On the irrationality of factorial series

by

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1. Introduction. Let \((b_n)_{n=1}^{\infty}\) be a sequence of integers. In the present paper we study the irrationality of \(R := \sum_{n=1}^{\infty} b_n/n!\) and, more generally, of

\[R^* := \sum_{N=1}^{\infty} \frac{b_N}{\prod_{n=1}^{N}(an+b)}\]

where \(a\) and \(b\) are given positive integers. In 1761 Lambert [10] proved the irrationality of \(e = 1 + \sum_{n=1}^{\infty} 1/n!\). In 1873 Hermite [9] established the transcendence of \(e\), which implies the irrationality of \(\sum_{n=1}^{\infty} m^n/n!\) for any nonzero integer \(m\). In 1869 G. Cantor [2] showed that if \(0 \leq b_n < n\), then \(R\) is irrational if and only if \(b_n > 0\) infinitely often and \(b_n < n - 1\) infinitely often. On the other hand, if \(b_n/(n-1)\) is constant for \(n\) larger than some \(n_0\), then \(R \in \mathbb{Q}\). This is an exceptional case in many results.

Oppenheim [12] showed that both the condition of \(b_n > 0\) and the condition of \(b_n < n - 1\) can be relaxed. For example, it follows from his results that if \(|b_n| < n\) for every \(n\), then \(R\) is rational if and only if \(b_n/(n-1)\) is ultimately a fixed integer. Thus if \(|b_n| < n - 1\) for every \(n\) and \(R \in \mathbb{Q}\), then \(b_n\) is ultimately equal to 0. The results of Oppenheim were extended by the authors [8] who showed that if \(n \nmid b_n\) for all \(n\), \(b_n = o(n^2)\) and \(\liminf_{n \to \infty} |b_n|/n = 0\), then \(R\) is irrational. They further proved that \(R\) is irrational if \((b_n)_{n=1}^{\infty}\) is a monotonic sequence of positive integers such that \(b_n = O(n^2)\) and \(\gcd(b_n, n-1) = o(b_n)\). Tijdeman and Yuan [14] extended another result of Oppenheim by showing that \(R\) is irrational if \(b_n = O(n)\) and the sequence \((b_n/n)_{n=1}^{\infty}\) has an irrational limit point. See also Hančl [6] and [7].

Erdős and Straus [5] started a series of results in which the size of the difference \(b_{n+1} - b_n\) is a relevant factor. They used such results to establish the

\[2000 \text{ Mathematics Subject Classification: Primary 11J72.} \]
\[\text{Key words and phrases: irrationality, linear independence, infinite series, factorial series.} \]
\[\text{The first-named author is supported by grants 201/04/0381 and MSM6198898701.} \]
irrationality of $R$ in case $(b_n)_{n=1}^\infty$ represents a multiplicative or other arithmetic function. It follows from their result that if $b_n > 0$ for all $n$, $b_{n+1} - b_n = o(n)$ and $\liminf_{n \to \infty} n/b_n = 0$, then $R$ is irrational. The authors [8] showed that the condition $\liminf_{n \to \infty} n/b_n = 0$ can be replaced with the necessary condition that $b_n/(n-1)$ is not ultimately constant. Tijdeman and Yuan [14] showed that, moreover, the condition $b_n > 0$ for all $n$ can be dropped: if $b_{n+1} - b_n = o(n)$, then $R \in \mathbb{Q}$ if and only if $b_n/(n-1)$ is ultimately a fixed integer. These results generalize Erdős’ result [3] that $\sum_{n=1}^\infty p_n/n! \notin \mathbb{Q}$, where $\{p_n\}_{n=1}^\infty$ is the sequence of consecutive prime numbers.

In fact Erdős claimed the irrationality of $\sum_{n=1}^\infty p_n^k/n! \notin \mathbb{Q}$ for $k = 1, 2, \ldots$, but unfortunately he proved only the case $k = 1$. Oppenheim [12] showed that $\sum_{n=1}^\infty \varepsilon_n d_n/n!$, $\sum_{n=1}^\infty \varepsilon_n \sigma_n/n!$ and $\sum_{n=1}^\infty \varepsilon_n \phi_n/n!$ are irrational for all choices of $\varepsilon_n \in \{-1, 1\}$, where $d(n), \sigma(n), \phi(n)$ denote the number of divisors, the sum of divisors, and the Euler function of $n$, respectively. A special case was treated by Erdős and Kac [4]. Erdős and Straus [5] proved that the numbers $1, \sum_{n=1}^\infty \sigma_n/n!$, $\sum_{n=1}^\infty \phi_n/n!$ and $\sum_{n=1}^\infty b_n/n!$, where $|b_n| < n^{1/2-\epsilon}$ for all large $n$ and $b_n \neq 0$ infinitely often, are linearly independent over the rationals. Most of the results mentioned were stated in greater generality in the original papers than above.

Tijdeman and Yuan [14] started to compare second order differences (cf. the proof of their Theorem 4.3). In the present paper we pursue this idea by studying $K$th order differences. For doing so we have to impose stronger regularity conditions on the numbers $b_n$. Nevertheless the results are valid for a wide class of sequences $(b_n)_{n=1}^\infty$. Corollary 3.1 precisely states for which polynomials $P(x)$ with integer coefficients $\sum_{n=1}^\infty P(n)/n!$ is rational. Section 3 further provides a method to establish the irrationality of a large class of numbers

$$\sum_{N=1}^\infty \frac{f(N)}{\prod_{n=1}^N (an + b)}$$

where $f(N)$ is an integer-valued function satisfying $f(N) = (aN + b)F(N) + O(1)$ and $F$ is a smooth function which does not grow faster than a polynomial. In particular it yields the irrationality of the following numbers:

$$\sum_{n=1}^\infty \frac{[n^\alpha]}{n!} (\alpha \geq 0), \quad \sum_{n=1}^\infty \frac{[\log^\beta n]}{n!} (\beta > 0), \quad \sum_{n=1}^\infty \frac{[\exp(\log n)]}{n!} (0 < \gamma < 1).$$

In Section 4 the linear independence over the rationals of such numbers is treated. For example, linear independence is shown for the numbers

$$1, e, \sum_{n=1}^\infty \frac{[n^\alpha]}{n!} \text{ for all } \alpha \in \mathbb{R}_+, \alpha \notin \mathbb{Z}.$$
who proved by Mahler’s method that $\sum_{n=1}^{\infty} [n\alpha] \beta^n$ is transcendental for $\alpha$ irrational and $\beta$ algebraic with $0 < |\beta| < 1$.

2. Basic lemmas. Let $a > 0$ and $b$ be integers such that $an + b \neq 0$ for every positive integer $n$. Let $\{b_n\}_{n=1}^{\infty}$ be a sequence of integers. We investigate under what conditions

$$R^* = \sum_{N=1}^{\infty} \frac{b_N}{\prod_{n=1}^{N}(an + b)}$$

is irrational. Since all terms are rational, we may neglect the terms with $an + b < 0$ and assume without loss of generality that $b \geq 0$.

The following lemma dealing with the sum

$$R_N^* := \sum_{m=N}^{\infty} \frac{b_m}{\prod_{n=N}^{m}(an + b)}$$

is crucial. We denote the set of positive integers by $\mathbb{N}$.

**Lemma 2.1.** If $R^* = p/q$ for some $p \in \mathbb{Z}$, $q \in \mathbb{N}$, then $qR_N^* \in \mathbb{Z}$ for all $N$.

**Proof.** We have

$$p \prod_{n=1}^{N-1}(an + b) = q \sum_{m=1}^{N-1} b_m \prod_{n=m+1}^{N}(an + b) + q \sum_{m=N}^{\infty} \prod_{n=N}^{\infty}(an + b)$$

and the first two terms are integers.

**Remark.** If $q$ divides $\prod_{n=1}^{N-1}(an + b)$, then we need not multiply by $q$ to obtain integers and can conclude that $R_N^*$ itself is an integer. If $q$ is coprime to $a$, this is the case for sufficiently large $N$. In particular, it is the case if $a = 1$, hence for sequences $\sum_{n=1}^{\infty} b_n/n!$.

The following consequence of a theorem of Oppenheim implies that $R^*$ is irrational if $b_n = o(n)$, but not ultimately constant 0.

**Lemma 2.2 (Oppenheim [12, Theorem 8]).** If $|b_n| < an + b$ for all $n > n_0$ and $\liminf_{n \to \infty} |b_n|/n = 0$, then $R^*$ is rational if and only if $b_n = 0$ for all $n > n_0$.

The next lemma displays some well known properties of Stirling numbers of the second kind.

**Lemma 2.3.** Let $K$ be a nonnegative integer. Put

$$S(r, K) = \frac{1}{K!} \sum_{j=0}^{K} (-1)^{K-j} \binom{K}{j} j^r.$$

Then $S(r, K) = 0$ if $r < K$, $S(r, K) = 1$ if $r = K$ and $S(r, K) \in \mathbb{N}$ if $r > K > 0$.

For a proof see [1, Section III.2].
The following lemma gives partial fractions of the denominator of $R^*$.

**Lemma 2.4.** Let $a > 0$, $b$, $N$ and $s \geq 0$ be integers such that $a(N+i)+b \neq 0$ for $i = 0, 1, \ldots, s$. Then

$$\prod_{i=0}^{s} \frac{1}{a(N+i)+b} = \frac{1}{s!a^s} \sum_{i=0}^{s} (-1)^i \binom{s}{i} \frac{1}{a(N+i)+b}. \quad (3)$$

**Proof.** We prove the identity by induction on $s$. For $s = 0$ identity (3) is trivial. Suppose that it holds for $s = n$. Then

$$\prod_{i=0}^{n+1} \frac{1}{a(N+i)+b} = \frac{1}{(n+1)a} \left( \prod_{i=0}^{n} \frac{1}{a(N+i)+b} - \prod_{i=1}^{n+1} \frac{1}{a(N+i)+b} \right)$$

$$= \frac{1}{(n+1)!a^{n+1}} \left( \sum_{i=0}^{n} (-1)^i \binom{n}{i} \frac{1}{a(N+i)+b} \right) - \sum_{i=1}^{n+1} (-1)^{i-1} \binom{n}{i-1} \frac{1}{a(N+i)+b}$$

$$= \frac{1}{(n+1)!a^{n+1}} \left( \frac{1}{aN+b} + \frac{(-1)^{n+1}}{a(N+n+1)+b} \right)$$

$$+ \sum_{i=1}^{n} (-1)^i \left( \binom{n}{i} + \binom{n}{i-1} \right) \frac{1}{a(N+i)+b}$$

$$= \frac{1}{(n+1)!a^{n+1}} \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} \frac{1}{a(N+i)+b}.$$

This completes the induction step. $\blacksquare$

The last lemma of the section will be used for all theorems except for Theorem 3.1 and Theorem 3.2. We use the convention that an empty product equals 1.

**Lemma 2.5.** Let $K \geq 0$, $a > 0$ and $b$ be given integers such that $an+b \neq 0$ for every $n \in \mathbb{N}$. Let $H : \mathbb{R} \to \mathbb{R}_+$ be a $K$ times continuously differentiable function such that $H(x) \neq 0$ for $x > x_0$. Suppose we have

$$H(N+j) = \sum_{r=0}^{K} \frac{H^{(r)}(N)}{r!} j^r + O\left( \frac{H(N)}{N^{K+1}} \right)$$

for $j = 0, 1, \ldots, K$ as $N \to \infty$. 

(4)
and
\[(5) \quad H^{(r)}(N) = O\left(\frac{H(N)}{N^r}\right) \quad \text{for } r = 0, 1, \ldots, K \text{ as } N \to \infty.\]

Put
\[R^K_*(N) = \sum_{j=0}^{K} (-1)^{j} \binom{K}{j} \sum_{s=0}^{\infty} \frac{H(N + j + s)}{\prod_{n=N+j}^{N+j+s-1} (an+b)}.\]

Then
\[(6) \quad R^K_*(N) = (-1)^K H^{(K)}(N) + O(H(N)N^{-K-1}) \quad \text{as } N \to \infty.\]

**Proof.** Let \(N\) be sufficiently large. Note that for \(j = 0, 1, \ldots, K\), by (4) and (5),
\[(7) \quad H(N + j) = O\left(H(N) \sum_{r=0}^{K+1} \frac{1}{r!} \left(\frac{j}{N}\right)^r\right) = O(H(N)e^{j/N})
= O(H(N)) \quad \text{as } N \to \infty.\]

Write
\[(8) \quad R^K_*(N) = R^K_{*1}(N) + R^K_{*2}(N) + R^K_{*3}(N)\]
where
\[R^K_{*1}(N) = \sum_{j=0}^{K} (-1)^{j} \binom{K}{j} H(N + j),\]
\[R^K_{*2}(N) = \sum_{j=0}^{K} (-1)^{j} \binom{K}{j} \sum_{s=1}^{K} \frac{H(N + j + s)}{\prod_{n=N+j}^{N+j+s-1} (an+b)},\]
\[R^K_{*3}(N) = \sum_{j=0}^{K} (-1)^{j} \binom{K}{j} \sum_{s=K+1}^{\infty} \frac{H(N + j + s)}{\prod_{n=N+j}^{N+j+s-1} (an+b)}.\]

By Lemma 2.3 and (4) we have
\[(9) \quad R^K_{*1}(N) = \sum_{j=0}^{K} (-1)^{j} \binom{K}{j} \sum_{r=0}^{K} \frac{H^{(r)}(N)}{r!} j^r + O(H(N)N^{-K-1})
= \sum_{r=0}^{K} \frac{H^{(r)}(N)}{r!} \sum_{j=0}^{K} (-1)^{j} \binom{K}{j} j^r + O(H(N)N^{-K-1})
= (-1)^K H^{(K)}(N) + O(H(N)N^{-K-1}) \quad \text{as } N \to \infty.\]

We now turn to \(R^K_{*2}(N)\). By Lemma 2.4, (4), (7), Lemma 2.3, and (5), we have
The combination of (8), (9), (10) and (11) yields (6).

Finally we estimate \( R_{K,2}(N) \). By (7) there exists a constant \( c > 1 \) such that \( H(n+1) < cH(n) \) for all sufficiently large \( n \). Hence \( H(N+s) < c^sH(N) \) for every positive integer \( s \). It follows that

\[
R_{K,2}^*(N) = O\left( H(N)\frac{1}{NK+1} \right).
\]

Thus

\[
(10) \quad R_{K,2}^*(N) = O\left( H(N)\frac{1}{NK+1} \right).
\]

Finally we estimate \( R_{K,3}^*(N) \). By (7) there exists a constant \( c > 1 \) such that \( H(n+1) < cH(n) \) for all sufficiently large \( n \). Hence \( H(N+s) < c^sH(N) \) for every positive integer \( s \). It follows that

\[
R_{K,3}^*(N) = O\left( \sum_{s=K+1}^{\infty} \frac{H(N+s)}{\prod_{n=N}^{N+s-1}(an+b)} \right) = O\left( \sum_{s=K+1}^{\infty} \frac{H(N)c^s}{Ns} \right).
\]

Hence

\[
(11) \quad R_{K,3}^*(N) = O\left( \frac{H(N)}{NK+1} \right) \quad \text{as} \quad N \to \infty.
\]

The combination of (8), (9), (10) and (11) yields (6). ■

**Remark.** In applications of Lemma 2.5 the integer \( K \) is usually chosen as the smallest nonnegative integer such that \( H(K)(N) \to 0 \) as \( N \to \infty \).
3. Irrationality. The next lemma implies that $\sum_{n=1}^{\infty} P(n)/n!$ for $P(x) \in \mathbb{Q}[x]$ is a linear combination over $\mathbb{Q}$ of 1 and $e$.

**Lemma 3.1.** Let $a > 0$ and $b$ be integers such that $an + b \neq 0$ for every $n \in \mathbb{N}$. Suppose that $P(x) = \sum_{i=0}^{T} a_i x^i$ is a polynomial with rational coefficients. Let $d$ be the least common denominator of $a_0, a_1, \ldots, a_T$. Then there exist rational numbers $Q_0$ and $Q_1$ such that

$$\sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(an + b)} = Q_0 + Q_1 \sum_{N=1}^{\infty} \frac{1}{\prod_{n=1}^{N}(an + b)}$$

where

(12) \[ a^T dQ_0 = \sum_{i=0}^{T} a_i \sum_{k=0}^{i} \sum_{h=k}^{i} \left( \begin{array}{c} i \\ h \end{array} \right) a^{T-i+h-k}((-b)^{i-h}S(h, k) \sum_{N=1}^{k} \prod_{n=0}^{N-1} (b - na) \right] \]

and

(13) \[ a^T dQ_1 = \sum_{i=0}^{T} a_i \sum_{k=0}^{i} \sum_{h=k}^{i} \left( \begin{array}{c} i \\ h \end{array} \right) a^{T-i+h-k}((-b)^{i-h}S(h, k) \right] \]

are integers.

**Proof.** We can write

(14) \[ x^i = \sum_{k=0}^{i} b_{i,k} \prod_{m=1}^{k}(a(x - m + 1) + b). \]

On substituting $x = -b/a + r$ for $r = 0, 1, \ldots, i$ into equation (14) we get

(15) \[ \left( \frac{-b}{a} + r \right)^i = \sum_{k=0}^{i} b_{i,k} \prod_{m=1}^{k}(a\left(\frac{-b}{a} + r - m + 1\right) + b) \]

\[ = \sum_{k=0}^{i} b_{i,k} a^k \frac{r!}{(r-k)!}. \]

We consider (15) as a system of $i + 1$ equations with $i + 1$ unknowns $b_{i,0}, b_{i,1}, \ldots, b_{i,i}$. Using the fact that $r \leq i$ we find that the solution of this system for $k = 0, 1, \ldots, i$ is given by

(16) \[ b_{i,k} = \frac{1}{k! a^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left( \frac{-b}{a} + k - j \right)^i. \]

We write

$$\sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(an + b)} = \sum_{i=0}^{T} a_i \sum_{N=1}^{\infty} \frac{N^i}{\prod_{n=1}^{N}(an + b)}.$$
From this, (14) and (16) we obtain

\[
\sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(an + b)} = \sum_{i=0}^{T} a_i \sum_{k=0}^{i} b_{i,k} \sum_{N=1}^{\infty} \frac{\prod_{m=1}^{k}(a(N - m + 1) + b)}{\prod_{n=1}^{N}(an + b)}
\]

\[
= \sum_{i=0}^{T} a_i \sum_{k=0}^{i} b_{i,k} \sum_{N=1}^{N-i-1} (b - na) + \sum_{i=0}^{T} a_i \sum_{k=0}^{i} b_{i,k} \sum_{N=k+1}^{\infty} \frac{1}{\prod_{n=1}^{N-k}(an + b)}
\]

\[=: Q_0 + Q_1 \sum_{N=1}^{\infty} \frac{1}{\prod_{n=1}^{N}(an + b)}\]

where \(Q_0\) and \(Q_1\) are rational numbers. We have, by (16) and Lemma 2.3,

\[
b_{i,k} = \frac{1}{k!a^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{h=0}^{i} \binom{i}{h} \left( -\frac{b}{a} \right)^{i-h} (k - j)^h
\]

\[
= \frac{1}{k!a^k} \sum_{h=0}^{i} \binom{i}{h} \left( -\frac{b}{a} \right)^{i-h} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k - j)^h
\]

\[
= \sum_{h=k}^{i} \binom{i}{h} \frac{1}{a^k} \left( -\frac{b}{a} \right)^{i-h} S(h,k)
\]

where \(S(h,k)\) is an integer. Note that the exponent of \(a\) in \((1/a^k)(-b/a)^{i-h}\) has minimal value \(-i\), viz. if \(h = k\). Hence \(a^i b_{i,k}\) is an integer for \(i = 0, 1, \ldots, k\). We deduce that \(da^T Q_0\) and \(da^T Q_1\) are the integers given by (12) and (13), respectively. 

**Theorem 3.1.** Let \(a > 0\) and \(b\) be integers such that \(an + b \neq 0\) for every \(n \in \mathbb{N}\). Let \(P(x) = \sum_{i=0}^{T} a_i x^i \in \mathbb{Z}[x]\). Then

\[
R^* := \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(an + b)}
\]

is rational if and only if

\[
(17) \quad Q_1 = \sum_{i=0}^{T} a_i \sum_{k=0}^{i} \frac{1}{k!a^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left( -\frac{b}{a} + k - j \right)^i = 0.
\]

**Proof.** It is obvious that \(R^*\) is absolutely convergent. It follows immediately from Lemma 3.1 that if (17) holds, then \(R^* \in \mathbb{Q}\). On the other hand, suppose (17) does not hold. By Oppenheim’s theorem (Lemma 2.2) we know that

\[
\sum_{N=1}^{\infty} \frac{1}{\prod_{n=1}^{N}(an + b)}
\]

is irrational. Hence, by Lemma 3.1, \(R^*\) is irrational. 

Corollary 3.1. Let \( P(x) = \sum_{i=0}^{T} a_i x^i \in \mathbb{Z}[x] \). Then \( \sum_{N=1}^{\infty} P(N)/N! \) is rational if and only if
\[
\sum_{N=1}^{\infty} a_i \frac{\sum_{k=0}^{N} S(i, k)}{N!} = 0.
\]
If \( a_T > 0 \) and \( a_i \geq 0 \) for \( i = 0, 1, \ldots, T-1 \), then \( \sum_{N=1}^{\infty} P(N)/N! \notin \mathbb{Q} \).

Proof. Apply Theorem 3.1 with \( a = 1, \ b = 0 \), and Lemma 2.3. \( \blacksquare \)

Remark. We recall that \( \sum_{n=1}^{\infty} (n-1)/n! = 1 \in \mathbb{Q} \). Hence the condition on the nonnegativity of the coefficients in the second statement cannot be dropped.

Corollary 3.2. Let \( a > 0 \) and \( b \) be integers such that \( an + b \neq 0 \) for every \( n \in \mathbb{N} \). Let \( P(x) = \sum_{i=0}^{T} a_i x^i \in \mathbb{Z}[x] \). If \( a \neq a_T(b-1)^T \) then \( R^* \) is irrational.

Proof. Observe that, by (13) and Lemma 2.3, the terms in \( a^T Q_1 \) are divisible by \( a \) unless \( h = k, \ i = T \). Hence, by Lemma 2.3 again,
\[
a^T Q_1 = a_T \sum_{k=0}^{T} \binom{T}{k} (-b)^{T-k} S(k, k) = a_T (1-b)^T \mod a. \blacksquare
\]

Theorem 3.2. Let \( a > 0 \) and \( b \) be integers such that \( an + b \neq 0 \) for every \( n \in \mathbb{N} \). Let \( P(x) = \sum_{i=0}^{T} a_i x^i \in \mathbb{Z}[x] \). Let \( f : \mathbb{N} \to \mathbb{Z} \) be a sequence such that \( f(N) = P(N) + o(N) \) as \( N \to \infty \). Suppose
\[
\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an+b)} \in \mathbb{Q}.
\]
Then
\[
f(N) = P(N) - Q_1 \quad \text{for all } N
\]
where \( Q_1 \) is given by (13).

Proof. We have, by Lemma 3.1,
\[
\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an+b)} = \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(an+b)} + \sum_{N=1}^{\infty} \frac{f(N) - P(N)}{\prod_{n=1}^{N}(an+b)}
\]
\[
= Q_0 + Q_1 \sum_{N=1}^{\infty} \frac{1}{\prod_{n=1}^{N}(an+b)} + \sum_{N=1}^{\infty} \frac{f(N) - P(N)}{\prod_{n=1}^{N}(an+b)}
\]
\[
= Q_0 + \sum_{N=1}^{\infty} \frac{Q_1 + (f(N) - P(N))}{\prod_{n=1}^{N}(an+b)}.
\]
The numerator of the last fraction is a rational number which is \( o(N) \) as \( N \to \infty \) and has a denominator which is independent of \( N \). Hence, by Lemma 2.2, \( Q_1 + f(N) - P(N) = 0 \) for \( N \geq N_0 \). It follows that \( f(N) = P(N) - Q_1 \). \( \blacksquare \)

Corollary 3.3. Under the conditions of Theorem 3.2 we have \( P(N) \equiv Q_1 \mod 1 \) for all \( N \) and therefore \( dQ_1 \in \mathbb{Z} \).
Theorem 3.3. Let $a > 0$ and $b$ be integers such that $an + b \neq 0$ for every $n \in \mathbb{N}$. Let $P(x) = \sum_{i=0}^{T} a_i x^i \in \mathbb{R}[x]$. Let $f: \mathbb{N} \to \mathbb{Z}$ be a sequence such that

$$f(N) = (aN + b)P(N) + O(1)$$

as $N \to \infty$. Suppose

$$\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} (an + b)} \in \mathbb{Q}.$$ 

Then $a_T, a_{T-1}, \ldots, a_1, a_0 \in \mathbb{Q}$.

Proof. Suppose

$$\sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} (an + b)} = \frac{p}{q}$$

where $p$ and $q > 0$ are coprime integers. Let $U$ be the largest index $i$ with $a_i \notin \mathbb{Q}$. Let $d$ be a common denominator of $a_{U+1}, \ldots, a_T$. Put $P_1(x) = \sum_{i=0}^{U} a_i x^i$ and $P_2(x) = (ax + b) \sum_{i=U+1}^{T} a_i x^i$. Then we have, by Lemma 3.1,

$$a^T dp = a^T dq \sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} (an + b)} = q \sum_{N=1}^{\infty} \frac{a^T d(f(N) - P_2(N) + P_2(N))}{\prod_{n=1}^{N} (an + b)}$$

$$= qa^T dQ_{2,0} + q \sum_{N=1}^{\infty} \frac{a^T d(f(N) - P_2(N)) + a^T dQ_{2,1}}{\prod_{n=1}^{N} (an + b)}$$

where $Q_{2,0}$ and $Q_{2,1}$ are rational numbers corresponding to $P_2$ according to Lemma 3.1. From this and Lemma 2.1 we deduce that for every positive integer $N$ the number

$$R^*_N := qa^T d \sum_{s=0}^{\infty} \frac{f(N + s) - P_2(N + s) + Q_{2,1}}{\prod_{n=N}^{N+s} (an + b)}$$

is an integer. From the definition of $P_1, P_2$ and the assumption on $f$ it follows that

$$R^*_N = qa^T d \sum_{s=0}^{\infty} \frac{P_1(N + s)}{\prod_{n=N}^{N+s-1} (an + b)} + O\left(\frac{1}{N}\right).$$

This combined with Lemma 2.5 applied to $H(X) = qa^T dP_1(X + j)$ implies that the number

$$R^*_U(N) := \sum_{j=0}^{U} (-1)^j \binom{U}{j} R^*_{N+j}$$

$$= (-1)^U qa^T dP_1^{(U)}(N) + O\left(\frac{1}{N}\right) = (-1)^U qa^T U! da_U + O\left(\frac{1}{N}\right)$$

is an integer. This is a contradiction for a sufficiently large number $N$. $lacksquare$

Corollary 3.4. Let $P(x) = \sum_{i=0}^{T} a_i x^i \in \mathbb{R}[x]$ with nonnegative coefficients and $a_T > 0$. If $\sum_{N=1}^{\infty} [P(N)]/N! \in \mathbb{Q}$, then $T = 0$ and $[a_0] = 0$. 
Proof. Suppose $\sum_{N=1}^{\infty} [P(N)]/N! \in \mathbb{Q}$. It follows from Theorem 3.3 with $[P(N)] = N\sum_{i=1}^{T} a_i N^{i-1} + (a_0 + O(1))$ that $a_T, a_{T-1}, \ldots, a_1$ are rational. Therefore we may assume without loss of generality that $a_0 \in \mathbb{Q}$. Let $d$ be the common denominator of $a_T, a_{T-1}, \ldots, a_0$. By Theorem 3.2 applied with $f(N) = [P(N)]$ we obtain $P(N) - [P(N)] = Q_1$ for all $N$ where $Q_1$ is a rational number. Hence $Q_1$ equals $a_0 - [a_0]$. It follows that $[P(x)] = \sum_{i=1}^{T} a_i x^i + [a_0] \in \mathbb{Q}[x]$. Since $[P(x)]$ has nonnegative rational coefficients and $a_T > 0$, we deduce by applying Corollary 3.1 to $d[P(x)] = \sum_{i=1}^{T} d a_i x^i + d[a_0] \in \mathbb{Z}[x]$ that $T = 0$ and $[a_0] = 0$. $lacksquare$

Remark. The corollary implies that for any positive integer $K$ the sum $\sum_{n=1}^{\infty} [\beta n^K]/n!$ is a strictly increasing function of $\beta > 0$ which does not take rational values. This phenomenon will be met several times later on.

Theorem 3.4. Let $K \geq 0$, $a > 0$ and $b$ be given integers such that $an+b \neq 0$ for every $n \in \mathbb{N}$. Let $F : \mathbb{R}_+ \to \mathbb{R}_+$ be a $K+1$ times continuously differentiable function such that

\begin{equation}
F(N+j) = \sum_{r=0}^{K} \frac{F^{(r)}(N)}{r!} j^r + O\left(\frac{F(N)}{N^{K+1}}\right) \quad \text{for } j = 0, 1, \ldots, K \text{ as } N \to \infty,
\end{equation}

\begin{equation}
F^{(r)}(N) = O\left(\frac{F(N)}{N^r}\right) \quad \text{for } r = 0, 1, \ldots, K \text{ as } N \to \infty,
\end{equation}

\begin{equation}
\lim_{N \to \infty} F^{(K)}(N) = 0, \quad \lim_{N \to \infty} N^{K+1} |F^{(K)}(N)| = \infty,
\end{equation}

\begin{equation}
\limsup_{N \to \infty} N |F^{(K)}(N)| = \infty.
\end{equation}

Let $f : \mathbb{N} \to \mathbb{Z}$ be a sequence such that

\begin{equation}
R^* := \sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} (an+b)}
\end{equation}

is absolutely convergent and $f(N) = (aN+b)F(N)+O(1)$ as $N \to \infty$. Then $R^*$ is irrational.

Proof. Let $N$ be sufficiently large. Suppose $R^* = p/q$ where $p$ and $q > 0$ are coprime integers. Put

\begin{equation}
R_K^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{s=0}^{\infty} \frac{f(N+j+s)}{\prod_{n=N+j}^{N+j+s} (an+b)}.
\end{equation}
By Lemma 2.1, \( qR_N^*(N) \) is an integer for every positive integer \( N \). We have, by Lemma 2.5,

\[
R_N^*(N) = \sum_{j=0}^{K} (-1)^j \binom{K}{j} \sum_{s=0}^{\infty} \frac{F(N + j + s)}{\prod_{n=N+j}^{\infty} (an + b)} + O\left(\frac{1}{N}\right)
\]

\[
= (-1)^K F^K(N) + O\left(\frac{F(N)}{NK+1} + \frac{1}{N}\right)
\]

\[
= (-1)^K F^K(N)\left(1 + O\left(\frac{F(N)}{NK+1|F^K(N)|}\right)\right) + O\left(\frac{1}{N}\right).
\]

By (20) the right-hand side tends to 0 as \( N \to \infty \). Since \( qR_N^*(N) \) is an integer, we infer that \( R_N^*(N) = 0 \) for \( N \geq N_0 \). It follows that

\[
0 = (1 + o(1))NF^K(N) + O(1) \quad \text{for} \quad N \geq N_0,
\]

which contradicts (21).

**Corollary 3.5.** Let \( \alpha \in \mathbb{R}_{\geq 0} \), \( \gamma \in \mathbb{R}_{\geq 0} \). If \( \sum_{N=1}^{\infty} [\gamma N^\alpha]/N! \in \mathbb{Q} \), then \( \alpha = 0 \) and \( \lceil \gamma \rceil = 0 \).

**Proof.** If \( \alpha \notin \mathbb{Z} \), then apply Theorem 3.4 with \( a = 1 \), \( b = 0 \), \( K = \lfloor \alpha \rfloor \), \( f(N) = [\gamma N^\alpha] \), \( F(N) = \gamma N^\alpha - 1 \). If \( \alpha \in \mathbb{Z} \), then apply Corollary 3.4 with \( T = \alpha \).

**Remark.** It is remarkable that Corollary 3.5 holds for all \( \alpha \geq 0 \) and \( \gamma > 0 \) so that \( \sum_{N=1}^{\infty} [\gamma N^\alpha]/N! \) is a strictly monotonic function of \( \alpha \) and \( \gamma \) missing all rational values.

**Corollary 3.6.** Let \( \alpha \in \mathbb{R}_{\geq 0} \setminus \mathbb{Z} \), \( \gamma \in \mathbb{R}_{+} \). Then

\[
\sum_{N=1}^{\infty} [\gamma N^\alpha \log N]/N! \notin \mathbb{Q}.
\]

**Proof.** Apply Theorem 3.4 with \( a = 1 \), \( b = 0 \), \( K = \lfloor \alpha \rfloor \), \( f(N) = [\gamma N^\alpha \log N] \), \( F(N) = \gamma N^{\lfloor \alpha \rfloor} \log N \).

Theorem 3.4 is not strong enough to prove that \( \sum_{N=1}^{\infty} [N \log N]/N! \) is irrational. For such series we provide a variant of Theorem 3.4 where condition (21) becomes weaker, but conditions (18) and (19) become stronger.

**Theorem 3.5.** Let \( K \geq 1 \), \( a > 0 \) and \( b \) be given integers such that \( an + b \neq 0 \) for every \( n \in \mathbb{N} \). Let \( F : \mathbb{R}_{+} \to \mathbb{R}_{+} \) be a function such that

\[
F(N + x) = \sum_{r=0}^{\infty} \frac{F^{(r)}(N)}{r!} x^r \quad \text{for} \quad x = o(N) \quad \text{as} \quad N \to \infty,
\]

(22)

\[
F^{(r)}(N) = O\left(r! \frac{F(N)}{N^r}\right) \quad \text{uniformly for} \quad r = 0, 1, \ldots \quad \text{as} \quad N \to \infty,
\]

(23)
(24) \[ \lim_{x \to \infty} F^{(K)}(x) = 0, \quad \lim_{x \to \infty} \frac{x^{K+1} |F^{(K)}(x)|}{F(x)} = \infty, \]

(25) \[ \lim_{x \to \infty} x^2 |F^{(K)}(x)| = \infty. \]

Let \( f : \mathbb{N} \to \mathbb{Z} \) be a sequence such that

\[ R^* := \sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an + b)} \]

is absolutely convergent and \( f(N) = (aN + b)F(N) + O(1) \) as \( N \to \infty \). Then \( R^* \) is irrational.

Proof. Let \( N \) be sufficiently large. Suppose \( R^* = p/q \) where \( p \) and \( q > 0 \) are coprime integers. Conditions (22) and (23) imply conditions (18) and (19). Hence, by Theorem 3.4, we may assume without loss of generality that

(26) \[ NF^{(K)}(N) = O(1). \]

Observe that by (23) and (22), for \( x = o(N) \),

(27) \[ F(N + x) \leq \sum_{r=0}^{\infty} |F^{(r)}(N)| \frac{x^r}{r!} = O \left( F(N) \sum_{r=0}^{\infty} \left( \frac{x}{N} \right)^r \right) = O(F(N)). \]

Put

\[ R_{K-1}^*(N) = \sum_{j=0}^{K-1} (-1)^j \binom{K-1}{j} \sum_{s=0}^{\infty} \frac{F(N + j + s)}{\prod_{n=N+j}^{N+s}(an + b)}. \]

By Lemma 2.1, \( qR_{K-1}^*(N) \) is an integer for every positive integer \( N \). We have, by Lemma 2.5,

(28) \[ R_{K-1}^*(N) = \sum_{j=0}^{K-1} (-1)^j \binom{K-1}{j} \sum_{s=0}^{\infty} \frac{F(N + j + s)}{\prod_{n=N+j}^{N+s}(an + b)} + O \left( \frac{1}{N} \right) = (-1)^{K-1} F^{(K-1)}(N) + O \left( \frac{F(N)}{NK} + \frac{1}{N} \right). \]

Put

(29) \[ t = 1 + \left[ \frac{N}{\min_{x \in [N,2N]} (x^2 |F^{(K)}(x)|)^{1/2}} + \frac{N}{\min_{x \in [N,2N]} (x^{K+1} |F^{(K)}(x)| / F(x))^{1/2}} \right], \]

It follows from (25) and (24) that \( t = o(N) \) as \( N \to \infty \). Hence, by (27), similarly to (28),

(30) \[ qR_{K-1}^*(N + t) = (-1)^{K-1} qF^{(K-1)}(N + t) + O \left( \frac{F(N)}{NK} + \frac{1}{N} \right) \]

is an integer.
We apply the Mean Value Theorem to (30) and (28). Hence there exists a real number $\tau$ with $0 < \tau < t$ such that

$$M(N) := qR_{K-1}^*(N+t) - qR_{K-1}^*(N) = (-1)^{K-1}qtF^{(K)}(N+\tau) + O\left(\frac{F(N)}{N^K} + \frac{1}{N}\right)$$

is an integer. It follows from (29), (27) and (24) that, for some positive constant $c$,

$$\frac{tN^K|F^{(K)}(N+\tau)|}{F(N)} \geq \frac{N^{K+1}|F^{(K)}(N+\tau)|}{F(N)\min_{x\in[N,2N]}(x^{K+1}|F^{(K)}(x)|/F(x))^{1/2}}$$

$$\geq c\left(\frac{(N+\tau)^{K+1}|F^{(K)}(N+\tau)|}{F(N+\tau)}\right)^{1/2} \rightarrow \infty \text{ as } N \rightarrow \infty.$$ 

Hence, by (26) and $t = o(N)$,

$$M(N) = (-1)^{K-1}(1 + o(1))qtF^{(K)}(N+\tau) + O(1/N) \rightarrow 0.$$  

Thus, since $M(N)$ represents an integer, $M(N) = 0$ for $N \geq N_1$. It follows from (31), (29) and (27) that

$$0 = Nt|F^{(K)}(N+\tau)| + O(1)$$

$$\geq \frac{N^2|F^{(K)}(N+\tau)|}{\min_{x\in[N,2N]}(x^2|F^{(K)}(x)|/F(x))^{1/2}} + O(1)$$

$$\geq ((N+\tau)^2|F^{(K)}(N+\tau)|)^{1/2} + O(1),$$

which contradicts (25).

**Corollary 3.7.** Let $\alpha \in \mathbb{R}_{\geq 0}$, $\beta \in \mathbb{R}$, $\beta \neq 0$, $\gamma \in \mathbb{R}_+$. Suppose $\beta > 0$ whenever $\alpha = 0$. Then

$$\sum_{N=1}^{\infty} \frac{[\gamma N^\alpha \log^\beta N]}{N!} \notin \mathbb{Q}.$$ 

**Proof.** Apply Theorem 3.5 with $a = 1$, $b = 0$, $K = [\alpha]$ if $\alpha \notin \mathbb{Z}$ or $\beta > 0$, and $K = \alpha - 1$ otherwise, $f(N) = [\gamma N^\alpha \log^\beta N]$, $F(N) = \gamma N^{\alpha-1} \log^\beta N$. ■

**Corollary 3.8.** Let $\alpha \in \mathbb{R}_{\geq 0}$, $0 < \beta < 1$, $\gamma \in \mathbb{R}_+$. Then

$$\sum_{N=1}^{\infty} \frac{[\gamma N^\alpha \exp(\log^\beta N)]}{N!} \notin \mathbb{Q}.$$ 

**Proof.** Apply Theorem 3.5 with $a = 1$, $b = 0$, $K = [\alpha]$, $f(N) = [\gamma N^\alpha \exp(\log^\beta N)]$, $F(N) = \gamma N^{\alpha-1} \exp(\log^\beta N)$. ■
4. Linear independence. The method from the previous section enables us to prove the linear independence of the considered sums. Recall that Theorem 3.1 precisely states when

\[ \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(an + b)} \]

for \( P(x) \in \mathbb{Z}[x] \) is rational.

**Theorem 4.1.** Let \( a > 0 \) and \( b \) be integers such that \( an + b \neq 0 \) for every \( n \in \mathbb{N} \). Suppose that \( P(x) = \sum_{t=0}^{T} a_t x^t \in \mathbb{Z}[x] \) and that

\[ \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N}(an + b)} \]

is irrational. Let \( W \) be a set of functions \( F : \mathbb{R}_+ \to \mathbb{R}_+ \) which have the following properties:

(i)

\[ F(N + x) = \sum_{r=0}^{\infty} \frac{F^{(r)}(N)}{r!} x^r \quad \text{for} \quad x = o(N) \quad \text{as} \quad N \to \infty, \]

(ii)

\[ F^{(r)}(N) = O(r!F(N)/N^r) \quad \text{uniformly for} \quad r = 0, 1, \ldots \quad \text{as} \quad n \to \infty, \]

(iii) either there exists a positive integer \( K \) such that

\[ F^{(K)}(x) = o(1), \quad \frac{F(x)}{x^{K+1}} = o(|F^{(K)}(x)|), \quad \lim_{x \to \infty} x^2|F^{(K)}(x)| = \infty \]

or

\[ K = 0, \quad \lim_{x \to \infty} F(x) = 0, \quad \lim_{x \to \infty} xF(x) = \infty, \]

(iv) for every pair of functions \( F, G \in W \) with corresponding integers \( K > L \) one has \( \lim_{x \to \infty} G^{(k)}(x)/F^{(k)}(x) = 0 \) for \( k = 0, 1, \ldots, K \); for every pair of functions \( F, G \in W \) with \( F \neq G \) and corresponding integers \( K = L \) one has either \( \lim_{x \to \infty} G^{(k)}(x)/F^{(k)}(x) = 0 \) for \( k = 0, 1, \ldots, K \) or \( \lim_{x \to \infty} F^{(k)}(x)/G^{(k)}(x) = 0 \) for \( k = 0, 1, \ldots, K \).

Suppose that for every function \( F \in W \) there exists a function \( f : \mathbb{N} \to \mathbb{Z} \) such that

\[ \sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N}(an + b)} \]

is absolutely convergent and

\[ f(N) = (aN + b)F(N) + O(1) \quad \text{as} \quad N \to \infty. \]
Then the numbers
\[ \sum_{N=1}^{\infty} \frac{f(N)}{\prod_{n=1}^{N} (an+b)} \quad (F \in W), \quad \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N} (an+b)}, \quad 1 \]
are linearly independent over the rationals.

Proof. Suppose that there exist functions \( f_0 := P, f_1, f_2, \ldots, f_M \in W \) such that the numbers
\[ \sum_{N=1}^{\infty} \frac{f_i(N)}{\prod_{n=1}^{N} (an+b)} \]
\((i = 0, 1, \ldots, M)\) and the number 1 are linearly dependent over the rationals. Then there exist integers \( A_0, A_1, \ldots, A_M, p \) and \( q > 0, \) not all zero, such that
\[ \frac{p}{q} = \sum_{m=0}^{M} A_m \sum_{N=1}^{\infty} \frac{f_m(N)}{\prod_{n=1}^{N} (an+b)}. \] 
It is excluded that \( A_1 = \cdots = A_M = 0. \) Without loss of generality we may assume that \( A_1, \ldots, A_M \) are all nonzero and \( M > 0. \) Lemma 3.1 and equation (36) imply
\[ \frac{p}{q} = A_0 \sum_{N=1}^{\infty} \frac{P(N)}{\prod_{n=1}^{N} (an+b)} + \sum_{m=1}^{M} A_m \sum_{N=1}^{\infty} \frac{f_m(N)}{\prod_{n=1}^{N} (an+b)} \]
\[ = \frac{p_1}{q_1} + \frac{p_2}{q_2} \sum_{N=1}^{\infty} \frac{1}{\prod_{n=1}^{N} (an+b)} + \sum_{N=1}^{\infty} \sum_{m=1}^{M} A_m f_m(N) \]
where \( p_1, p_2, q_1 > 0 \) and \( q_2 > 0 \) are suitable integers which do not depend on \( N. \) Thus
\[ A := q_1 q_2 q \left( \frac{p}{q} - \frac{p_1}{q_1} \right) = q_1 q_2 \sum_{N=1}^{\infty} \frac{p_2 + q_2 \sum_{m=1}^{M} A_m f_m(N)}{\prod_{n=1}^{N} (an+b)} \]
is an integer.

By Lemma 2.1, for every positive integer \( N \) the number
\[ S_N := q_1 q_2 \sum_{s=0}^{\infty} \frac{p_2 + q_2 \sum_{m=1}^{M} A_m f_m(N+s)}{\prod_{n=N}^{N+s} (an+b)} \]
is an integer. Moreover
\[ S_N = q_1 q_2 \sum_{s=0}^{\infty} \frac{\sum_{m=1}^{M} A_m F_m(N+s)}{\prod_{n=N}^{N+s-1} (an+b)} + O \left( \frac{1}{N} \right). \]
Without loss of generality we may assume by (iv) and (iii) that if $K$ is the integer corresponding to $F_M$, then

$$
\lim_{x \to \infty} \frac{F_m^{(k)}(x)}{F_M^{(k)}(x)} = 0 \quad \text{for } k = 0, 1, \ldots, K \text{ and } m = 0, 1, \ldots, M - 1.
$$

Let $L$ be a nonnegative integer with $L \leq K$. We deduce from (37) that

$$
R^*_L(N) := \sum_{j=0}^{L} (-1)^j \binom{L}{j} S_{N+j}
$$

$$
= q_1 q_2 q \sum_{j=0}^{L} (-1)^j \binom{L}{j} \sum_{s=0}^{\infty} \left( \sum_{m=1}^{M} A_m F_m(N + s + j) \right) + O\left( \frac{1}{N} \right)
$$

is an integer as well. From Lemma 2.5, (32), and (33) we deduce that

$$
R^*_L(N) = (-1)^L q_1 q_2 q \sum_{m=1}^{M} A_m F_m^{(L)}(N) + \sum_{m=1}^{M} O(F_m(N)N^{-L-1}) + O\left( \frac{1}{N} \right).
$$

Hence, by (38),

$$
R^*_L(N) = (-1)^L q_1 q_2 q \sum_{m=1}^{M} A_m F_m^{(L)}(N) + O(F_M(N)N^{-L-1}) + O(1/N).
$$

We distinguish two cases.

(a) Suppose $\limsup_{N \to \infty} N|F_M^{(K)}(N)| = \infty$. Then we put $L = K$ and find by (38) that

$$
R^*_K(N) = (-1)^K q_1 q_2 q A_K F_M^{(K)}(N)(1 + o(1)) + O(F_M(N)N^{-K-1}) + O\left( \frac{1}{N} \right).
$$

Hence we derive a contradiction in the same way as we did in the last lines of the proof of Theorem 3.4.

(b) Suppose $\limsup_{N \to \infty} N|F_M^{(K)}(N)| < \infty$, i.e. $F_M^{(K)}(N) = O(1/N)$ as $N \to \infty$. Note that $K \geq 1$ in view of (35). By (27) and (38) we see, for $m = 1, \ldots, M$ and $x = o(N)$, that $F_m(N + x) = O(F_m(N)) = O(F_M(N))$. Put $L = K - 1$ in (39). Hence

$$
R^*_{K-1}(N) = (-1)^{K-1} q_1 q_2 q \sum_{m=1}^{M} A_m F_m^{(K-1)}(N) + O(F_M(N)N^{-K}) + O\left( \frac{1}{N} \right).
$$

Define $t$ as in (29). We apply the Mean Value Theorem to the integer $M(N) := R^*_{K-1}(N + t) - R^*_K(N)$. We obtain

$$
M(N) = (-1)^{K-1} q_1 q_2 q t \sum_{m=1}^{M} A_m F_m^{(K)}(N + \tau) + O(F_M(N)N^{-K}) + O\left( \frac{1}{N} \right)
$$
for some \( \tau \) with \( 0 < \tau < t \). By (38) we have

\[
M(N) = (-1)^{K-1}q_1q_2\tau(1 + o(1))A_M F_M^{(K)}(N + \tau) + O\left(\frac{F_M(N)}{N^K} + \frac{1}{N}\right).
\]

The further proof proceeds as that of Theorem 3.5 from the introduction of \( M(N) \) on. \( \blacksquare \)

**Remark.** It follows from a repeated use of de l’Hôpital’s rule that condition (iv) can be relaxed. If

\[
\lim_{x \to \infty} F_m(x) = \infty \quad \text{and} \quad \lim_{x \to \infty} G^{(K-1)}(x) = \infty,
\]

then \( \lim_{x \to \infty} F^{(k)}(x)/G^{(k)}(x) = 0 \) for \( k = 0, 1, \ldots, K \). (See Theorem 5.13 in [13], with \( A = 0 \) and \( a = \infty \).)

**Corollary 4.1.** The numbers 1, e and \( \sum_{n=1}^{\infty} [n^\alpha]/n! \) (\( \alpha \in \mathbb{R}_+, \alpha \notin \mathbb{Z} \)) are linearly independent over the rationals.

**Corollary 4.2.** Let \( \alpha_1, \ldots, \alpha_M \) be positive real numbers and let \( P_1, \ldots, P_M \) be nonzero polynomials with integer coefficients such that the numbers \( \alpha_m \deg P_m \) are distinct and nonintegral. Then the numbers 1, e and \( \sum_{N=1}^{\infty} [N^\alpha P_m(N)]/N! \) (\( m = 1, \ldots, M \)) are linearly independent over the rationals.

**Proof.** Apply Theorem 4.1 with \( P(x) = 1 \) and \( F_m(X) = x^{\alpha_m-1}P_m(x) \) for \( m = 1, \ldots, M \). \( \blacksquare \)

**Corollary 4.3.** The numbers 1 and \( \sum_{n=1}^{\infty} [(\log n)^\alpha]/n! \) (\( \alpha \in \mathbb{R} \)) are linearly independent over the rationals.

**Remark.** Conditions (i), (ii) and (iii) of Theorem 4.1 are satisfied by a large class of functions comprising

- \( \gamma x^\alpha \quad (\alpha > -1, \alpha \notin \mathbb{Z}, \gamma \in \mathbb{R}_+) \) with \( K = [\alpha] + 1 \),
- \( \gamma e^{\beta(\log x)^\alpha} \quad (0 < \alpha < 1, \beta \in \mathbb{R}_+, \gamma \in \mathbb{R}_+) \) with \( K = 1 \),
- \( \gamma (\log x)^\alpha \quad (\alpha \neq 0, \gamma \in \mathbb{R}_+) \) with \( K = 1 \) if \( \alpha > 0, K = 0 \) if \( \alpha < 0 \),
- \( \gamma (\log \log x)^\alpha \quad (\alpha \neq 0, \gamma \in \mathbb{R}_+) \) with \( K = 1 \) if \( \alpha > 0, K = 0 \) if \( \alpha < 0 \).

It is therefore possible to apply Theorem 4.1 to sums and products of such functions and polynomials provided that condition (iv) is satisfied.

**Example 4.1.** The numbers

\[
1, \sum_{n=1}^{\infty} \frac{[(\log n)^{1/2}]}{n!}, \sum_{n=1}^{\infty} \frac{[e(\log n)^{1/2}]}{n!}
\]

are linearly independent over the rationals.
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Received on 15.11.2004