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Generalized modular forms representable as eta products

by

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1. Introduction. Let τ be in the upper half plane H and $n \in \mathbb{Z}$. The Dedekind eta function is defined by

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau})$$

and the generalized Dedekind eta function [5, 6] is defined by

$$\eta_{\delta,g}(\tau) = e^{\pi i P_2(g/\delta)\delta\tau} \prod_{\substack{m>0 \\ m\equiv g \bmod \delta}} (1 - x^m) \prod_{\substack{m>0 \\ m\equiv -g \bmod \delta}} (1 - x^m),$$

where $x = e^{2\pi i\tau}$, $\tau \in H$, $g \in \mathbb{Z}$ is such that $0 \le g < \delta$,

$$P_2(t) = \{t\}^2 - \{t\} + 1/6$$

is the second Bernoulli function, and $\{t\} = t - [t]$ is the fractional part of t. Note that

$$\eta_{\delta,0}(\tau) = \eta(\delta\tau)^2$$

and that

$$\eta_{\delta,\delta/2}(\tau) = \frac{\eta^2((\delta/2)\tau)}{\eta^2(\delta\tau)}.$$

Consider

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

Consider also

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\},$$

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$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N,\ a \equiv d \equiv 1 \bmod N \right\},$$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \bmod N,\ b \equiv c \equiv 0 \bmod N \right\},$$

which are congruence subgroups of the full modular group.

DEFINITION 1. A *generalized modular form* (GMF) of weight k on Γ is a function $f(\tau)$ meromorphic throughout the complex upper half plane H, which is also meromorphic at the cusps and satisfies the transformation law

$$f(M\tau) = \nu(M)(c\tau + d)^k f(\tau)$$

for all $M \in \Gamma$. Here we allow the possibility that $|\nu(M)| \neq 1$ where ν is a character of the group Γ.

In [3], Kohnen and Mason proved the following theorem.

THEOREM 1. *Let f be a GMF of weight 0. Assume that f has no poles or zeroes in $H \cup \mathbb{Q} \cup \infty$. Assume furthermore that Γ is a congruence subgroup and that the Fourier coefficients at $i\infty$ are rational and are p-integral for all but a finite number of primes p. Then f is constant.*

Afterwards, they considered the subgroup $\Gamma_0(N)$ and proved that a GMF with zeroes and poles supported at the cusps, and such that the order of the function at the cusp is independent of the numerator of that cusp with the above conditions on the Fourier coefficients, is a classical eta product. Their result is given in Theorem 2 below. In this paper, we replace the condition imposed by Kohnen and Mason on the order of the function at the cusp by a condition on N. We then prove a theorem with conditions at the cusps which are similar to those of Kohnen and Mason, but on $\Gamma_1(N)$ instead of $\Gamma_0(N)$. It will turn out that functions with such conditions upon the order of the function at the cusps are also representable as eta products and generalized eta products on $\Gamma_1(N)$. Finally, we deduce similar theorems on $\Gamma(N)$.

The theorem of Kohnen and Mason on the subgroup $\Gamma_0(N)$ is as follows. Note that a complete set of representatives of the cusps of $\Gamma_0(N)$ [3] is given by a/c where c divides N and a is taken modulo N, with $(a, N) = 1$ and the a's are inequivalent modulo $(c, N/c)$.

THEOREM 2. *Let f be a GMF of integral weight k on $\Gamma_0(N)$. Suppose that the poles and zeroes of f are supported at the cusps. Suppose that the Fourier coefficients at $i\infty$ are rational and are p-integral for all but a finite number of primes p. Suppose further that the order of f at each cusp of $\Gamma_0(N)$ is independent of a. Then f is an eta quotient, i.e. there are integers*

$M \neq 0$ and m_t for $t \mid N$ such that

$$f^M(\tau) = c \prod_{t|N} \Delta(t\tau)^{m_t},$$

where $\Delta(\tau) = \eta(\tau)^{24}$.

Notice that the Fourier coefficients of $f^M / \prod_{t|N} \Delta(t\tau)^{m_t}$ are rational and p-integral for all but a finite number of primes p. This is due to the fact that the product in the denominator has integer coefficients with 1 as leading coefficient. Theorem 2 then easily follows from Theorem 1.

To modify the condition on the order of the cusps we define a class of functions which is a form on $\Gamma_1(N)$ and then lift it by applying a coset operator.

We now present another class of functions called the generalized Dedekind eta products. Consider

$$f(\tau) = \prod_{\substack{\delta|N \\ g}} \eta_{\delta,g}^{r_{\delta,g}}(\tau), \tag{1}$$

where $0 \le g < \delta$ and $r_{\delta,g}$ are integers and may be half integers only if $g = 0$ or $g = \delta/2$ (we allow half integers in order to include the ordinary eta products).

In [5], S. Robins proved that (1) is a modular function on $\Gamma_1(N)$ under certain conditions on the $r_{\delta,g}$'s. It will be sufficient for our purposes to note that the above function is a classical modular form on $\Gamma_1(N)$ with a multiplier system. For $A \in \Gamma_1(N)$, the transformation law of $f(\tau)$ is given by

$$f(A\tau) = f(\tau) e^{\pi i \sum \mu_{\delta,g} r_{\delta,g}}, \tag{2}$$

where

$$\mu_{\delta,g} = \frac{\delta a}{c} P_2\left(\frac{g}{\delta}\right) + \frac{\delta d}{c} P_2\left(\frac{ag}{\delta}\right) - 2s\left(a, \frac{c}{\delta}, 0, \frac{g}{\delta}\right)$$

and $s(h,k,x,y)$ is the Meyer sum, a generalized Dedekind sum, defined by

$$s(h,k,x,y) = \sum_{\mu \bmod k} \left(\left(h\left(\frac{\mu+y}{k}\right) + x\right)\right)\left(\left(\frac{\mu+y}{k}\right)\right).$$

As usual $((x)) = x - [x] - 1/2$ if x is not an integer and 0 otherwise.

2. GMF's on $\Gamma_0(N)$ representable as generalized eta products. As we have already pointed out, a complete set of representatives of the cusps of $\Gamma_0(N)$ is given by

$$a/c \tag{3}$$

where c is a positive divisor of N and a runs through integers with $1 \le a \le N$, $(a, N) = 1$ that are inequivalent modulo $(c, N/c)$. The width of the cusp a/c in (3) is given by

$$w_{a/c} = N/(c^2, N).$$

THEOREM 3. *Let f be a GMF of rational weight k' on $\Gamma_0(N)$. Suppose that the rank of*

$$((\delta, c)^2 P_2(ag/(\delta, c)))_{(\delta|N, 0 \le g < \delta), (c|N, a)} \tag{4}$$

is equal to the number of cusps, where the columns of the matrix corresponds to the cusps a/c of $\Gamma_0(N)$. Suppose further that the Fourier coefficients at $i\infty$ are rational and are p-integral for all but a finite number of primes p and that the poles and zeroes of f are supported at the cusps. Then f is a classical modular form.

REMARK. Note that the rank of the above matrix is less than or equal to the number of cusps. I will mention several examples where the rank turned out to be equal to the number of cusps while in another example this will fail to happen. For $N = 9$ the above matrix has rank less than the number of cusps, while for $N = 16, 20, 24, 28$ and also for any square free integer, the matrix has rank equal to the number of cusps.

Proof of Theorem 3. For given integers $r_{\delta,g}$ put

$$F(\tau) = \prod_{\delta|N} \prod_{0 \le g < \delta} \eta_{\delta,g}(\tau)^{r_{\delta,g}}. \tag{5}$$

Then $F(\tau)$ is a modular form on $\Gamma_1(N)$ of weight $k = \sum r_{\delta,0}$ and by [5],

$$\operatorname{ord}_{a/c} F = \frac{w_{a/c}}{2} \sum_{\delta|N} \sum_{0 \le g < \delta} \frac{(\delta, c)^2}{\delta} P_2\left(\frac{ag}{(\delta, c)}\right) r_{\delta,g}. \tag{6}$$

We now consider the cosets of $\Gamma_1(N)$ in $\Gamma_0(N)$. By applying the operator defined below, we lift the generalized eta product in (5) from a modular form on $\Gamma_1(N)$ to a modular form on $\Gamma_0(N)$. For

$$\beta_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$$

and F a function on H, we define the following operator:

$$F|_k \beta_j = (c_j \tau + d_j)^{-k} F(\beta_j \tau). \tag{7}$$

Let

$$H(F) = \prod_j F|_k \beta_j \tag{8}$$

where $\{\beta_j\}$ are coset representatives. We see from (5) and (7) that

$$F(\tau)|_k\beta_j = (c_j\tau + d_j)^{-k} \prod_{\delta|N} \prod_{0\le g<\delta} \eta_{\delta,g}(\beta_j\tau)^{r_{\delta,g}}.$$

Recall that F is a modular form on $\Gamma_1(N)$ of weight k. It follows that $H(F)$ in (8) is a modular form on the larger group $\Gamma_0(N)$ of weight $k_1 = |\Gamma_1(N)\backslash\Gamma_0(N)|k$. We have to determine the order of $H(F)$ at any cusp of $\Gamma_0(N)$. We have to show first that after applying the operator we again get an eta product and that the operator will not affect the order of the function at the cusps as calculated in [5]. Recall that Robins [5] found the transformation of $\eta_{\delta,g}$ under $A \in \Gamma_0(N)$. For $g \neq 0$ and $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ we have, by (2),

$$\eta_{\delta,g}(A\tau) = e^{\pi i \mu_{\delta,g}} \eta_{\delta,ag}(\tau). \tag{9}$$

Thus if $\beta_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right)$ and for a given δ where $(a_j, \delta) = 1$, if $0 \le g < \delta$, then $a_j g$ will run through a complete set of representatives modulo δ, and also for a given δ, if $g_1 \equiv -g_2 \bmod \delta$ then

$$\prod_{\substack{m>0 \\ m\equiv g_1 \bmod \delta}} (1 - x^m) \prod_{\substack{m>0 \\ m\equiv -g_1 \bmod \delta}} (1 - x^m) = \prod_{\substack{m>0 \\ m\equiv g_2 \bmod \delta}} (1 - x^m) \prod_{\substack{m>0 \\ m\equiv -g_2 \bmod \delta}} (1 - x^m)$$

and

$$P_2\left(\frac{g_1}{\delta}\right) = P_2\left(\frac{k\delta - g_2}{\delta}\right) = P_2\left(1 - \frac{g_2}{\delta}\right) = P_2\left(\frac{g_2}{\delta}\right).$$

Hence

$$\eta_{\delta,g_1} = \eta_{\delta,g_2}.$$

Recall that $\beta_j \in \Gamma_0(N)$. Also $\eta_{\delta,0}$ and $\eta_{\delta,\delta/2}$ are forms on $\Gamma_0(\delta)$ and hence on $\Gamma_0(N)$. As a result, using (9) we obtain

$$\begin{aligned} F(\tau)|_k\beta_j &= \nu_j(c_j\tau + d_j)^{-k} \prod_{\delta|N} \prod_{0\le g<\delta} \eta_{\delta,g}(\beta_j\tau)^{r_{\delta,g}} \\ &= \nu_j(c_j\tau + d_j)^{-k} \prod_{\delta|N} \eta_{\delta,0}(\beta_j\tau)^{r_{\delta,0}} \prod_{\delta|N} \prod_{0<g<\delta} \eta_{\delta,g}(\beta_j\tau)^{r_{\delta,g}} \\ &= \nu_j^* \prod_{\delta|N} \eta_{\delta,0}(\tau)^{r_{\delta,0}} \prod_{\delta|N} \prod_{0<g<\delta} \eta_{\delta,g}(\tau)^{r_{\delta,g}}, \end{aligned} \tag{10}$$

where the exponents $r_{\delta,g}$ are renamed according to the new values of g, and

ν_j is a constant depending on β_j. Thus by (10) we obtain

$$H(F) = \nu \prod_{\delta|N} \prod_{0\le g<\delta} \eta_{\delta,g}(\tau)^{r'_{\delta,g}},$$

where again $r'_{\delta,g} = \sum_j r_{\delta,a_j g}$ are new exponents and ν is a constant depending on β_j for all j. Using the condition on the matrix (4), we have to solve now for $r'_{\delta,g}$. To do this we have to determine the order of $H(F)$ at the cusps. In [5], it is shown that

$$\text{(11)} \qquad \operatorname{ord}_{a/c} \prod_{\delta|N} \eta_{\delta,g}^{r_{\delta,g}} = \frac{w_{a/c}}{2} \sum_{\delta|N} \frac{(\delta,c)^2}{\delta} P_2\left(\frac{ag}{(\delta,c)}\right) r_{\delta,g}.$$

Thus using (11), we get

$$\operatorname{ord}_{a/c} H(F) = \frac{w_{a/c}}{2} \sum_{\delta|N} \sum_{0\le g<\delta} \frac{(\delta,c)^2}{\delta} P_2\left(\frac{ag}{(\delta,c)}\right) r'_{\delta,g}.$$

Notice now that the product of two expressions whose Fourier coefficients are rational and p-integral for all but a finite number of primes p has rational Fourier coefficients that are p-integral for all but a finite number of primes p. As $\eta_{\delta,g}(\tau)$ has Fourier coefficients that are rational and p-integral for all but a finite number of primes, so also are the Fourier coefficients of $H(F)$ since $H(F)$ has turned to be a generalized eta product. We still want to show that $r'_{\delta,g}$ can be chosen so that

$$\text{(12)} \qquad \operatorname{ord}_{a/c} H(F) = m h_{a/c}$$

for all cusps a/c of $\Gamma_0(N)$. Here $h_{a/c}$ is the order of f at a/c and m is an appropriate non-zero integer depending only on f. By hypothesis, the rank of

$$((\delta,c)^2 P_2(ag/(\delta,c)))_{(\delta|N, 0\le g<\delta),(c|N,a)}$$

is equal to the number of cusps. Therefore we can choose $r'_{\delta,g}$ so that (12) is satisfied, with an appropriate m. Since η does not vanish on H, we find from the valence formula applied to $H(F)$ that the sum of the orders of $H(F)$ at the different cusps of $\Gamma_0(N)$ is equal to

$$\frac{k_1}{12} [\Gamma(1) : \Gamma_0(N)].$$

On the other hand, the valence formula is also valid for the GMF f of weight k' (see [2]). We then deduce from (12) that

$$k_1 = mk'.$$

We see that $f^m/H(F)$ is a GMF satisfying all the assumptions of Theorem 1. We conclude that $f^m = cH(F)$, as required.

Note that for $N = p_1 \cdots p_n$ square-free, the cusps of $\Gamma_0(N)$ are $1/1$, $1/p_1, \ldots, 1/p_n$ and $1/N$. Hence $\Gamma_0(N)$ satisfies the condition on the order of the cusps given in the paper of Kohnen and Mason [3].

For $N = p^2$, the condition of Theorem 3 fails. This happens because $P_2(a_1 g/(\delta, c)) = P_2(a_2 g/(\delta, c))$ for $c = p$ and for all $a_1 \equiv -a_2 \bmod p$ for all g and thus

$$((\delta, c)^2 P_2(ag(\delta, c)))_{(\delta|N, 0 \le g < \delta), (c|N, a)}$$

has a rank smaller than the number of cusps.

3. GMF's on $\Gamma_1(N)$ representable as eta products. In this section, we derive theorems similar to those derived in the previous section but on the congruence subgroup $\Gamma_1(N)$. We impose different conditions on the order of the function at the cusps in one theorem and then relax the condition in the following theorem. Every cusp of $\Gamma_1(N)$ is equivalent to

$$a/c \tag{13}$$

where c is taken modulo N and a is taken modulo $d = (N, c)$, and $(a, d) = 1$. Moreover, for every cusp of $\Gamma_1(N)$ there exist precisely two fractions a/c of the above form that are equivalent to that cusp. The width of every cusp in (13) is given by

$$w_{a/c} = N/(c, N).$$

THEOREM 4. *Let f be a GMF of integral weight k on $\Gamma_1(N)$. Suppose that the poles and zeroes of f are supported at the cusps and that the Fourier coefficients at $i\infty$ are rational and are p-integral for all but a finite number of primes p. Suppose further that the order of the function f at each cusp of $\Gamma_1(N)$ is independent of a and for the cusps a_1/c_1 whose denominator does not divide N, the function has the same order at a_1/c_1 as at those cusps whose denominators are (c_1, N). Then f is an eta quotient, i.e., there are integers $M \neq 0$ and m_t for all $t \mid N$ such that*

$$f^M(\tau) = c \prod_{t|N} \Delta(t\tau)^{m_t}.$$

Proof. We have

$$\Delta(\tau) = q \prod_{n \ge 1} (1 - q^n)^{24}.$$

For given integers m_t put

$$F(\tau) = \prod_{t|N} \Delta(t\tau)^{m_t}.$$

Then F is a modular form on $\Gamma_1(N)$ and

$$\mathrm{ord}_{a/c} F = w_{a/c}\biggl(\sum_{t|N} \frac{(t,c)^2}{t}\, m_t\biggr).$$

Note that the order at every cusp a/c is independent of a and hence F itself satisfies the order condition given by Theorem 4. Moreover the conditions imposed in the theorem are important since the sum in the above expression for $\mathrm{ord}_{a/c}$ runs only over the divisors of N. We want to show that m_t can be chosen so that

$$\mathrm{ord}_{a/c} F = m h_{a/c} \tag{14}$$

for all cusps a/c of $\Gamma_1(N)$. Here $h_{a/c}$ is the order of f^{12} at a/c and m is an appropriate non-zero integer depending only on f. Note that by assumption $h_{a/c}$ is independent of a. Note that in the case of $\Gamma_1(N)$, the denominator of the cusp is taken modulo N, not as a divisor of N as in the case of $\Gamma_0(N)$. Since we know that the order of the function at the cusp a/c whose denominator does not divide N is equal to the order of the function at the cusp whose denominator is (c,N), there are $\sigma_0(N)$ equations. So as in the proof of Mason and Kohnen, it will be sufficient to prove that the square matrix

$$A_N = ((t,c)^2)_{t|N,c|N}$$

of size $\sigma_0(N) \times \sigma_0(N)$ is invertible. Now using [1], we see that

$$A'_N = ((t,c))_{t|N,c|N}$$

is positive definite and hence invertible. The Oppenheim inequality [4] states that if two matrices A and B are positive definite, then

$$|A \circ B| \ge |B| \prod_i a_{ii},$$

where $\circ$ denotes the Hadamard product of matrices. As a result,

$$|A'_N \circ A'_N| = |A_N| \ge |A'_N| \prod_i a_{ii}.$$

Thus our matrix is invertible. We have thus established formula (14), with an appropriate m.

Let k_1 be the weight of F. Since Δ does not vanish on H, we find from the valence formula applied to F that the sum of the orders of F at the different cusps of $\Gamma(N)$ is equal to

$$\frac{k_1}{12}\,[\Gamma(1) : \Gamma_1(N)].$$

On the other hand, the valence formula is also valid for the GMF f^{12} of weight $12k$ (see [2]). We then deduce from (14) that

$$k_1 = 12mk.$$

Letting $M = 12m$ we see that f^m/F is a GMF satisfying all the assumptions of Theorem 1. We conclude that $f^M = cF$, as required.

We now change, in the above theorem, a condition on the order of the function at the cusps to a condition on the level N of the congruence subgroup.

THEOREM 5. *Let f be a GMF of integral weight k on $\Gamma_1(N)$, and suppose that the poles and zeroes of f are supported at the cusps. Suppose that the Fourier coefficients at $i\infty$ are rational and are p-integral for all but a finite number of primes p. Suppose further that for the cusps a/c whose denominator does not divide N, the function has the same order at a/c as at any of those cusps whose denominators are (c,N), and that the rank of*

$$((\delta,c)^2 P_2(ag/(\delta,c)))_{(\delta|N,0\le g<\delta),(c|N,a)} \tag{15}$$

is equal to the number of cusps whose denominator divides N. Then f is a classical modular form.

REMARK. For $N = 9$ as well, the above matrix has rank less than the number of cusps whose denominator divides N while for $N = 16$ it has rank equal to the number of cusps whose denominators divide N.

Proof of Theorem 5. For given integers $r_{\delta,g}$ put

$$F(\tau) = \prod_{\delta|N}\prod_{0\le g<\delta} \eta_{\delta,g}(\tau)^{r_{\delta,g}}.$$

We want to find $r_{\delta,g}$ such that $f^m = cF$ for some constant c. Now, F is a modular form on $\Gamma_1(N)$ of weight $k_1 = \sum r_{\delta,0}$ and by [5],

$$\operatorname{ord}_{a/c} F = \frac{w_{a/c}}{2}\sum_{\delta|N}\sum_{0\le g<\delta}\frac{(\delta,c)^2}{\delta}\,P_2\biggl(\frac{ag}{(\delta,c)}\biggr) r_{\delta,g}.$$

Using the condition on the matrix (15), we have to solve now for $r_{\delta,g}$. Notice that (by arguing as in the proof of Theorem 3) the Fourier coefficients of F are rational and p-integral for all but a finite number of primes p. We still want to show that $r_{\delta,g}$ can be chosen so that

$$\operatorname{ord}_{a/c} F = m h_{a/c} \tag{16}$$

for all cusps a/c of $\Gamma_0(N)$. Here $h_{a/c}$ is the order of f at a/c and m is an appropriate non-zero integer depending only on f. By assumption, the rank of (15) is equal to the number of cusps whose denominator divides N. Thus we have a non-trivial solution. Thus we have established formula (16), with an appropriate m. Since η does not vanish on H, we find from the valence formula applied to F that the sum of the orders of F at the different cusps of $\Gamma_1(N)$ is equal to

$$\frac{k_1}{12}\,[\Gamma(1) : \Gamma_1(N)].$$

On the other hand, the valence formula is also valid for the GMF f of weight k (see [2]). We then deduce from (16) that

$$k_1 = mk.$$

We see that f^m/F is a GMF satisfying all the assumptions of Theorem 1. We conclude that $f^m = cF$, as required.

4. GMF's on $\Gamma(N)$ representable as eta products. A complete set of representatives of the cusps of $\Gamma(N)$ is given by

$$a/c \tag{17}$$

where c is taken modulo N and a is taken modulo N and $(a,\nu) = 1$ where $\nu = (c,N)$. In this set of representatives, the cusps pair up. The width of every cusp a/c in (17) is given by

$$w_{a/c} = N.$$

In the case of $\Gamma(N)$, we can also derive a theorem with strong restrictions on the order of the function at the cusps and then in a following theorem, we relax those conditions by imposing a condition on N as in the case of $\Gamma_1(N)$.

Theorem 6. *Let f be a GMF of integral weight k on $\Gamma(N)$. Suppose that the poles and zeroes of f are supported at the cusps and that the Fourier coefficients at $i\infty$ are rational and are p-integral for all but a finite number of primes p. Suppose further that the order of f at each cusp a/c of $\Gamma(N)$ is independent of a and for the cusps a_1/c_1 whose denominator does not divide N, the function has the same order as at those cusps whose denominators are (c_1,N). Then f is an eta quotient, i.e. there are integers $M \neq 0$ and m_t for $t \mid N$ such that*

$$f^M(\tau) = c\prod_{t|N} \Delta(t\tau)^{m_t}.$$

Proof. We follow exactly the proof of Theorem 4.

Theorem 7. *Let f be a GMF of integral weight k on $\Gamma(N)$, and suppose that the poles and zeroes of f are supported at the cusps. Suppose that the Fourier coefficients at $i\infty$ are rational and are p-integral for all but a finite number of primes p. Suppose further that for the cusps a/c whose denominator does not divide N, the function has the same order at a/c as at any of those cusps whose denominators are (c,N), and that the rank of*

$$((\delta,c)^2 P_2(ag/(\delta,c)))_{(\delta|N,0\le g<\delta),(c|N,a)} \tag{18}$$

is equal to the number of cusps whose denominator divides N. Then f is a classical modular form.

Proof. Since every modular form on $\Gamma_1(N)$ is a modular form on $\Gamma(N)$, follow exactly the proof of Theorem 5.

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