alpha)G(1-s-i\alpha)\,ds. \tag{4.1}$$

From the functional equation we have

$$\frac{\zeta'}{\zeta}(1-s)=\frac{\chi'}{\chi}(1-s)-\frac{\zeta'}{\zeta}(s).$$

Hence the contribution from the left edge, upon replacing s by $1-s$, is

$$\frac{1}{2\pi i}\int_{a-i}^{a-iT}\frac{\zeta'}{\zeta}(1-s)\zeta(1-s+i\alpha)\zeta(s-i\alpha)G(1-s+i\alpha)G(s-i\alpha)\,ds$$
$$=\frac{1}{2\pi i}\int_{a-i}^{a-iT}\Big(\frac{\chi'}{\chi}(1-s)-\frac{\zeta'}{\zeta}(s)\Big)$$
$$\zeta(1-s+i\alpha)\zeta(s-i\alpha)G(1-s+i\alpha)G(s-i\alpha)\,ds$$
$$=-\overline{\mathcal N_2}+\overline{\mathcal N_1}+O_\varepsilon(yT^{1/2+\varepsilon}),$$

where

(4.2)
$$\mathcal{N}_2=\frac{1}{2\pi i}\int\limits_{a+i}^{a+iT}\frac{\chi'}{\chi}(1-s)\zeta(1-s+i\alpha)\zeta(s-i\alpha)G(1-s+i\alpha)G(s-i\alpha)\,ds.$$

Thus

$$\sum_{0<\gamma\leq T}H(\rho+i\alpha)H(1-\rho-i\alpha)=2\mathrm{Re}\big(\mathcal{N}_1\big)-\overline{\mathcal{N}_2}+O_\varepsilon(yT^{1/2+\varepsilon}).\tag{4.3}$$

4.1. Evaluation of $\mathcal{N}_2$. We move the line of integration in (4.2) to the $\frac{1}{2}$-line. As before, this produces an error of size $O_\varepsilon(yT^{1/2+\varepsilon})$. Hence we get

$$\mathcal{N}_2=\frac{1}{2\pi}\int\limits_{1-\alpha}^{T-\alpha}\frac{\chi'}{\chi}\big(\tfrac{1}{2}-it-i\alpha\big)\big|H\big(\tfrac{1}{2}+it\big)\big|^2\,dt+O_\varepsilon(yT^{1/2+\varepsilon}).$$

From Stirling's approximation we have

$$\frac{\chi'}{\chi}\big(\tfrac{1}{2}-it\big)=-\log\frac{t}{2\pi}+O(t^{-1})\qquad(t\geq 1).$$

Combining this with Lemma 1.2 and integration by parts, we easily obtain

$$\begin{aligned}\mathcal{N}_2=&-\frac{ATL(\log y)^9}{12\pi}\\&\times\int\limits_0^1(1-x)^3(\vartheta^{-1}P_1(x)^2-2P_1(x)P_2(x))\,dx+O(TL^9).\end{aligned}\tag{4.4}$$

4.2. Evaluation of $\mathcal{N}_1$. It is easier to start with a more general sum

$$\begin{aligned}\mathcal{N}_1(\beta,\gamma)=\frac{1}{2\pi i}\int\limits_{a+i(1+\alpha)}^{a+i(T+\alpha)}&\chi(1-s)\frac{\zeta'}{\zeta}(s+\beta)\zeta(s+\gamma)\zeta(s)\\&\times\bigg(\sum_{m\leq y}\frac{h(m)P([m]_y)}{m^s}\bigg)\bigg(\sum_{n\leq y}\frac{h(n)P([n]_y)}{n^{1-s}}\bigg)\,ds,\end{aligned}$$

so that $\mathcal{N}_1=\mathcal{N}_1(-i\alpha,0)$. From Lemma 2.1, we obtain

$$\mathcal{N}_1(\beta,\gamma)=\sum_{n\leq y}\frac{h(n)P([n]_y)}{n}\sum_{m\leq nT/2\pi}a(m)e\Big(-\frac{m}{n}\Big)+O_\varepsilon(yT^{1/2+\varepsilon}),$$

where the arithmetic function $a(m)$ is defined by

$$\frac{\zeta'}{\zeta}(s+\beta)\zeta(s+\gamma)\zeta(s)\sum_{m\leq y}\frac{h(m)P([m]_y)}{m^s}=\sum_{m=1}^{\infty}\frac{a(m)}{m^s}.\tag{4.5}$$

Following the methods of Conrey, Ghosh and Gonek [9, Sections 5–6 and (8.2)], and of Bui and Heath-Brown [5], we can write

$$\mathcal{N}_1(\beta,\gamma)=\mathcal{Q}(\beta,\gamma)+E+O_\varepsilon(yT^{1/2+\varepsilon}),$$

where

$$\mathcal{Q}(\beta,\gamma)=\sum_{ln\le y}\frac{h(ln)P([ln]_y)}{ln}\frac{\mu(n)}{\varphi(n)}\sum_{\substack{m\le nT/2\pi\\(m,n)=1}}a(lm)\tag{4.6}$$

and

$$E\ll_{B,\varepsilon}TL^{-B}+y^{1/3}T^{5/6+\varepsilon}$$

for any $B>0$.

Let

$$\frac{\zeta'}{\zeta}(s+\beta)\zeta(s+\gamma)\zeta(s)=\sum_{n=1}^{\infty}\frac{g(n)}{n^s}.\tag{4.7}$$

From (4.5) and Lemma 2.2 we have

$$a(lm)=\sum_{\substack{l=l_1l_2\\m=m_1m_2\\l_1m_1\le y\\(m_2,l_1)=1}}h(l_1m_1)P([l_1m_1]_y)g(l_2m_2).$$

Hence

$$\begin{aligned}\mathcal{Q}(\beta,\gamma)=&\sum_{l_1l_2n\le y}\frac{h(l_1l_2n)P([l_1l_2n]_y)}{l_1l_2n}\frac{\mu(n)}{\varphi(n)}\\&\times\sum_{\substack{l_1m_1\le y\\(m_1,n)=1}}h(l_1m_1)P([l_1m_1]_y)\sum_{\substack{m_2\le nT/2\pi m_1\\(m_2,l_1n)=1}}g(l_2m_2).\end{aligned}\tag{4.8}$$

Lemma 4.1. *Suppose a and b are coprime, square-free integers. Then*

$$\begin{aligned}G(x;a,b):=&\sum_{\substack{n\le x\\(n,b)=1}}g(an)\\=&-\frac{x^{1-\beta}}{1-\beta}\sum_{a=a_2a_3}\frac{1}{a_2^{\gamma}}\zeta(1-\beta+\gamma)\zeta(1-\beta)F(b,-\beta+\gamma)F(a_2b,-\beta)\\&+\frac{x^{1-\gamma}}{1-\gamma}\sum_{a=a_2a_3}\frac{1}{a_2^{\gamma}}\bigg(\frac{\zeta'}{\zeta}(1+\beta-\gamma)+\sum_{p\mid b}\frac{\log p}{p^{1+\beta-\gamma}-1}\bigg)\\&\qquad\times\zeta(1-\gamma)F(b)F(a_2b,-\gamma)\end{aligned}$$

$$- \frac{x^{1-\gamma}}{1-\gamma} \sum_{a=pa_2a_3} \frac{1}{p^\beta a_2^\gamma} \frac{\log p}{1-p^{-(1+\beta-\gamma)}} \zeta(1-\gamma) F(pb) F(pa_2 b, -\gamma)$$
$$+ x \sum_{a=a_2a_3} \frac{1}{a_2^\gamma} \left(\frac{\zeta'}{\zeta}(1+\beta) + \sum_{p|b} \frac{\log p}{p^{1+\beta}-1} \right) \zeta(1+\gamma) F(b,\gamma) F(a_2 b)$$
$$- x \sum_{a=pa_2a_3} \frac{1}{p^\beta a_2^\gamma} \frac{\log p}{1-p^{-(1+\beta)}} \zeta(1+\gamma) F(pb,\gamma) F(pa_2 b)$$
$$+ O_{B,\varepsilon}((\log ab)^{1+\varepsilon} x (\log x)^{-B}).$$

Proof. An argument similar to that for the prime number theorem implies that, up to an error term of size $O_{B,\varepsilon}((\log ab)^{1+\varepsilon} x(\log x)^{-B})$ for any $B > 0$, $G(x; a, b)$ is the sum of the residues at $s = 1-\beta$, $s = 1-\gamma$ and $s = 1$ of

$$\frac{x^s}{s} \sum_{(n,b)=1} \frac{g(an)}{n^s}.$$

Combining (4.7) and Lemma 2.2 shows that the above expression is

$$\frac{x^s}{s} \sum_{a=a_1a_2a_3} \left(- \sum_{(n,b)=1} \frac{\Lambda(a_1 n)}{(a_1 n)^\beta n^s} \right) \left(\sum_{(n,a_1 b)=1} \frac{1}{(a_2 n)^\gamma n^s} \right) \left(\sum_{(n,a_1a_2b)=1} \frac{1}{n^s} \right)$$
$$= \frac{x^s}{s} \sum_{a=a_1a_2a_3} \frac{1}{a_1^\beta a_2^\gamma} \left(- \sum_{(n,b)=1} \frac{\Lambda(a_1 n)}{n^{s+\beta}} \right)$$
$$\times \zeta(s+\gamma)\zeta(s) F(a_1 b, s+\gamma-1) F(a_1 a_2 b, s-1).$$

We have

$$- \sum_{(n,b)=1} \frac{\Lambda(a_1 n)}{n^{s+\beta}} = \begin{cases} \frac{\zeta'}{\zeta}(s+\beta) + \sum_{p|b} \frac{\log p}{p^{s+\beta}-1} & \text{if } a_1 = 1, \\ -\frac{\log p}{1-p^{-(s+\beta)}} & \text{if } a_1 = p, \\ 0 & \text{otherwise.} \end{cases}$$

The result follows. ∎

In view of the above definition, the innermost sum in (4.8) is

$$G(nT/2\pi m_1; l_2, l_1 n).$$

We then write

$$\mathcal{Q}(\beta,\gamma) = \sum_{j=1}^{6} \mathcal{Q}_j(\beta,\gamma)$$

corresponding to the decomposition of $G(x; a, b)$ in Lemma 4.1.

We begin with $\mathcal{Q}_1(\beta,\gamma)$. Writing l_2l_3 for l_2, and m for m_1, we have

$$\begin{aligned}\mathcal{Q}_1(\beta,\gamma) = -&\frac{(T/2\pi)^{1-\beta}}{1-\beta}\zeta(1-\beta+\gamma)\zeta(1-\beta)\\ &\times \sum_{\substack{l_1l_2l_3\le y\\ l_1m\le y}} \frac{h(l_1l_2l_3)h(l_1m)P([l_1m]_y)}{l_1l_2^{1+\gamma}l_3m^{1-\beta}} F(l_1,-\beta+\gamma)F(l_1l_2,-\beta)\\ &\times \sum_{\substack{n\le y/l_1l_2l_3\\ (n,l_1l_2l_3m)=1}} \frac{\mu(n)h(n)P([l_1l_2l_3n]_y)}{\varphi(n)n^{\beta}} F(n,-\beta+\gamma)F(n,-\beta).\end{aligned}$$

From Lemma 2.6, the innermost sum is

$$\begin{aligned}U_1V_1(l_1l_2l_3m)&\biggl(\frac{P''([l_1l_2l_3]_y)}{(\log y)^2}+\frac{2\beta P'([l_1l_2l_3]_y)}{\log y}+\beta^2P([l_1l_2l_3]_y)\biggr)\\ &+O(F_{\tau_0}(l_1l_2l_3m)L^{-3})+O_\varepsilon\biggl(F_{\tau_0}(l_1l_2l_3m)\Bigl(\frac{y}{l_1l_2l_3}\Bigr)^{-\nu}L^{-2+\varepsilon}\biggr).\end{aligned}$$

By Lemma 2.5, the contribution of the O-terms to $\mathcal{Q}_1(\beta,\gamma)$ is $O_\varepsilon(TL^{9+\varepsilon})$. Hence

$$\begin{aligned}\mathcal{Q}_1(\beta,\gamma) &= -U_1(T/2\pi)^{1-\beta}\zeta(1-\beta+\gamma)\zeta(1-\beta)\\ &\quad\times\sum_{l_1l_2\le y}\frac{F(l_1,-\beta+\gamma)F(l_1l_2,-\beta)}{l_1l_2^{1+\gamma}}\biggl(\frac{\mathcal{A}_2(y,P,P'';l_1,l_2,-\beta)}{(\log y)^2}\\ &\quad+\frac{2\beta\mathcal{A}_2(y,P,P';l_1,l_2,-\beta)}{\log y}+\beta^2\mathcal{A}_2(y,P,P;l_1,l_2,-\beta)\biggr)+O_\varepsilon(TL^{9+\varepsilon}).\end{aligned}$$

Using Lemmas 2.8–2.9 we obtain

$$\begin{aligned}\text{(4.9)}\qquad \mathcal{Q}_1(\beta,\gamma) &= -\frac{A(T/2\pi)^{1-\beta}(\log y)^{10}}{6}\zeta(1-\beta+\gamma)\zeta(1-\beta)\\ &\quad\times\int_0^1\int_0^x\int_0^x(1-x)^3y^{\beta t-\gamma t_1}tt_1P(x-t)\\ &\quad\times\biggl(\frac{P(x-t_1)}{(\log y)^2}+\frac{2\beta P_0(x-t_1)}{\log y}+\beta^2P_1(x-t_1)\biggr)\,dt\,dt_1\,dx+O_\varepsilon(TL^{9+\varepsilon}).\end{aligned}$$

Here we have used the easily verified fact that $U_1U_2W=A$.

For $\mathcal{Q}_2(\beta,\gamma)$, we write the sum $\sum_{p|l_1n}$ as $\sum_{p|l_1}+\sum_{p|n}$, since the function $h(n)$ is supported on square-free integers. In doing so we have

$$\begin{aligned}
\mathcal{Q}_2(\beta,\gamma)&=\frac{(T/2\pi)^{1-\gamma}}{1-\gamma}\zeta(1-\gamma)\\
&\qquad\times\sum_{\substack{l_1l_2l_3\leq y\\ l_1m\leq y}}\frac{h(l_1l_2l_3)h(l_1m)P([l_1m]_y)}{l_1l_2^{1+\gamma}l_3m^{1-\gamma}}F(l_1)F(l_1l_2,-\gamma)\\
&\qquad\times\bigg(\frac{\zeta'}{\zeta}(1+\beta-\gamma)+\sum_{p|l_1}\frac{\log p}{p^{1+\beta-\gamma}-1}\bigg)\\
&\qquad\times\sum_{\substack{n\leq y/l_1l_2l_3\\ (n,l_1l_2l_3m)=1}}\frac{\mu(n)h(n)P([l_1l_2l_3n]_y)}{\varphi(n)n^{\gamma}}F(n)F(n,-\gamma)\\
&\quad+\frac{(T/2\pi)^{1-\gamma}}{1-\gamma}\zeta(1-\gamma)\sum_{\substack{l_1l_2l_3\leq y\\ l_1m\leq y}}\frac{h(l_1l_2l_3)h(l_1m)P([l_1m]_y)}{l_1l_2^{1+\gamma}l_3m^{1-\gamma}}F(l_1)F(l_1l_2,-\gamma)\\
&\qquad\times\sum_{\substack{p|n\\ n\leq y/l_1l_2l_3\\ (n,l_1l_2l_3m)=1}}\frac{\log p}{p^{1+\beta-\gamma}-1}\frac{\mu(n)h(n)P([l_1l_2l_3n]_y)}{\varphi(n)n^{\gamma}}F(n)F(n,-\gamma).
\end{aligned}\tag{4.10}$$

We consider the contribution from the terms $\sum_{p|l_1}$. From Lemma 2.6, the sum over n is

$$\ll L^{-2}+F_{\tau_0}(l_1l_2l_3m)L^{-3}+O_\varepsilon\bigg(F_{\tau_0}(l_1l_2l_3m)\Big(\frac{y}{l_1l_2l_3}\Big)^{-\nu}L^{-2+\varepsilon}\bigg).$$

Hence the contribution of the terms $\sum_{p|l_1}$ to $\mathcal{Q}_2(\beta,\gamma)$ is

$$\begin{aligned}
&\ll_\varepsilon TL^{-1}\sum_{\substack{p|l_1\\ l_1l_2l_3\leq y\\ l_1m\leq y}}\frac{\log p}{p-1}\frac{d_4(l_1)d(l_2)d(l_3)d(m)}{l_1l_2l_3m}\\
&\qquad\qquad\times\bigg(1+F_{\tau_0}(l_1l_2l_3m)L^{-1}+F_{\tau_0}(l_1l_2l_3m)\Big(\frac{y}{l_1l_2l_3}\Big)^{-\nu}L^{\varepsilon}\bigg)\\
&\ll_\varepsilon TL^5\sum_{\substack{p|l_1\\ l_1\leq y}}\frac{\log p}{p-1}\frac{d_4(l_1)}{l_1}\big(1+F_{\tau_0}(l_1)L^{-1+\varepsilon}\big)\ll_\varepsilon TL^{9+\varepsilon}.
\end{aligned}$$

The same argument shows that the last term in (4.10) is also $O_\varepsilon(TL^{9+\varepsilon})$. The remaining terms are

$$\frac{(T/2\pi)^{1-\gamma}}{1-\gamma}\,\frac{\zeta'}{\zeta}(1+\beta-\gamma)\zeta(1-\gamma)\sum_{\substack{l_1l_2l_3\le y\\ l_1m\le y}}\frac{h(l_1l_2l_3)h(l_1m)P([l_1m]_y)}{l_1l_2^{1+\gamma}l_3m^{1-\gamma}}$$
$$\times F(l_1)F(l_1l_2,-\gamma)\sum_{\substack{n\le y/l_1l_2l_3\\ (n,l_1l_2l_3m)=1}}\frac{\mu(n)h(n)P([l_1l_2l_3n]_y)}{\varphi(n)n^{\gamma}}F(n)F(n,-\gamma).$$

Similarly to $\mathcal{Q}_1(\beta,\gamma)$, we thus obtain

$$\mathcal{Q}_2(\beta,\gamma)=\frac{A(T/2\pi)^{1-\gamma}(\log y)^{10}}{6}\,\frac{\zeta'}{\zeta}(1+\beta-\gamma)\zeta(1-\gamma)$$
$$\times\int_0^1\int_0^x\int_0^x(1-x)^3y^{\gamma(t-t_1)}tt_1P(x-t)$$
$$\times\left(\frac{P(x-t_1)}{(\log y)^2}+\frac{2\gamma P_0(x-t_1)}{\log y}+\gamma^2P_1(x-t_1)\right)dt\,dt_1\,dx+O_\varepsilon(TL^{9+\varepsilon}). \tag{4.11}$$

The fourth term $\mathcal{Q}_4(\beta,\gamma)$ is in the same form as $\mathcal{Q}_2(\beta,\gamma)$. Similar calculations then yield

$$\mathcal{Q}_4(\beta,\gamma)=\frac{A(T/2\pi)(\log y)^{8}}{6}\,\frac{\zeta'}{\zeta}(1+\beta)\zeta(1+\gamma)$$
$$\times\int_0^1\int_0^x(1-x)^3y^{-\gamma t_1}t_1P_1(x)P(x-t_1)\,dt_1\,dx+O_\varepsilon(TL^{9+\varepsilon}). \tag{4.12}$$

To evaluate $\mathcal{Q}_3(\beta,\gamma)$, we rearrange the sums and write $\mathcal{Q}_3(\beta,\gamma)$ in the form

$$-\frac{(T/2\pi)^{1-\gamma}}{1-\gamma}\zeta(1-\gamma)\sum_{\substack{pl_1l_2l_3\le y\\ l_1m\le y}}\frac{\log p}{(p^{1+\beta-\gamma}-1)p^{\gamma}}\,\frac{h(pl_1l_2l_3)h(l_1m)P([l_1m]_y)}{l_1l_2^{1+\gamma}l_3m^{1-\gamma}}$$
$$\times F(pl_1)F(pl_1l_2,-\gamma)\sum_{\substack{n\le y/pl_1l_2l_3\\ (n,pl_1l_2l_3m)=1}}\frac{\mu(n)h(n)P([pl_1l_2l_3n]_y)}{\varphi(n)n^{\gamma}}F(n)F(n,-\gamma).$$

By Lemma 2.6, the innermost sum is

$$U_1V_1(pl_1l_2l_3m)\left(\frac{P''([pl_1l_2l_3]_y)}{(\log y)^2}+\frac{2\gamma P'([pl_1l_2l_3]_y)}{\log y}+\gamma^2P([pl_1l_2l_3]_y)\right)$$
$$+O(F_{\tau_0}(pl_1l_2l_3m)L^{-3})+O_\varepsilon\left(F_{\tau_0}(pl_1l_2l_3m)\left(\frac{y}{pl_1l_2l_3}\right)^{-\nu}L^{-2+\varepsilon}\right).$$

The contribution of the O-terms is $O_\varepsilon(TL^{9+\varepsilon})$, by Lemma 2.5. The remaining terms contribute

$$-\frac{U_1(T/2\pi)^{1-\gamma}}{(1-\gamma)}\zeta(1-\gamma)\sum_{pl_1l_2\leq y}\frac{\log p}{(p^{1+\beta-\gamma}-1)p^\gamma}\frac{F(pl_1)F(pl_1l_2,-\gamma)}{l_1l_2^{1+\gamma}}$$
$$\times\bigg(\frac{\mathcal{A}_2(y,P,P'';l_1,pl_2,-\gamma)}{(\log y)^2}+\frac{2\gamma\mathcal{A}_2(y,P,P';l_1,pl_2,-\gamma)}{\log y}$$
$$+\gamma^2\mathcal{A}_2(y,P,P;l_1,pl_2,-\gamma)\bigg).$$

In view of Lemma 2.8, this equals

$$-U_1U_2(T/2\pi)^{1-\gamma}(\log y)^4\zeta(1-\gamma)\sum_{pl_1l_2\leq y}\frac{\log p}{(p^{1+\beta-\gamma}-1)p^\gamma}\frac{h(pl_1l_2)h(l_1)}{l_1l_2^{1+\gamma}}$$
$$\times F(pl_1)F(pl_1l_2,-\gamma)V_1(pl_1l_2)V_2(l_1)V_3(pl_2)V_4(pl_1l_2)$$
$$\times\int_0^{[l_1]_y}y^{\gamma t}tP([l_1]_y-t)\bigg(\frac{P([pl_1l_2]_y)}{(\log y)^2}+\frac{2\gamma P_0([pl_1l_2]_y)}{\log y}$$
$$+\gamma^2P_1([pl_1l_2]_y)\bigg)\,dt+O(TL^9).$$

From Lemma 2.9(ii) we obtain

$$\mathcal{Q}_3(\beta,\gamma)=-\frac{A(T/2\pi)^{1-\gamma}(\log y)^{11}}{3}\zeta(1-\gamma)\tag{4.13}$$
$$\times\int_0^1\int_{\substack{t,t_j\geq0\\ t\leq x\\ t_1+t_2\leq x}}(1-x)^3y^{\gamma(t-t_1)-\beta t_2}tt_1P(x-t)$$
$$\times\bigg(\frac{P(x-t_1-t_2)}{(\log y)^2}+\frac{2\gamma P_0(x-t_1-t_2)}{\log y}$$
$$+\gamma^2P_1(x-t_1-t_2)\bigg)\,dt\,dt_1\,dt_2\,dx+O_\varepsilon(TL^{9+\varepsilon}).$$

The term $\mathcal{Q}_5(\beta,\gamma)$ is in the same form as $\mathcal{Q}_3(\beta,\gamma)$. Therefore calculations give

$$\mathcal{Q}_5(\beta,\gamma)=-\frac{A(T/2\pi)(\log y)^9}{3}\zeta(1+\gamma)\int_0^1\int_{\substack{t_j\geq0\\ t_1+t_2\leq x}}(1-x)^3y^{-\gamma t_1-\beta t_2}t_1\tag{4.14}$$
$$\times P_1(x)P(x-t_1-t_2)\,dt_1\,dt_2\,dx+O_\varepsilon(TL^{9+\varepsilon}).$$

Finally, we have $\mathcal{Q}_6(\beta,\gamma) = O_B(TL^{-B})$ for any $B > 0$.

Collecting the estimates (4.3), (4.4), (4.9), (4.11)–(4.14), and letting

$$\beta = -i\alpha, \quad \gamma \to 0,$$

we obtain Lemma 1.3. ■

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References

[1] R. Balasubramanian, J. B. Conrey and D. R. Heath-Brown, *Asymptotic mean square of the product of the Riemann zeta-function and a Dirichlet polynomial*, J. Reine Angew. Math. 357 (1985), 161–181.

[2] J. Bredberg, *On large gaps between consecutive zeros, on the critical line, of some zeta-functions*, DPhil. thesis, Univ. of Oxford, 2011; see also arXiv:1101.3197.

[3] H. M. Bui, *Large gaps between consecutive zeros of the Riemann zeta-function*, J. Number Theory 131 (2011), 67–95.

[4] H. M. Bui, J. B. Conrey and M. P. Young, *More than 41% of the zeros of the zeta function are on the critical line*, Acta Arith. 150 (2011), 35–64.

[5] H. M. Bui and D. R. Heath-Brown, *On simple zeros of the Riemann zeta-function*, Bull. London Math. Soc. 45 (2013), 953–961.

[6] H. M. Bui, M. B. Milinovich and N. C. Ng, *A note on the gaps between consecutive zeros of the Riemann zeta-function*, Proc. Amer. Math. Soc. 138 (2010), 4167–4175 .

[7] J. B. Conrey, A. Ghosh and S. M. Gonek, *A note on gaps between zeros of the zeta function*, Bull. London Math. Soc. 16 (1984), 421–424.

[8] J. B. Conrey, A. Ghosh and S. M. Gonek, *Large gaps between zeros of the zeta-function*, Mathematika 33 (1986), 212–238.

[9] J. B. Conrey, A. Ghosh and S. M. Gonek, *Simple zeros of the Riemann zeta-function*, Proc. London Math. Soc. 76 (1998), 497–522.

[10] S. Feng and X. Wu, *On large spacing of the zeros of the Riemann zeta-function*, J. Number Theory 133 (2013), 2538–2566.

[11] R. R. Hall, *A new unconditional result about large spaces between zeta zeros*, Mathematika 52 (2005), 101–113.

[12] H. L. Montgomery and A. M. Odlyzko, *Gaps between zeros of the zeta function*, in: Topics in Classical Number Theory, Colloq. Math. Soc. János Bolyai 34, North-Holland, 1984, 1079–1106.

[13] J. Mueller, *On the difference between consecutive zeros of the Riemann zeta function*, J. Number Theory 14 (1982), 327–331.

[14] N. Ng, *Large gaps between the zeros of the Riemann zeta function*, J. Number Theory 128 (2008), 509–556.

[15] A. Selberg, *The zeta-function and the Riemann hypothesis*, in: C. R. Dixième Congrès Math. Scandinaves 1946, Jul. Gjellerups Forlag, København, 1947, 187–200.

[16] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., Clarendon Press, 1986.

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