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On the equation $n(n+d)\cdots(n+(i_0-1)d)(n+(i_0+1)d)\cdots(n+(k-1)d)=y^l$ with $0<i_0<k-1$

by

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Dedicated to the memory of Professor S. Srinivasan

1. Introduction. In 1975, Erdős and Selfridge [2] resolved an old conjecture that a product of two or more consecutive positive integers is never a perfect power. In other words, the equation

$$\Delta_0 = n(n+1)\cdots(n+k-1) = y^l \tag{1.1}$$

in positive integers n, y, $k \geq 2$, $l \geq 2$ has no solution. Erdős and Selfridge observed at the end of their paper [2, p. 300] that

$$\frac{4!}{3} = 2^3, \qquad \frac{6!}{5} = 12^2, \qquad \frac{10!}{7} = 720^2. \tag{1.2}$$

They conjectured that these are the only cases in which a product of $k-1$ distinct integers taken out of k (≥ 3) consecutive positive integers can be a perfect power. In other words, the conjecture says that the equation

$$\begin{aligned}\Delta_0(i_0) &= n(n+1)\cdots(n+i_0-1)(n+i_0+1)\cdots(n+k-1)\\ &= y^l, \quad 0 \leq i_0 < k,\end{aligned} \tag{1.3}$$

in positive integers n, y, $k \geq 3$, $l \geq 2$ has only the solutions given by (1.2). We note that $\Delta_0(i_0)$ is the product Δ_0 with one term missing. This conjecture was confirmed by the present authors in [6, Theorem 1] and [8, Theorem 1].

In [6] and [7], we considered equations analogous to (1.1) and (1.3) when the terms of the product are taken from an arithmetic progression with common difference greater than 1. For any integer $n > 1$, we write $P(n)$ for the greatest prime divisor of n and $\omega(n)$ for the number of distinct prime

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divisors of n. We put $P(1) = 1$ and $\omega(1) = 0$. We consider

$$\Delta = n(n+d)\cdots(n+(k-1)d) = by^l, \tag{1.4}$$

$$\begin{aligned}\Delta(i_0) &= n(n+d)\cdots(n+(i_0-1)d)(n+(i_0+1)d)\cdots(n+(k-1)d)\\ &= by^l, \quad 0 < i_0 < k-1,\end{aligned} \tag{1.5}$$

in positive integers b, n, $d > 1$, $k \geq 3$, y and $l \geq 2$ such that $P(b) \leq k$ and $\gcd(n, d) = 1$. These conditions on b, n, d, k, y and l will be assumed from now on. There is no loss of generality in assuming that l is prime, which we suppose throughout the paper. A well known conjecture in combinatorial diophantine analysis states that (1.4) never holds.

Let $l = 2$. Then Shorey and Tijdeman [14] proved that (1.4) implies that k is bounded by an effectively computable number depending only on $\omega(d)$. It has been proved in [7], [4], [5] and [13] that (1.5) with $b = \omega(d) = 1$ and $k \geq 6$ does not hold. Further the authors proved in [7] that (1.4) with $\omega(d) = 1$ and $k \geq 4$ is not possible.

Let $k = 3$. Then (1.4) implies that

$$n = 2y_0^2, \quad n + d = y_1^2, \quad n + 2d = 2y_2^2,$$

which gives $y_2 - y_0 = 1$, $y_2 + y_0 = d$ and hence $n = (d-1)^2/2$. Since $n + d = y_1^2$, we get $d^2 - 2y_1^2 = -1$. It is not known whether this Pell's equation has infinitely many solutions in d, y_1 with d prime. Thus the case $k = 3$ remains open.

For $l \geq 3$, we define $D_1 > 0$ as the maximal divisor of d with all prime factors of D_1 congruent to $1 \pmod{l}$ and we put

$$d = D_1 D_2.$$

The following result for $k \geq 4$ was shown by the authors in [6, Theorem 2]. The result for $k = 3$ is due to Győry [3].

Theorem A. *Suppose* (1.4) *holds with* $k \geq 4$ *or* (1.5) *holds with* $k \geq 9$. *Let* $l \geq 3$ *and* $d > 1$. *Then* $D_1 > 1$. *Further* (1.4) *with* $k = 3$ *and* $P(b) \leq 2$ *does not hold.*

Thus under the hypothesis of Theorem A, equations (1.4) and (1.5) imply that $P(d) \geq 2l + 1 \geq 7$. Thus equations (1.4) and (1.5) have no solution if $d = 2^\alpha 3^\beta 5^\gamma$ for positive integers α, β, γ. Our aim in this paper is to cover the small values $4 \leq k \leq 8$ in the above result for (1.5) when $b = 1$. Thus we prove

Theorem. *Equation* (1.5) *with* $4 \leq k \leq 8$, $l \geq 3$, $b = 1$ *and* $d > 1$ *implies that* $D_1 > 1$.

When $k = 3$, equation (1.5) with $b = 1$ becomes $n(n + 2d) = y^l$. We see that $(n, d) = (1, (y^l - 1)/2)$ with odd $y > 1$ are all solutions to (1.5) with $D_1 > 1$. Thus there are infinitely many values of d satisfying (1.5) with $D_1 > 1$.

Now we give a plan of the proof of the Theorem. We assume that (1.5) holds with $b = D_1 = 1$. For $0 \le j < k$ and $j \ne i$, we write

$$n + jd = a_j x_j^l \quad \text{where } a_j \text{ is } l\text{th power free and } P(a_j) < k. \tag{1.6}$$

The main thrust of the paper lies in analyzing the properties of a_j's. Since $k \le 8$, we see that a_j's are composed only of the primes 2, 3, 5 and 7. A careful analysis enables us to determine the divisibility of a_j's by these primes. In the majority of cases we find that one of the a_j's equals 1. In these cases we use a fundamental and elementary approach of Erdős and Selfridge (Corollary 1). When none of the a_j's equals 1, we use identities (2.9) or (2.10) to form equations of the form

$$Ax^l + By^l = Cz^l \quad \text{or} \quad Ax^l + By^l = Cz^2$$

in x, y, z with A, B, C involving only a_j's. Now we apply results on several generalized Fermat equations resulting from contributions on Fermat equations (see Lemmas 1–3) to bound $l \le 7$. We exclude these small values of l by a congruence argument and by Lemma 5. Thus the elementary method of Erdős and Selfridge combines well with contributions on Fermat equations. This feature appeared for the first time in [6, pp. 385–387] and it has been considerably developed in the present paper. For the case $l = 3$, we use an old result of Selmer [9] where equations of the form

$$x^3 + m_1 y^3 + m_2 z^3 = 0$$

for several integral values of m_1, m_2 are solved (see Lemma 4). Also in some cases, we bound x, y, z using Lemma 5 and then exclude them by computation (see Lemma 6).

We refer to [10]–[13] for information on equations (1.1), (1.3), (1.4), (1.5) and their generalizations. We thank Professors M. A. Bennett, K. Győry and L. Hajdu for sending us a copy of [1], from which Lemma 1 is taken. We also thank Professor L. Hajdu for bringing to our attention the right use of Selmer's result. Finally, we thank the referee for his useful comments.

2. Preliminaries. We shall always assume from now on that $4 \le k \le 8$, $l \ge 3$, $b = 1$ and $d > 1$. Let $2 = p_1 < p_2 < \cdots$ be the sequence of all primes. By [6, Theorem 4], we see that $\Delta(i)$ is divisible by a prime $> k$. Thus

$$n + (k-1)d \ge p^l_{\pi(k)+1}. \tag{2.1}$$

We assume from now on that (1.5) holds with $b = 1$. By (1.6), we write

$$a_j = p_1^{\alpha_{j,1}} \cdots p_{\pi(k)-1}^{\alpha_{j,\pi(k)-1}} \quad \text{with} \tag{2.2}$$
$$0 \le \alpha_{j,r} < l,\ 0 \le j < k,\ 1 \le r < \pi(k) \text{ and } j \ne i_0,$$

$$(2.3)\qquad A_j = p_1^{\beta_{j,1}} \cdots p_{\pi(k)-1}^{\beta_{j,\pi(k)-1}} \quad \text{with}$$
$$\beta_{j,r} = \mathrm{ord}_{p_r}(n+jd),\ 0 \le j < k, 1 \le r < \pi(k) \text{ and } j \ne i_0.$$

We note that $\beta_{j,r} \equiv \alpha_{j,r} \pmod{l}$ for $0 \le j < k, 1 \le r < \pi(k)$ and $j \ne i_0$. Thus $A_j = a_j t_j^l$ for some integer $t_j > 0$ with $0 \le j < k$ and $j \ne i_0$. We observe the following distribution of the powers of the primes 2, 3, 5, 7 among the a_j's. If $k = 7, 8$ and there is a j such that 2 divides only A_j, A_{j+2}, A_{j+4} and A_{j+6}, then

$$(2.4)\qquad (\alpha_{j,1}, \alpha_{j+2,1}, \alpha_{j+4,1}, \alpha_{j+6,1}) \in \begin{cases} \{(1,2,1,2),(2,1,2,1)\} & \text{if } l = 3,\\ \{(2,1,1,1),(1,2,1,1),(1,1,2,1),(1,1,1,2)\} & \text{if } l = 5,\\ \{(1,3,1,2),(1,2,1,3),(3,1,2,1),(2,1,3,1)\} & \text{if } l = 7.\end{cases}$$

If $5 \le k \le 8$ and 2 divides only A_j, A_{j+2} and A_{j+4} for some j, then

$$(2.5)\qquad (\alpha_{j,1}, \alpha_{j+2,1}, \alpha_{j+4,1}) \in \begin{cases} \{(0,1,2),(2,1,0),(1,1,1)\} & \text{if } l = 3,\\ \{(1,3,1),(2,1,2)\} & \text{if } l = 5,\\ \{1,5,1),(2,1,4),(4,1,2)\} & \text{if } l = 7.\end{cases}$$

If $k = 7, 8$ and 2 divides only A_j, A_{j+4} and A_{j+6} for some j, then

$$(2.6)\qquad (\alpha_{j,1}, \alpha_{j+4,1}, \alpha_{j+6,1}) \in \begin{cases} \{(2,0,1),(0,2,1),(1,1,1)\} & \text{if } l = 3,\\ \{(2,2,1),(1,1,3)\} & \text{if } l = 5,\\ \{(2,4,1),(4,2,1),(1,1,5)\} & \text{if } l = 7.\end{cases}$$

If $k = 7, 8$ and 3 divides only A_j, A_{j+3} and A_{j+6} for some j, then

$$(2.7)\qquad (\alpha_{j,2}, \alpha_{j+3,2}, \alpha_{j+6,2}) \in \begin{cases} \{(1,1,1)\} & \text{if } l = 3,\\ \{(3,1,1),(1,3,1),(1,1,3)\} & \text{if } l = 5,\\ \{(5,1,1),(1,5,1),(1,1,5)\} & \text{if } l = 7.\end{cases}$$

If $4 \le k \le 8$ and 3 divides only A_j and one of A_{j+3} or A_{j+6}, then

$$(2.8)\qquad (\alpha_{j,2}, \alpha_{j+3,2}) \text{ or } (\alpha_{j,2}, \alpha_{j+6,2}) \in \begin{cases} \{(2,1),(1,2)\} & \text{if } l = 3,\\ \{(4,1),(1,4)\} & \text{if } l = 5,\\ \{(6,1),(1,6)\} & \text{if } l = 7.\end{cases}$$

We define

$$S(i) = \prod_{\substack{j=0\\ j\ne i}}^{k-1} a_j$$

and let $T(i)$ be the set of primes dividing $S(i)$. We follow some notation used in [1]. We denote the identity

$$(2.9)\qquad (i_3-i_2)(n+i_1d)+(i_2-i_1)(n+i_3d) = (i_3-i_1)(n+i_2d) \text{ with } i_1 < i_2 < i_3$$

by $[i_1, i_2, i_3]$. If p, q, r, s are non-negative integers with $qr \neq ps$ and $p + s = q + r$, then we denote the identity

$$(n + qd)(n + rd) - (n + pd)(n + sd) = (qr - ps)d^2 \neq 0 \tag{2.10}$$

by $\{p, q, r, s\}$.

3. Lemmas. The first lemma is part of [1, Proposition 3.1].

LEMMA 1. *Let $l \geq 7$ be prime and A, B co-prime positive integers. Then the following equations have no solution in non-zero co-prime integers (x, y, z) with $xy \neq \pm 1$:*

(i) $Ax^l + By^l = z^2$, $P(AB) \leq 3$, $p \,|\, xy$ *for each* $p \in \{5, 7\}$.
(ii) $Ax^l + By^l = z^2$, $P(AB) \leq 5$, $7 \,|\, xy$ *and* $l \geq 11$.
(iii) $x^l + 2^\alpha y^l = 3z^2$ *with* $p \,|\, xy$ *for each* $p \in \{5, 7\}$.

The next lemma is [6, Lemma 13].

LEMMA 2. *Let $l \geq 5$. Let a, b, c be non-zero integers such that either $P(abc) \leq 3$ or a, b, c are composed of 2 and 5. Then the equation*

$$ax^l - by^l = cz^l$$

has no solution, in non-zero integers x, y, z with

$$\gcd(ax^l, by^l, cz^l) = 1, \qquad \mathrm{ord}_2(by^l) \geq 4.$$

The following result is [1, Proposition 6.1], which is based on classical arguments.

LEMMA 3. *Let C be a positive integer with $P(C) \leq 5$. If the equation*

$$x^5 + y^5 = Cz^5$$

has solutions in non-zero co-prime integers x, y, z, then $C = 2$ and $x = y = \pm 1$.

It is a well known old result that the cubic equations

$$x^3 + y^3 = z^3 \quad \text{and} \quad x^3 + y^3 = 2z^3$$

have no non-trivial solution. Selmer [9] made an extensive study of several cubic equations. Lemma 4 is a part of his work. We refer to [9, Tables 2^a and 4^a].

LEMMA 4. *Let m_1 and m_2 be positive integers such that $m_1 = m_2 = 1$ or $m_1 < m_2$. Then the equation*

$$x^3 + m_1 y^3 + m_2 z^3 = 0 \tag{3.1}$$

has no solution in non-zero integers x, y, z *with* $\gcd(x, y, z) = 1$ *whenever* (m_1, m_2) *belongs to*

$$\begin{aligned} H_1 = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,10),(1,25),(1,45),(1,100),\\ &(2,9),(3,7),(4,7),(4,9),(5,9),(5,12),(5,18),(5,21),(5,28),(5,36),\\ &(6,25),(7,9),(7,36),(9,20),(9,25),(12,25),(25,28),(25,36)\}. \end{aligned}$$

For a proof of the next lemma, see [6, Lemmas 4–6].

LEMMA 5. *Suppose* (1.5) *holds with* $b = 1$. *Let* l' *be an integer with* $1 \le l' < l$ *and*

$$\theta = \begin{cases} 1 & \text{if } l \nmid d, \\ 1/l & \text{if } l \mid d. \end{cases}$$

For any $\kappa > 0$, *define*

$$\kappa_0 = \min\left(\frac{l}{l'(\kappa+1)^{(l-l')/l}}, \frac{\kappa}{(\kappa+1)(l')^{1/l}}\right)$$

and assume that

$$D_1 \le \kappa_0 \theta \, \frac{(n + (k-1)d)^{1-l'/l}}{k}. \tag{3.2}$$

Then for no distinct l'*-tuples* $(i_1, \dots, i_{l'})$ *and* $(j_1, \dots, j_{l'})$ *with* $i_1 \le \dots \le i_{l'}$ *and* $j_1 \le \dots \le j_{l'}$, *the ratio of the two products* $a_{i_1} \cdots a_{i_{l'}}$ *and* $a_{j_1} \cdots a_{j_{l'}}$ *is an* l*th power of a rational number.*

Further let $H(d, k, p_{r_1}, \dots, p_{r_m})$ *denote the number of* a_j*'s composed of* $p_{r_1} < \dots < p_{r_m}$. *Then*

$$\binom{H(d, k, p_{r_1}, \dots, p_{r_m}) + l' - 1}{l'} \le l^m \tag{3.3}$$

where the left hand side is zero if $H(d, k, p_{r_1}, \dots, p_{r_m}) < 1$.

REMARK 1. For several values of l and l' that we come across, we can choose κ suitably so that $\kappa_0 > .7$. We give here the values of κ for the following pairs (l, l') so that $\kappa_0 > .7$:

- 5.5 if $(l, l') = (l, 1)$;
- 7.5 if $(l, l') = (3, 2)$;
- 7.25 if $(l, l') = (l, 2)$ with $l \ge 5$;
- 7 if $(l, l') = (5, 3)$ or $(l, l') = (7, 4)$;
- 15.5 if $(l, l') = (5, 4)$;
- 9 if $(l, l') = (7, 5)$.

Further for $(l, l') = (7, 6)$, we take $\kappa = 23$ to get $\kappa_0 > .73$.

REMARK 2. Suppose (1.5) holds and $b = D_1 = 1$. For $l \ge 3$ and $k \ge 4$, we see that

$$(n + (k-1)d)^{(l-1)/l} \ge (k+1)^{l-1} > 1.5kl$$

by (2.1). Thus (3.2) is satisfied with $l' = 1$ and $\kappa_0 = .7$ by Remark 1. Hence by Lemma 5, we conclude that all a_j's are distinct. This fact will be used throughout the paper.

As a consequence of Lemma 5, we get

COROLLARY 1. *Suppose* (1.5) *holds with* $b = 1$, *and one of the* a_j*'s is equal to* 1. *Assume that for some integers* r, s *with* $1 \le r \le s \le l - 1$, *there exist tuples* $(i_1, \ldots, i_r)$ *and* $(j_1, \ldots, j_s)$ *with* $i_1 \le \cdots \le i_r$ *and* $j_1 \le \cdots \le j_s$ *such that* $a_{i_1} \cdots a_{i_r} = a_{j_1} \cdots a_{j_s} t^l$ *for some rational number* t. *Then*

$$D_1 > \kappa_0 \theta (n + (k-1)d)^{1-s/l}/k \tag{3.4}$$

where κ_0 *is calculated with* $l' = s$. *In particular, if* $k \le l + 1$, *then* (3.4) *holds with* $s = k - 2$.

Proof. Let $a_{j_0} = 1$ for some j_0 with $0 \le j_0 < k$. Then we see that $a_{i_1} \cdots a_{i_r} a_{j_0} \cdots a_{j_0} = a_{j_1} \cdots a_{j_s} t^l$ where a_{j_0} occurs $s - r$ times. Hence (3.4) follows by Lemma 5 with $l' = s$. This proves the first statement. For the second, we write

$$\{0, 1, \ldots, k-1\} - \{i\} = \{h_1, \ldots, h_{k-2}\} \cup \{j_0\}.$$

Since $a_{j_0} = 1$ we find by (1.5) that $a_{h_1} \cdots a_{h_{k-2}}$ is also an lth power. Thus $a_{h_1} \cdots a_{h_{k-2}} = a_{j_0}^{k-2} t^l$ for some positive integer t. Hence (3.4) follows as above with $l' = k - 2$ since $k \le l + 1$. ∎

Whenever there exist integers r, s with the property mentioned in Corollary 1, we say that an (r, s)-*product* exists. Thus if an (r, s)-product exists, then (3.4) holds.

COROLLARY 2. *Suppose* (1.5) *holds with* $b = 1$, *and*

$$H(d, k, p_{r_1}, p_{r_2}) > [(\sqrt{(1 + 8l^2)} - 1)/2]. \tag{3.5}$$

Then

$$n + (k-1)d < \left(\frac{kD_1}{.7\theta}\right)^{l/(l-2)}.$$

Proof. By (3.5) we see that (3.3) does not hold with $l' = 2$. Thus by Lemma 5, we have

$$D_1 > \kappa_0 \theta (n + (k-1)d)^{(l-2)/l}/k.$$

By Remark 1, $\kappa_0 > .7$, which gives the assertion. ∎

LEMMA 6. *Assume that* (1.5) *holds with* $4 \le k \le 8$ *and* $b = 1$. *Let* $D_1 = 1$. *Suppose there exists* j *with* $0 \le j < k$, $j \ne i_0$, *such that either* a_j, a_{j+1}, a_{j+2} *or* a_j, a_{j+2}, a_{j+3} *or* a_j, a_{j+2}, a_{j+4} *are all composed of* 2 *and* 3.

Further assume that one of the following properties is satisfied:

(i) (3.5) *holds for some* p_{r_1} *and* p_{r_2}.
(ii) *There exist distinct tuples* (i_1, i_2) *and* (j_1, j_2) *with* $i_1 \le i_2$ *and* $j_1 \le j_2$ *such that* $a_{i_1}a_{i_2} = a_{j_1}a_{j_2}t^l$ *for some rational number* t.

Then $l \neq 3$.

Proof. Suppose (1.5) holds, $D_1 = 1$ and $l = 3$. Let a_j, a_{j+1}, a_{j+2} be composed of 2 and 3 for some j with $0 \le j < k$, $j \neq i_0$. Then

$$a_jx_j^3 + a_{j+2}x_{j+2}^3 = 2a_{j+1}x_{j+1}^3. \tag{3.6}$$

We use the facts that

$$\gcd(n + jd, n + (j + 2)d) = 1 \text{ or } 2,$$
$$\gcd(n + jd, n + (j + 1)d) = \gcd(n + (j + 1)d, n + (j + 2)d) = 1,$$

a_j's are distinct and cube free. Further if $(a_j, a_{j+2}) = (a_{j+2}, a_j)$, then the above cubic equation remains the same due to symmetry. Thus we assume $a_j < a_{j+2}$ to list the triples (a_j, a_{j+1}, a_{j+2}) as follows:

$$(a_j, a_{j+1}, a_{j+2}) \in \{(1, 2^\alpha, 3^\beta), (1, 3^\beta, 2^\alpha), (2, 1, 2^2), (2^2, 1, 2 \cdot 3^\beta),$$
$$(2, 1, 3^\beta), (2, 1, 2^2 3^\beta), (2, 1, 2 \cdot 3^\beta), (2, 3, 2^2), (2, 3^2, 2^2)\}$$

with $1 \le \alpha, \beta \le 2$. For these values, we divide the terms in (3.6) by their gcd, say g, to get equations of the form (3.1) with the three terms pairwise co-prime and (m_1, m_2) from the set

$$\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (1, 9), (1, 12), (1, 18), (1, 36),$$
$$(2, 3), (2, 9), (3, 4), (4, 9)\}.$$

Note that $g = 1, 2$. In the other two cases we form equations

$$a_jx_j^3 + 2a_{j+3}x_{j+3}^3 = 3a_{j+2}x_{j+2}^3, \quad a_jx_j^3 + a_{j+4}x_{j+1}^3 = 2a_{j+2}x_{j+2}^3 \tag{3.7}$$

and dividing out by the gcd, say g, we get cubic equations as in (3.1) with (m_1, m_2) listed above. We note that in these cases $g \in \{1, 2, 3, 6\}$ or $g \in \{1, 2, 4\}$. Further we may assume that in the cubic equations formed as in (3.1), two terms are positive and one term is negative.

On applying Lemma 4 we see that we need to consider only those (m_1, m_2) from

$$H_2 = \{(1, 6), (1, 9), (1, 12), (1, 18), (1, 36), (2, 3), (3, 4)\}.$$

For each of the above pairs, we write equation (3.1) where we observe that every term is bounded by $n + (k - 1)d$. Now we use Corollary 2 if (i) holds and Lemma 5 with $l' = 2$ if (ii) holds to get

$$\max(|x|, |y|, |z|) < 30k/7.$$

For $4 \le k \le 8, |x| < 30k/7$, $|y| < 30k/7$ with $\gcd(x, y) = 1$, we check that (3.1) is satisfied only when

$$(m_1, m_2) \in \{(1, 6), (1, 9), (2, 3), (3, 4)\}.$$

Further we see that

$$9 \max(|x|, |y|, |z|)^3 g \ge n + (j + 2)d \ge \frac{j + 2}{k - 1}(n + (k - 1)d) \ge \frac{2}{k - 1} p^3_{\pi(k)+1}.$$

Hence we find that $\max(|x|, |y|, |z|) > 1$. Thus we have

- $(m_1, m_2) = (1, 6)$, $(x, y, z) = (37, 17, -21)$;
- $(m_1, m_2) = (1, 9)$, $(x, y, z) = (17, -20, 7), (1, 2, -1)$;
- $(m_1, m_2) = (2, 3)$, $(x, y, z) = (5, -4, 1)$;
- $(m_1, m_2) = (3, 4)$, $(x, y, z) = (7, -5, 2)$.

Let $(m_1, m_2) = (1, 6), (x, y, z) = (37, 17, -21)$. By (3.1), we see that we need to consider only the first equation in (3.7) and we get $a_{j+2}x^3_{j+2} = 2g21^3$. Then $n + (j + 2)d$ is divisible by 6 and hence we get $a_{j+3} = 1, n + (j + 3)d$ odd and $2x^3_{j+3} = g17^3$ or $g37^3$. Hence $g = 2$. Since $a_j x^3_j < a_{j+3}x^3_{j+3}$, we see that $a_j x^3_j = 2 \cdot 17^3$ and $a_{j+3}x^3_{j+3} = 37^3$. Thus

$$(n + jd, n + (j + 2)d, n + (j + 3)d) = (2 \cdot 17^3, 2^2 21^3, 37^3),$$

giving $d = 13609 = 31 \cdot 439$. Thus $D_1 > 1$, a contradiction. By a similar argument we find that if $(m_1, m_2) = (1, 9)$, $(x, y, z) = (17, -20, 7)$ then we have

- $n + jd = 9 \cdot 7^3, n + (j + 1)d = 4 \cdot 10^3, n + (j + 2)d = 17^3$ with $d = 913 = 11 \cdot 83$;
- $n + jd = 2 \cdot 9 \cdot 7^3, n + (j + 2)d = 20^3, n + (j + 4)d = 2 \cdot 17^3$ with $d = 913 = 11 \cdot 83$.

Then we check that there exists a term of $\Delta(i)$ having a prime factor $> k$ which divides the term to a power which is not a multiple of 3. This contradicts (1.5). For instance, in the latter case we find that $n + (j + 1)d = 19 \cdot 373$ and $n + (j + 3)d = 3 \cdot 2971$. Since one of these terms is certainly a term of $\Delta(i)$ we get a contradiction to (1.5). We check that the case $(x, y, z) = (1, 2, -1)$ does not give rise to any possibility. Let $(m_1, m_2) = (2, 3), (x, y, z) = (5, -4, 1)$. Then we get

- $n + jd = 3$, $n + (j + 1)d = 4^3$, $n + (j + 2)d = 3 \cdot 5^3$ with $d = 61$;
- $n + jd = 6$, $n + (j + 2)d = 2 \cdot 4^3$, $n + (j + 4)d = 2 \cdot 5^3$ with $d = 61$;
- $n + jd = 18$, $n + (j + 2)d = 4 \cdot 4^3$, $n + (j + 3)d = 3 \cdot 5^3$ with $d = 119 = 7 \cdot 17$.

Hence $D_1 > 1$. Let $(m_1, m_2) = (3, 4), (x, y, z) = (7, -5, 2)$. Then we have

- $n + jd = 2^6$, $n + (j + 1)d = 3 \cdot 5^3$, $n + (j + 2)d = 2 \cdot 7^3$ with $d = 311$;
- $n + jd = 2^7$, $n + (j + 2)d = 6 \cdot 5^3$, $n + (j + 4)d = 2^2 \cdot 7^3$ with $d = 311$;
- $n + jd = 2^6$, $n + (j + 2)d = 2 \cdot 5^3$, $n + (j + 3)d = 7^3$ with $d = 93 = 3 \cdot 31$.

The last case is excluded since $D_1 > 1$. In the other two cases, as before we find a prime $> k$ dividing $\Delta(i)$ to the power not divisible by 3.

This proves the lemma. ∎

4. Listing A_j's. Fix $4 \le k \le 8$ and suppose (1.5) holds with $b = 1$. For a prime p we define

$$C_p(r) = \{A_j \mid 0 \le j < k,\ j \ne i_0,\ j \equiv r \pmod{p}\} \quad \text{for } 0 \le r < p.$$

We observe that p divides either all $A_j \in C_p(r)$ or none. Let $\{q_1, \ldots, q_h\} \subseteq \{p_1, \ldots, p_{\pi(k)}\}$ with $q_1 < \cdots < q_h$ and $0 \le r_t < q_t$, $1 \le t \le h$. We call the set $C_{q_1}(r_1) \cup \cdots \cup C_{q_h}(r_h)$ the *class* $C_{q_1,\ldots,q_h}(r_1, \ldots, r_h)$. Thus if an A_j is in this class, then $j \ne i_0$ and $j \equiv r_t \pmod{q_t}$ for some t with $1 \le t \le h$. We denote by L_{i_0} the set of classes $C_{q_1,\ldots,q_h}(r_1, \ldots, r_h)$ for all $\{q_1, \ldots, q_h\} \subseteq \{p_1, \ldots, p_{\pi(k)}\}$ and for all $0 \le r_t < q_t, 1 \le t \le h, 1 \le h \le \pi(k)$ satisfying the following conditions:

- (i) Either each A_j with $j \ne i_0$ occurs in some class $C_{q_1,\ldots,q_h}(r_1, \ldots, r_h)$, or $A_{j_0} = 1$ for some j_0 with $0 \le j_0 < k$ and each A_j with $j \ne i_0, j_0$ occurs in some class $C_{q_1,\ldots,q_h}(r_1, \ldots, r_h)$. Further every $C_{q_u}(r_u)$ with $|C_{q_u}(r_u)| = 1$ is contained in $C_{q_v}(r_v)$ for some $v \ne u, 1 \le v \le h$.
- (ii) No class $C_{q_1,\ldots,q_h}(r_1, \ldots, r_h)$ contains t (≥ 4) consecutive A_j's with their greatest prime factor $\le t$. Also no class contains three consecutive A_j's composed of only 2. By t consecutive A_j's we mean $A_{j_0}, A_{j_0+1}, \ldots, A_{j_0+t-1}$ for some j_0.

From now on we suppose that $a_1, \ldots, a_{i_0-1}, a_{i_0+1}, \ldots, a_{k-1}$ are all distinct. This implies that $A_1, \ldots, A_{i_0-1}, A_{i_0+1}, \ldots, A_{k-1}$ are all distinct. Further we see that

(4.1) at most one A_j with $0 \le j < k, j \ne i_0$ is an lth power.

Suppose $\{q_1, \ldots, q_h\}$ is the set of *all* primes dividing A_j's. We observe that this set is non-empty and q_j's are co-prime to d. For a prime q_u, the set of A_j's divisible by q_u is given by $C_{q_u}(r_u^{(0)})$ for some $0 \le r_u^{(0)} < q_u$ with $1 \le u \le h$. Thus it is clear that all A_j's greater than 1 can be put into a class $C = C_{q_1,\ldots,q_h}(r_1^{(0)}, \ldots, r_h^{(0)})$ for some $0 \le r_u^{(0)} < q_u, 1 \le u \le h$. In this class, if an A_j is omitted, then it must be 1 as it is not divisible by any of the q_u's. If one A_j is omitted in C and $|C_{q_u}(r_u^{(0)})| = 1$ for some $0 \le r_u^{(0)} < q_u$ with $1 \le u \le h$, then $C_{q_u}(r_u^{(0)})$ is contained in $C_{q_v}(r_v^{(0)})$ for some $v \ne u$ and $1 \le v \le h$ by equation (1.5) with $b = 1$ and (4.1). Suppose C contains t (≥ 4) consecutive A_j's with $P(A_j) \le t$, say $A_s, \ldots, A_{s+t-1}$. Then we observe that

$$(n + sd) \cdots (n + (s + t - 1)d) = by^l \quad \text{with } P(b) \le t.$$

Now we apply Theorem A to get $D_1 > 1$. If C contains three consecutive A_j's with $P(A_j) \le 2$, then as above we get an equation (1.4) with $P(b) \le 2$, which is impossible by Theorem A. Thus in these cases the Theorem is true and we may exclude them from our consideration. So we see that $C \in L_{i_0}$.

We illustrate the construction of L_i by an example. We take $k = 6, i_0 = 1$. We have $\{q_1, \dots, q_h\} \subseteq \{2, 3, 5\}$. It is clear that $h > 1$. Suppose $\{q_1, \dots, q_h\} = \{3, 5\}$. Then there are at least two A_j's which are equal to 1, contradicting their distinctness. Thus $\{q_1, \dots, q_h\} \ne \{3, 5\}$. By Theorem A, $\{q_1, \dots, q_h\} \ne \{2, 3\}$ or $\{2, 5\}$. Thus $h \ne 2$. Now we take $h = 3$. We check that only

$$C_{2,3,5}(0,0,0); \quad C_{2,3,5}(0,2,0) \text{ and } A_3 = 1; \quad C_{2,3,5}(1,2,0) \text{ and } A_4 = 1 \tag{4.2}$$

satisfy (i) and (ii). Thus L_1 consists of three elements given by (4.2).

Suppose (1.5) holds with $[(k-1)/2] < i_0 < k-1$. Then we set

$$b_j = a_{k-1-j} \quad \text{for } 0 \le j < k,\ j \ne k-1-i_0.$$

We write $k - 1 = t_0 + t_1 p$ with $0 \le t_0 < p$ and $t_1 \ge 0$. Let $0 \le r \le t_0$. Then we see that

$$C_p(r) = \{A_r, A_{r+p}, \dots, A_{r+t_1 p}\} - \{A_{i_0}\}.$$

We define

$$\begin{aligned} C'_p(r) &= \{B_{k-1-r}, B_{k-1-r-p}, \dots, B_{k-1-r-t_1 p}\} \\ &= \{B_{t_0-r}, B_{t_0-r+p}, \dots, B_{t_0-r+t_1 p}\}. \end{aligned}$$

We observe that $C_p(r)$ is transformed to $C'_p(r)$. Thus both $C_p(r)$ and $C'_p(t_0 - r)$ have the same set of suffixes. Let $t_0 < r < p$. Then $C_p(r) = \{A_r, A_{r+p}, \dots, A_{r+(t_1-1)p}\}$ and this is transformed to $C'_p(r) = \{B_{t_0-r+p}, B_{t_0-r+2p}, \dots, B_{t_0-r+t_1 p}\}$. Thus $C_p(r)$ and $C'_p(t_0 - r + p)$ will have the same set of suffixes. This shows that the set of $C_p(r)$ for $0 \le r < p$ is in 1-1 correspondence with the set of $C'_p(r)$ for $0 \le r < p$. Hence the list L'_{k-1-i_0} formed by the procedure above with the b_j's satisfies $L'_{k-1-i_0} = L_{i_0}$. On the other hand, we see that there is a 1-1 correspondence between the lists L'_{k-1-i_0} and L_{k-1-i_0} by replacing b with a. Further the suffix of the missing term $a_{i_0} = b_{k-1-i_0}$ is

$$k - 1 - i_0 \le k - 1 - \left[\frac{k-1}{2}\right] \le \left[\frac{k-1}{2}\right].$$

Thus while preparing the lists we may assume that

$$1 \le i_0 \le \left[\frac{k-1}{2}\right]. \tag{4.3}$$

We recall from Section 2 that for any i with $0 \le i < k$, $T(i)$ denotes the set of primes dividing the product $S(i)$ of all a_j's with $j \ne i$. We now use (4.3) to find $T(i)$. Thus if $k = 4$, then $i_0 = 1$ and $T(1) \in \{\{2\}, \{3\}, \{2, 3\}\}$.

If $k = 5$, then $i_0 \leq 2$ and $T(1) \in \{\{2\}, \{2,3\}\}$ and $T(2) = \{2,3\}$. If $k = 6$, then $i_0 \leq 2$ and $|T(i_0)| \geq 2$. If $k = 7$, then $i_0 \leq 3$ and $|T(i_0)| \geq 2$. If $k = 8$, then $i_0 \leq 3$ with $|T(i_0)| \geq 3$, by Theorem A.

We use these facts while preparing the list L_{i_0}. We present the list L_{i_0} with i_0 satisfying (4.3) and $4 \leq k \leq 8$ in Tables 1–5.

The tables should be read as follows. Let $k = 6$, $i_0 = 1$. We have three elements of L_{i_0} given by (4.2). Consider the second element in (4.2), viz. $C_{2,3,5}(0,2,0)$ and $A_3 = 1$. This is the possibility of 2 dividing A_0, A_2, A_4, 3 dividing A_2, A_5, 5 dividing A_0, A_5 and $A_3 = 1$. This is tabulated under the columns of primes 2, 3 and 5 in Table 2. Further $A_3 = 1$ is given in the last column of Table 2. For convenience, we write this element as $2 : A_0, A_2, A_4$; $3 : A_2, A_5$; $5 : A_0, A_5$; $A_3 = 1$. We will also be using this notation for all other cases. If in some case a prime does not divide any of the A_j's we put $-$ in the column under this prime. If no A_j equals 1, we put $-$ in the last columns in Tables 1–4. We refer to "Assertions on the tables" for $*$ and $**$ appearing in the last column of the tables, and to Section 6 for an explanation of the last two columns in Table 5.

Table 1

$-$	$-$	$k=4$	$-$	$-$	$-$	$k=5$	$-$
i_0	2	3	$-$	i_0	2	3	$-$
1	A_0, A_2	$-$	$A_3 = 1$ $**$	1	A_0, A_2, A_4	A_0, A_3	$-$
1	$-$	A_0, A_3	$A_2 = 1$ $**$	2	A_0, A_4	A_0, A_3	$A_1 = 1$
1	A_0, A_2	A_0, A_3	$-$ $**$	2	A_0, A_4	A_1, A_4	$A_3 = 1$
$-$	$-$	$-$		2	A_1, A_3	A_0, A_3	$A_4 = 1$
$-$	$-$	$-$		2	A_1, A_3	A_1, A_4	$A_0 = 1$

Table 2. $k = 6$

i_0	2	3	5	$-$
1	A_0, A_2, A_4	A_2, A_5	A_0, A_5	$A_3 = 1$ $*$
1	A_3, A_5	A_2, A_5	A_0, A_5	$A_4 = 1$ $*$
1	A_0, A_2, A_4	A_0, A_3	A_0, A_5	$-$ $**$
2	A_0, A_4	A_0, A_3	A_0, A_5	$A_1 = 1$ $*$
2	A_1, A_3, A_5	A_0, A_3	A_0, A_5	$A_4 = 1$ $*$
2	A_0, A_4	A_1, A_4	A_0, A_5	$A_3 = 1$ $*$
2	A_1, A_3, A_5	$-$	A_0, A_5	$A_4 = 1$
2	$-$	A_1, A_4	A_0, A_5	$A_3 = 1$
2	A_1, A_3, A_5	A_1, A_4	$-$	$A_0 = 1$
2	A_1, A_3, A_5	A_0, A_3	$-$	$A_4 = 1$
2	A_1, A_3, A_5	A_1, A_4	A_0, A_5	$-$ $**$

Table 3. $k = 7$

No.	i_0	2	3	5	–
1	2	A_0, A_4, A_6	A_1, A_4	A_0, A_5	$A_3 = 1$
2	2	A_0, A_4, A_6	A_0, A_3, A_6	A_0, A_5	$A_1 = 1$
3	2	A_0, A_4, A_6	A_0, A_3, A_6	A_1, A_6	$A_5 = 1$
4	2	A_1, A_3, A_5	A_1, A_4	A_0, A_5	$A_6 = 1$
5	2	A_1, A_3, A_5	A_0, A_3, A_6	A_0, A_5	$A_4 = 1$
6	2	A_1, A_3, A_5	A_0, A_3, A_6	A_1, A_6	$A_4 = 1$
7	2	A_1, A_3, A_5	A_1, A_4	A_1, A_6	$A_0 = 1$
8	3	A_0, A_2, A_4, A_6	A_2, A_5	A_0, A_5	$A_1 = 1$
9	3	A_0, A_2, A_4, A_6	A_1, A_4	–	$A_5 = 1$
10	3	A_0, A_2, A_4, A_6	A_2, A_5	–	$A_1 = 1$
11	3	A_0, A_2, A_4, A_6	–	A_0, A_5	$A_1 = 1$
12	3	A_0, A_2, A_4, A_6	–	A_1, A_6	$A_5 = 1$
13	3	A_0, A_2, A_4, A_6	A_0, A_6	A_0, A_5	$A_1 = 1$
14	3	A_0, A_2, A_4, A_6	A_0, A_6	A_1, A_6	$A_5 = 1$
15	3	A_0, A_2, A_4, A_6	A_1, A_4	A_1, A_6	$A_5 = 1$
16	3	A_0, A_2, A_4, A_6	A_1, A_4	A_0, A_5	–
17	3	A_0, A_2, A_4, A_6	A_2, A_5	A_1, A_6	–

Table 4. $k = 8$

No.	i_0	2	3	5	7	–
1	2	A_1, A_3, A_5, A_7	–	A_1, A_6	A_0, A_7	$A_4 = 1$
2	2	A_0, A_4, A_6	A_0, A_3, A_6	A_0, A_5	A_0, A_7	$A_1 = 1$
3	2	A_0, A_4, A_6	A_0, A_3, A_6	A_1, A_6	A_0, A_7	$A_5 = 1$ *
4	2	A_0, A_4, A_6	A_1, A_4, A_7	A_0, A_5	A_0, A_7	$A_3 = 1$
5	2	A_1, A_3, A_5, A_7	A_0, A_3, A_6	A_0, A_5	A_0, A_7	$A_4 = 1$
6	2	A_1, A_3, A_5, A_7	A_1, A_4, A_7	A_0, A_5	A_0, A_7	$A_6 = 1$
7	2	A_1, A_3, A_5, A_7	A_0, A_3, A_6	A_1, A_6	A_0, A_7	$A_4 = 1$ **
8	3	A_1, A_5, A_7	A_1, A_4, A_7	A_1, A_6	A_0, A_7	$A_2 = 1$ *
9	3	A_1, A_5, A_7	A_1, A_4, A_7	A_2, A_7	A_0, A_7	$A_6 = 1$
10	3	A_0, A_2, A_4, A_6	A_1, A_4, A_7	A_1, A_6	A_0, A_7	$A_5 = 1$ **
11	3	A_0, A_2, A_4, A_6	A_1, A_4, A_7	A_2, A_7	A_0, A_7	$A_5 = 1$
12	3	A_1, A_5, A_7	A_2, A_5	A_1, A_6	A_0, A_7	$A_4 = 1$ *
13	3	A_1, A_5, A_7	A_0, A_6	A_2, A_7	–	$A_4 = 1$
14	3	A_0, A_2, A_4, A_6	A_2, A_5	A_1, A_6	–	$A_7 = 1$
15	3	A_0, A_2, A_4, A_6	A_2, A_5	A_2, A_7	–	$A_1 = 1$
16	3	A_0, A_2, A_4, A_6	A_2, A_5	–	A_0, A_7	$A_1 = 1$
17	3	A_0, A_2, A_4, A_6	A_1, A_4, A_7	–	A_0, A_7	$A_5 = 1$

Table 4 (cont.). $k = 8$

No.	i_0	2	3	5	7	–
18	3	A_0, A_2, A_4, A_6	–	A_0, A_5	A_0, A_7	$A_1 = 1$
19	3	A_0, A_2, A_4, A_6	–	A_1, A_6	A_0, A_7	$A_5 = 1$
20	3	A_1, A_5, A_7	A_0, A_6	A_2, A_7	A_0, A_7	$A_4 = 1$
21	3	A_0, A_2, A_4, A_6	A_0, A_6	A_0, A_5	A_0, A_7	$A_1 = 1$
22	3	A_0, A_2, A_4, A_6	A_0, A_6	A_1, A_6	A_0, A_7	$A_5 = 1$
23	3	A_0, A_2, A_4, A_6	A_2, A_5	A_0, A_5	A_0, A_7	$A_1 = 1$
24	3	A_0, A_2, A_4, A_6	A_2, A_5	A_2, A_7	A_0, A_7	$A_1 = 1$
25	3	A_0, A_2, A_4, A_6	A_1, A_4, A_7	A_1, A_6	–	$A_5 = 1$
26	3	A_0, A_2, A_4, A_6	A_1, A_4, A_7	A_2, A_7	–	$A_5 = 1$
27	3	–	A_2, A_5	A_1, A_6	A_0, A_7	$A_4 = 1$

Table 5. $k = 8$

No.	i_0	2	3	5	7	$\{p, q, r, s\}$	–
1	1	A_0, A_2, A_4, A_6	A_0, A_3, A_6	A_0, A_5	A_0, A_7	$\{0, 2, 5, 7\}$	(ii)
2	2	A_1, A_3, A_5, A_7	A_1, A_4, A_7	A_1, A_6	A_0, A_7	$\{0, 1, 6, 7\}$	(i) $**$
3	3	A_0, A_2, A_4, A_6	A_1, A_4, A_7	A_0, A_5	A_0, A_7	$\{0, 1, 6, 7\}$	(ii)
4	3	A_0, A_2, A_4, A_6	A_1, A_4, A_7	A_0, A_5	–	$\{0, 1, 4, 5\}$	(i)
5	3	A_0, A_2, A_4, A_6	A_2, A_5	A_1, A_6	A_0, A_7	$\{0, 1, 6, 7\}$	(iii) $**$

Assertions on the tables. (i) The combinations marked $*$ and $**$ in Tables 1–5 are the only cases with $H(d, k, p_{r_1}, p_{r_2}) \le 3$ for every $(p_{r_1}, p_{r_2}) \in \{(2,3), (2,5), (3,5)\}$ while for all other combinations we have $H(d, k, p_{r_1}, p_{r_2}) > 3$ with $(p_{r_1}, p_{r_2}) = (2,3)$ or $(2,5)$ or $(3,5)$.

(ii) Let $l = 3$. For the combinations marked $**$ in Tables 1–5 we check using (2.4)–(2.8) that property (ii) of Lemma 6 holds. For instance, take the combination

$$\{2 : A_0, A_2, A_4, A_6;\ 3 : A_2, A_5;\ 5 : A_1, A_6;\ 7 : A_0, A_7\}$$

from Table 5. Then $a_1 = 5^{\alpha_{1,3}}$, $a_2 = 2^{\alpha_{2,1}}3^{\alpha_{2,2}}$, $a_5 = 3^{\alpha_{5,2}}$, $a_6 = 2^{\alpha_{6,1}}5^{\alpha_{6,3}}$. We use (2.4) with $l = 3$ to get $\alpha_{2,1} = \alpha_{6,1}$, $\alpha_{2,2} + \alpha_{5,2} = \alpha_{1,3} + \alpha_{6,3} = 3$. This gives $a_2 a_5 = a_1 a_6 t^l$.

(iii) One can check easily that for all the combinations listed in Tables 1–5, there exists j with $0 \le j < k$ such that either a_j, a_{j+1}, a_{j+2} or a_j, a_{j+2}, a_{j+3} or a_j, a_{j+2}, a_{j+4} are all composed of 2 and 3. Here no suffix of a's equals i_0.

LEMMA 7. *Suppose* (1.5) *holds with* $4 \le k \le 8$, *and* $b = D_1 = 1$. *Then* $l \ne 3$.

Proof. Suppose $l = 3$. From Tables 1–5, Assertions (i)–(iii) and Lemma 6, we find that we need to consider only the combinations marked $*$. We proceed as follows. We consider the combination

$$2 : A_0, A_2, A_4;\ 3 : A_2, A_5;\ 5 : A_0, A_5;\ A_3 = 1$$

in Table 2. Then $a_3 = 1$. Since a_4 is divisible only by 2, we have $\alpha_{4,1} \neq 0$. Also $(\alpha_{0,1}, \alpha_{2,1}, \alpha_{4,1}) \in \{(1,1,1),(0,1,2)\}, (\alpha_{2,2}, \alpha_{2,5}) \in \{(1,2),(2,1)\}$. First we consider $(\alpha_{0,1}, \alpha_{2,1}, \alpha_{4,1}) = (1,1,1)$. We use (2.9) with $[i_1, i_2, i_3] = [2,3,4]$. This gives a cubic equation (3.1) with $(m_1, m_2) = (1,3)$ or $(1,9)$. The case $(1,3)$ is not possible by Lemma 4. The case $(1,9)$ occurs when $\alpha_{2,2} = 2$. Then we consider (2.9) with $[0,2,3]$ to get (3.1) with $(m_1, m_2) = (1,5)$ or $(1,25)$ both of which are excluded by Lemma 4. Below we depict this sequence pictorially:

$$[2,3,4] \to (1,3) \text{ or } \{(1,9) \to [0,2,3] \to (1,5) \text{ or } (1,25)\}.$$

Let $(\alpha_{0,1}, \alpha_{2,1}, \alpha_{4,1}) = (0,1,2)$. Then $[i_1, i_2, i_3] = [2,3,4]$ gives the equation (3.1) with $(m_1, m_2) = (2,9)$ or $(2,3)$. By Lemma 4, $(2,9)$ is excluded. When $(2,3)$ occurs, we have $\alpha_{2,2} = 1$. In this case we continue with $[0,2,3]$ which gives (3.1) with $(m_1, m_2) = (9,20)$ or $(9,100)$. The former is excluded by Lemma 4. In the latter case, we take $[3,4,5]$, which gives (3.1) with $(m_1, m_2) = (1,45)$, which is not possible. We depict this sequence pictorially as

$$[2,3,4] \to (2,9) \text{ or}$$
$$\{(2,3) \to [0,2,3] \to (9,20) \text{ or } \{(9,100) \to [3,4,5] \to (1,45)\}\}.$$

We give such sequences for all other combinations marked $*$. Also we take from (2.4)–(2.8) only the right choices for α's.

$$\underline{2 : A_3, A_5;\ 3 : A_2, A_5;\ 5 : A_0, A_5;\ A_4 = 1}$$
$$[2,3,4] \to (1,3) \text{ or } (4,9) \text{ or } \{(1,9) \to [0,2,3] \to (1,5) \text{ or } (1,25)\} \text{ or}$$
$$\{(3,4) \to [0,3,5] \to (1,10) \text{ or } \{(2,5) \to [0,4,5] \to (5,18)\}\}.$$

$$\underline{2 : A_0, A_4;\ 3 : A_0, A_3;\ 5 : A_0, A_5;\ A_1 = 1}$$
$$[1,3,4] \to (1,1) \text{ or } (1,4) \text{ or } (4,9) \text{ or}$$
$$\{(1,9) \to [1,3,5] \to (6,25) \text{ or } \{(5,6) \to [0,1,4] \to (1,100)\}\}.$$

$$\underline{2 : A_1, A_3, A_5;\ 3 : A_0, A_3;\ 5 : A_0, A_5;\ A_4 = 1}$$

If $(\alpha_{1,1}, \alpha_{3,1}, \alpha_{5,1}) = (1,1,1)$, then

$$[1,3,4] \to (1,1) \text{ or } \{(1,9) \to [0,1,4] \to (1,5) \text{ or } (1,25)\}.$$

If $(\alpha_{1,1}, \alpha_{3,1}, \alpha_{5,1}) = (2,1,0)$, then $[1,3,4] \to (1,2)$ or $(2,9)$.

$$\underline{2 : A_0, A_4;\ 3 : A_1, A_4;\ 5 : A_0, A_5;\ A_3 = 1}$$
$$[1,3,4] \to (1,3) \text{ or } \{(1,12) \to [0,1,3] \to (5,9) \text{ or } (9,25)\} \text{ or}$$
$$\{(3,4) \to [0,1,3] \to (1,5) \text{ or } (1,25)\}.$$

$$\underline{2 : A_0, A_4, A_6;\ 3 : A_0, A_3, A_6;\ 5 : A_1, A_6;\ 7 : A_0, A_7;\ A_5 = 1}$$

If $(\alpha_{0,1}, \alpha_{4,1}, \alpha_{6,1}) = (0, 2, 1)$, then $[3, 4, 5] \to (1, 3)$. If $(\alpha_{0,1}, \alpha_{4,1}, \alpha_{6,1}) = (1, 1, 1)$, then

$$[0, 1, 3] \to (5, 28) \text{ or } (25, 28) \text{ or } \{(5, 196) \to [3, 4, 7] \to (7, 9)\} \text{ or } \{(25, 196) \to [3, 4, 7] \to (7, 9)\}.$$

$$\underline{2 : A_1, A_5, A_7;\ 3 : A_1, A_4, A_7;\ 5 : A_1, A_6;\ 7 : A_0, A_7;\ A_2 = 1}$$

If $(\alpha_{1,1}, \alpha_{5,1}, \alpha_{7,1}) = (0, 2, 1)$, then

$$[1, 2, 4] \to (1, 10) \text{ or } \{(1, 50) \to [0, 2, 5] \to (5, 21) \text{ or } (5, 147) \to [1, 2, 7] \to (4, 7)\}.$$

If $(\alpha_{1,1}, \alpha_{5,1}, \alpha_{7,1}) = (1, 1, 1)$, then $[2, 4, 5] \to (4, 9)$.

$$\underline{2 : A_1, A_5, A_7;\ 3 : A_2, A_5;\ 5 : A_1, A_6;\ 7 : A_0, A_7;\ A_4 = 1}$$

If $(\alpha_{1,1}, \alpha_{5,1}, \alpha_{7,1}) = (0, 2, 1)$, then $[2, 4, 5] \to (1, 3)$. If $(\alpha_{1,1}, \alpha_{5,1}, \alpha_{7,1}) = (2, 0, 1)$, then

$$[4, 5, 6] \to (6, 25) \text{ or } (5, 18) \text{ or } \{(5, 6) \to [0, 4, 7] \to (3, 7) \text{ or } \{(1, 21) \to [4, 5, 7] \to (7, 36)\}\} \text{ or } \{(18, 25) \to [0, 4, 7] \to (3, 7) \text{ or } \{(1, 21) \to [4, 5, 7] \to (4, 7)\}\}.$$

If $(\alpha_{1,1}, \alpha_{5,1}, \alpha_{7,1}) = (1, 1, 1)$, then

$$[4, 5, 6] \to (5, 12) \text{ or } (12, 25) \text{ or } (5, 36) \text{ or } (25, 36). \ \blacksquare$$

5. Proof of the Theorem when one A_j equals 1. We suppose throughout this section that (1.5) holds and $b = D_1 = 1$. By Lemma 7, we have $l \geq 5$. Further we suppose that one of the A_j's is equal to 1. We know that all a_j's are distinct by Remark 2. First we show that

$$k \geq 6,\ l = 5 \text{ if } k = 6, 7;\ l = 5, 7 \text{ if } k = 8;\ l = 5 \text{ if } k = 8 \text{ and } 7 \nmid d. \tag{5.1}$$

Let $k = 4$. Then $k \leq l + 1$. Hence (3.4) is valid with $s = k - 2 = 2$. Thus using (2.1) we get

$$4 > .7\theta(n + (k-1)d)^{1-2/l} \geq .7\theta \cdot 5^{l-2}.$$

This is not possible. Similarly $k = 5$ is also excluded. Next we consider $k = 8$, $7 \nmid d$. Then $\theta = 1$. Suppose $l \geq 7$. By (2.1), Remark 1 and Corollary 1, we have $s = k - 2 = 6$ and

$$8 > .7\theta \cdot 11^{l-6} \quad \text{for } l \geq 11; \qquad 8 > .73 \cdot 11 \quad \text{for } l = 7.$$

This is not possible. Thus $l = 5$. The assertion follows similarly in the other cases.

Let $k = 6$ and $l = 5$. First we consider the two cases in Table 2 where 5 does not divide any A_i. We give the details for the case

$$2 : A_1, A_3, A_5;\ 3 : A_1, A_4;\ A_0 = 1.$$

By (2.5), we see that (a_3, a_5) takes the values from $\{(2^3, 2), (2, 2^2)\}$. Hence $a_3 = a_5^3$ or $a_5 = a_3^2$. Thus the assumptions of Corollary 1 are satisfied with $r = 1$ and $s = 3$ or 2. Hence by (3.4), we get

$$k = 6 > \kappa_0\theta \cdot 7^{5-s}$$

with $s = 2, 3$. This is not possible. In the other case we have $a_1 = a_5$, which contradicts the distinctness of a_j's. Next we take the remaining cases in Table 2 where 5 divides A_0, A_5. Hence $\theta = 1$. Further since $k \le l + 1$, (3.4) is valid with $s = k - 2 = 4$. Let us consider the case

$$2 : A_0, A_2, A_4;\ 3 : A_2, A_5;\ 5 : A_0, A_5;\ A_3 = 1.$$

Suppose $P(\Delta(i)) = 7$. Then we find that $7 \,|\, (n + 3d)$ since otherwise $n + 3d = 1$ as $n + 3d$ is not divisible by 2, 3 or 5. Further

$$(n, n+2d, n+3d, n+4d, n+5d) = (2^{\beta_{0,1}}5^{\beta_{0,3}}, 2^{\beta_{2,1}}3^{\beta_{2,2}}, 7^{\beta_{3,4}}, 2^{\beta_{4,1}}, 3^{\beta_{5,2}}5^{\beta_{5,3}}).$$

We find that $16 \le n + 4d = 2^{\beta_{4,1}}$, giving $\beta_{0,1} = \alpha_{0,1} = 2$, $\beta_{2,1} = \alpha_{2,1} = 1$ and hence $\alpha_{4,1} = 2$. Since $n + 2d = 2 \cdot 3^{\beta_{2,2}} > 6$, we get $\beta_{2,2} \ge 2$, giving $\beta_{5,2} = \alpha_{5,2} = 1$ and hence $\alpha_{2,2} = 4$. Thus

$$(a_0, a_2, a_3, a_4, a_5) \in \{(2^2 \cdot 5, 2 \cdot 3^4, 1, 2^2, 3 \cdot 5^4), (2^2 \cdot 5^4, 2 \cdot 3^4, 1, 2^2, 3 \cdot 5)\}.$$

We use (2.9) with $[2, 3, 4]$ to obtain

$$3^4x_2^5 + 2x_4^5 = x_3^5.$$

Now we observe that $x^5 \equiv 0, \pm 1 \pmod{11}$ and 11 divides at most one of x_2, x_3, x_4. Hence this equation is impossible by congruence mod 11. Thus we have $P(\Delta(i)) \ge 11$. Then we find that (3.4) does not hold. This is a contradiction. All the cases in Table 2 are excluded similarly.

Let $k = 7$ and $l = 5$. We need to consider all possibilities in Table 3 except the 16th and 17th cases. First we consider all the cases from 7 to 15. Then we find that there exist at least two a_j's which are powers of 2 only. We take one case for illustration, say the 8th:

$$2 : A_0, A_2, A_4, A_6;\ 3 : A_2, A_5;\ 5 : A_0, A_5;\ A_1 = 1.$$

Then $(a_4, a_6) \in \{(2, 2^2), (2^2, 2)\}$ by (2.4). Thus either $a_6 = a_4^2$ or $a_4 = a_6^2$. Hence (3.4) is valid with $s = 2$, which is not possible since

$$7 < .7\theta \cdot 11^{5-s} \quad \text{for} \quad s \le 3.$$

The other cases are excluded similarly. Next we consider cases 1–6 in Table 3. Then we have $5 \nmid d$. Hence $\theta = 1$. We find that in these cases the following equalities hold:

$$a_0a_1a_4a_5 = a_6^4t^l; \quad a_0a_3a_5a_6 = a_4^4t^l; \quad a_0a_1a_3a_6 = a_4^4t^l;$$
$$a_0a_1a_4a_5 = a_3^4t^l; \quad a_0a_3a_5a_6 = a_1^4t^l; \quad a_0a_1a_3a_6 = a_5^4t^l,$$

respectively, which satisfy the assumption of Corollary 1. But (3.4) is not satisfied with $s = 4$, a contradiction.

Let $k = 8$ *with* $l = 5, 7$. In cases 13 to 26 of Table 4, we find that there exists an (r, s)-product with $s \leq 3$ if $l = 5$ and $s \leq 4$ if $l = 7$ since there exist at least two a_j's which are powers of 2 only. This is also true for the 27th case, since then there exist at least two a_j's which are powers of 3 only. On the other hand, we see that

$$8 < .7\theta \cdot 11^{l-s} \quad \text{for } l = 5, s \leq 3 \text{ and } l = 7,\ s \leq 4.$$

This contradicts (3.4). Now we consider the combinations numbered 1 to 12 in Table 4. By (5.1), we see that $l = 5$ for all these cases, since $7 \nmid d$. We consider the 10th case in Table 4,

$$2 : A_0, A_2, A_4, A_6;\ 3 : A_1, A_4, A_7;\ 5 : A_1, A_6;\ 7 : A_0, A_7;\ A_5 = 1.$$

First we use (2.9) with $[i_1, i_2, i_3] = [2, 4, 5]$ to get

$$2^{\alpha_{2,1}-1}x_2^5 + x_5^5 = 3^{\alpha_{4,2}+1}2^{\alpha_{4,1}-1}x_4^5.$$

By (2.4) and (2.7), we see that $\alpha_{2,1}, \alpha_{4,1} \in \{1, 2\}$ and $\alpha_{4,2} \in \{1, 3\}$. Suppose $\alpha_{2,1} = 1$. Then we get an equation as in Lemma 3 with $C = 3^{\alpha_{4,2}+1}$, $2^{\alpha_{4,1}-1} \neq 2$, which is a contradiction. Thus $\alpha_{2,1} = 2$. Then by (2.4), $\alpha_{0,1} = \alpha_{4,1} = \alpha_{6,1} = 1$. Now we apply (2.9) with $[4, 5, 6]$ to get

$$3^{\alpha_{4,2}}x_4^5 + 5^{\alpha_{6,3}}x_6^5 = x_5^5.$$

Using congruence mod 11, we see that $(\alpha_{4,2}, \alpha_{6,3}) \in \{(1, 4), (3, 1)\}$. Thus

$(a_0, a_1, a_2, a_4, a_5, a_6, a_7) \in \{(2 \cdot 7^{\alpha_{0,4}}, 3 \cdot 5, 2^2, 2 \cdot 3, 1, 2 \cdot 5^4, 3^3 \cdot 7^{\alpha_{7,4}}),$
$(2\cdot 7^{\alpha_{0,4}}, 3^3\cdot 5, 2^2, 2\cdot 3, 1, 2\cdot 5^4, 3\cdot 7^{\alpha_{7,4}}), (2\cdot 7^{\alpha_{0,4}}, 3\cdot 5^4, 2^2, 2\cdot 3^3, 1, 2\cdot 5, 3\cdot 7^{\alpha_{7,4}})\}.$

In these cases we find that

$$a_1a_6 = a_45^l; \quad a_0a_7 = a_47^l; \quad a_0a_7 = a_1a_6(7/5)^l,$$

respectively. This contradicts Corollary 1 as earlier. The other cases are excluded similarly. ∎

6. Proof of the Theorem when no A_j equals 1. We suppose throughout this section that (1.5) holds and $b = D_1 = 1$ and none of the A_j's is 1. We know that all a_j's are distinct by Remark 2. First we use Lemmas 1 and 2 to bound l. Then for the small values of l we use the same strategy as in Section 5. Further by Lemma 7, we have $l \neq 3$.

Let $k = 4$. Then from Table 1 we have

$$2 : A_0, A_2;\ 3 : A_0, A_3.$$

We use (2.9) with $[0, 2, 3]$ to get an equation as in Lemma 2 with $\mathrm{ord}_2(By^l) \geq l - 2$. Thus by Lemma 2, we conclude that $l = 5$. Then we get

$$x^5 + y^5 = 2^3 3^3 z^5 \quad \text{or} \quad x^5 + 2^3 y^5 = 3^3 z^5.$$

The first equation has no solution by Lemma 3. The second equation is impossible by using congruence mod 11.

Let $k = 5$. Then from Table 1 we have

$$2 : A_0, A_2, A_4;\ 3 : A_0, A_3.$$

We apply (2.9) with $[0, 2, 4]$ to get an equation as in Lemma 2 with $\mathrm{ord}_2(by^l) \geq l-5$. Hence by Lemma 2, we get $l \leq 7$. We observe that (3.2) with $D_1 = 1$ is satisfied for $l = 5$ only when $l' \leq 3$, and for $l = 7$ only when $l' \leq 4$. On the other hand, by (2.5), we get

$$a_0a_3 = a_2^2t^l \text{ if } l = 5;\ a_0a_3a_2^2 = a_4^4t^l \text{ or } a_0a_3 = a_2^2t^l \text{ or } a_0a_3 = a_4^2t^l \text{ if } \ l = 7.$$

This contradicts Lemma 5.

Let $k = 6$. We have

$$\left\{\begin{array}{l} 2 : A_0, A_2, A_4;\ 3 : A_0, A_3;\ 5 : A_0, A_5, \\ 2 : A_1, A_3, A_5;\ 3 : A_1, A_4;\ 5 : A_0, A_5. \end{array}\right. \tag{6.1}$$

For the first case we apply (2.10) with $\{0, 2, 3, 5\}$ to get an equation of the form (i) of Lemma 1 and hence we have $l = 5$. In the second case we first apply (2.10) with $\{0, 1, 4, 5\}$ to conclude that $\alpha_{1,1} = \alpha_{5,1} = 1, \alpha_{3,1} = l - 2$. Then we apply Lemma 2 to conclude that $l = 5$. Since 5 divides A_0, A_5, we have $5 \nmid d$ and hence $\theta = 1$. Suppose $P(\Delta(i)) = 7$. Let us consider

$$2 : A_0, A_2, A_4;\ 3 : A_0, A_3;\ 5 : A_0, A_5.$$

Since not both $n+2d$ and $n+4d$ can be high powers of 2 we see that 7 divides either $n+2d$ or $n+4d$. Then $n+3d = 3^{\beta_{3,2}} > 3$ implies that $\alpha_{3,2} = 4$. Similarly $n + 5d = 5^{\beta_{5,3}} > 5$ gives $\alpha_{5,3} = 4$. Suppose $7 \mid (n + 2d)$. Then $n + 4d = 2^{\beta_{4,1}}$, implying that $\alpha_{4,1} = 2$, by (2.5). Thus $(a_3, a_4, a_5) = (3^4, 2^2, 5^4)$. We use (2.9) with $[3, 4, 5]$ and a congruence argument mod 11 to exclude this possibility. If $7 \mid (n + 4d)$, then $n + 2d = 2^{\beta_{2,1}}$ implies that $a_2 = 2$ or 2^3, by (2.5). Thus

$$(a_2, a_3, a_5) \in \{(2, 3^4, 5^4), (2^3, 3^4, 5^4)\}.$$

We use (2.9) with $[2, 3, 5]$ and a congruence argument mod 11 to exclude these possibilities. Thus we have $P(\Delta(i)) \geq 11$. Then (3.2) is valid with $l' = 4$. We use (2.5) to see that $a_0a_2a_3a_5 = a_4^4t^l$, a contradiction to Lemma 5. The other case in (6.1) is excluded similarly.

Let $k = 7$. In Table 3, we take the last two possibilities where no A_j equals 1. For these cases we apply (2.10) with $\{0, 1, 4, 5\}, \{1, 2, 5, 6\}$, respectively, to get an equation of the form (i) of Lemma 1. Hence we conclude that $l = 5$. Then $\theta = 1$. Hence (3.2) is satisfied with $l' \le 4$. We find that in these two cases

$$a_0 a_5 = a_1 a_4 t^l \quad \text{and} \quad a_1 a_6 = a_2 a_5 t^l,$$

respectively, which contradicts Lemma 5 when $l = 5$.

Let $k = 8$. We give in Table 5 the choice of $\{p, q, r, s\}$ in (2.10) and the equation we get in Lemma 1 to conclude that $l \le 7$ in cases 1, 3 and $l = 5$ in cases 2, 4, 5. We consider the first three cases in Table 5. We show that $P(\Delta(i)) \ge 13$ arguing as in the case $k = 6$. Thus (3.2) is valid for all $l' \le 4$ if $l = 5$ and $l' = 6$ if $l = 7$.

We give the details for excluding the first case in Table 5. The other cases follow similarly. Let $l = 5$. We have

$$\begin{aligned}(a_0, a_2, a_3, a_4, a_5, a_6, a_7) \in \{&(2\cdot 3^3 5^{\alpha_{0,3}} 7^{\alpha_{0,4}}, 2^2, 3, 2, 5^{\alpha_{5,3}}, 2\cdot 3, 7^{\alpha_{7,4}}),\\ &(2\cdot 3^3 5^{\alpha_{0,3}} 7^{\alpha_{0,4}}, 2, 3, 2^2, 5^{\alpha_{5,3}}, 2\cdot 3, 7^{\alpha_{7,4}}), (2\cdot 3\cdot 5^{\alpha_{0,3}} 7^{\alpha_{0,4}}, 2^2, 3^3, 2, 5^{\alpha_{5,3}}, 2\cdot 3, 7^{\alpha_{7,4}}),\\ &(2\cdot 3\cdot 5^{\alpha_{0,3}} 7^{\alpha_{0,4}}, 2, 3^3, 2^2, 5^{\alpha_{5,3}}, 2\cdot 3, 7^{\alpha_{7,4}}), (2\cdot 3\cdot 5^{\alpha_{0,3}} 7^{\alpha_{0,4}}, 2^2, 3, 2, 5^{\alpha_{5,3}}, 2\cdot 3^3, 7^{\alpha_{7,4}}),\\ &(2\cdot 3\cdot 5^{\alpha_{0,3}} 7^{\alpha_{0,4}}, 2, 3, 2^2, 5^{\alpha_{5,3}}, 2\cdot 3^3, 7^{\alpha_{7,4}})\}.\end{aligned}$$

Then we find that

$$\begin{aligned}a_0a_4a_5a_7 &= a_2a_3^3t^l; \quad a_0a_2a_5a_7 = a_3^3a_4t^l; \quad a_0a_5a_7 = a_3^2a_4t^l;\\ a_0a_5a_7 &= a_2a_3^2t^l; \quad a_0a_2a_5a_7 = a_3a_4^3t^l; \quad a_0a_4a_5a_7 = a_2^3a_3t^l,\end{aligned}$$

respectively. This contradicts Lemma 5.

Let $l = 7$. Then we find that $a_0a_3a_4a_5a_6a_7 = a_2^6t^l$, contradicting Lemma 5 with $l' = 6$.

Next we consider the 4th and 5th cases in Table 5. Then $l = 5$ and (3.2) is valid with $l' \le 3$. Using (2.4)–(2.7), we find that in the 4th case $a_1a_4a_7 = a_2^3t^l$ or $a_6^3t^l$ and in the 5th case $a_1a_6 = a_2a_5t^l$ or $a_1a_6 = a_0a_7t^l$, contradicting Lemma 5 with $l' = 3, 2$, respectively. ∎

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