The first negative Hecke eigenvalue of a Siegel cusp form of genus two

by

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1. Introduction and statement of result. Recently there have been several works on sign changes of Fourier coefficients and Hecke eigenvalues of elliptic cusp forms (cf. e.g. [4, 8, 10, 12, 13]).

Notably in [8] it was shown that if $f$ is a normalized Hecke eigenform of integral weight $k \geq 2$ and level $N \in \mathbb{N}$, and $\lambda(n) \ (n \in \mathbb{N})$ denote its Hecke eigenvalues, then there exists $n \in \mathbb{N}$ with

$$n \ll (k^2 N)^{29/60}$$

such that $\lambda(n) < 0$. Here the constant implied in $\ll$ is absolute and effectively computable. The proof uses convexity estimates for the Hecke $L$-function of $f$ and exploits the Hecke relations satisfied by the $\lambda(n)$.

Let $S_k(I_2)$ be the space of Siegel cusp forms of integral weight $k$ on the group $I_2 := \text{Sp}_2(\mathbb{Z}) \subset \text{GL}_4(\mathbb{Z})$ and let $F$ be a non-zero eigenfunction of all the Hecke operators $T(n) \ (n \in \mathbb{N})$ (cf. e.g. [2, 7] for details). Denote by $\lambda(n) \ (n \in \mathbb{N})$ the corresponding eigenvalues.

If $k$ is even and $F$ is contained in the Maass subspace $S_k^*(I_2) \subset S_k(I_2)$ (cf. e.g. [5]), it was proved in [3] that $\lambda(n) > 0$ for all $n$. On the other hand, if either $k$ is odd, or $k$ is even and $F$ is in the orthogonal complement of $S_k^*(I_2)$, then under the validity of the Ramanujan–Petersson conjecture for $F$ (a proof of which was announced in [15]) it was recently shown in [11] that the sequence $(\lambda(n))_{n \in \mathbb{N}}$ indeed changes sign infinitely often.

In the present paper we shall prove

Theorem. Let $F$ be a non-zero Siegel–Hecke eigenform in $S_k(I_2)$ and suppose that either $k$ is odd, or $k$ is even and $F$ is in the orthogonal complement of $S_k^*(I_2)$. Assume that $F$ satisfies the Ramanujan–Petersson conjecture (cf. Sect. 2). Denote by $\lambda(n) \ (n \in \mathbb{N})$ the eigenvalues of $F$. Then there

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exists \( n \in \mathbb{N} \) with
\[
n \ll k^2 \log^{20} k
\]
such that \( \lambda(n) < 0 \). Here the constant implied in \( \ll \) is absolute and effectively computable.

We note that the first case where a form \( F \) as above exists is \( k = 35 \) if \( k \) is odd and \( k = 20 \) if \( k \) is even.

The proof of the Theorem follows a similar pattern to that in [8], with the Hecke \( L \)-function replaced by the spinor zeta function. However, since the Hecke relations for \( \lambda(n) \) are more involved in genus 2 than in the elliptic case, exploiting them naturally turns out to be more difficult.

**Notations.** If in an estimate we write \( \ll \), it is always understood that the implied constant is absolute.

2. Preliminaries on Siegel modular forms. For basic facts on Siegel modular forms we refer to [2, 7, 9]. For \( n \in \mathbb{N} \) there is a Hecke operator \( T(n) \) on \( S_k(\Gamma_2) \) given by

\[
F|T(n) = \sum_{\gamma \in \Gamma_2 \setminus \mathcal{O}_{2,n}} F|_{k \gamma}
\]

where \( \mathcal{O}_{2,n} \) is the set of integral symplectic similitudes of size 4 and scale \( n \) and

\[
(F|_{k \gamma})(Z) := (\det \gamma)^{k/2} \det(CZ + D)^{-k} F((AZ + B)(CZ + D)^{-1})
\]

for
\[
\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad Z \in \mathcal{H}_2 = \text{Siegel upper half space of genus 2}.
\]

Note that our choice of normalization in (2.1) differs from the usual one by the scalar factor \( n^{k-3/2} \).

The space \( S_k(\Gamma_2) \) has a basis consisting of common eigenfunctions of all the \( T(n) \). The Maass subspace \( S_k^\sigma(\Gamma_2) \) \( (k \text{ even}) \) is invariant under all Hecke operators.

Let \( F \) be a non-zero eigenfunction of all \( T(n) \), with \( F|T(n) = \lambda(n)F \). Then \( \lambda(n) \) is real for all \( n \).

One has

\[
\sum_{n \geq 1} \lambda(n)n^{-s} = \frac{1}{\zeta(2s + 1)} Z_F(s) \quad (\sigma := \Re(s) \gg 1)
\]

where \( Z_F(s) \) is the spinor zeta function of \( F \), i.e.

\[
Z_F(s) = \prod_p Z_{F,p}(p^{-s})^{-1} \quad (\sigma \gg 1)
\]
with

\[(2.4) \quad Z_{F,p}(X) := (1 - \alpha_{0,p}X)(1 - \alpha_{0,p}\alpha_{1,p}X)(1 - \alpha_{0,p}\alpha_{2,p}X)(1 - \alpha_{0,p}\alpha_{1,p}\alpha_{2,p}X)\]

and where \(\alpha_{0,p}, \alpha_{1,p}, \alpha_{2,p}\) are “the” Satake \(p\)-parameters attached to \(F\). For details we refer to [1].

Note that due to our normalization one has

\[(2.5) \quad \alpha_{0,p}^2\alpha_{1,p}\alpha_{2,p} = 1.\]

Indeed, in comparison to the “classical” normalization we have replaced the variable \(s\) by \(s + k - 3/2\).

The function

\[Z_F^*(s) := (2\pi)^{-s}\Gamma(s + k - 3/2)\Gamma(s + 1/2)Z_F(s)\]

has meromorphic continuation to \(\mathbb{C}\) and is \((-1)^k\)-invariant under \(s \mapsto 1 - s\) (see [1]). It is entire if and only if either \(k\) is odd, or \(k\) is even and \(F\) is in the orthogonal complement of \(S_k^*(I_2)\) [6, 14].

In the latter case the Ramanujan–Petersson conjecture says that

\[(2.6) \quad |\alpha_{1,p}| = |\alpha_{2,p}| = 1 \quad (\forall p)\]

(a proof was announced in [15]). By (2.5) we then also have

\[(2.7) \quad |\alpha_{0,p}| = 1 \quad (\forall p).\]

3. Convexity estimates. Let \(F \in S_k(I_2)\) be a non-zero Hecke eigenform with normalized eigenvalues \(\lambda(n)\) \((n \in \mathbb{N})\). We assume that either \(k\) is odd, or \(k\) is even and \(F\) is not contained in \(S_k^*(I_2)\). We also assume (2.6).

The purpose of this section is to derive estimates uniform with respect to \(k\) for \(Z_F(s)\) on lines \(s = \delta + it\) \((t \in \mathbb{R})\) where \(0 < \delta < 1/2\). The arguments will be analogous to those given in Sect. 3 of [12] and therefore we will be brief.

Let us write

\[Z_F(s) = \sum_{n \geq 1} a(n)n^{-s} \quad (\sigma \gg 1).\]

Then from (2.3), (2.4), (2.6) and (2.7) we obtain

\[|a(n)| \leq d_4(n) \quad (n \geq 1)\]

where \(d_4(n)\) is the \(n\)th coefficient of \(\zeta^4(s)\). Since \(\zeta^4(s)\) has a pole at \(s = 1\) of order 4, a standard Tauberian argument gives

\[(3.1) \quad \sum_{x_0 \leq n \leq x} |a(n)| \ll x \log^3 x \quad (x_0 > 1).\]
Using integration by parts for Stieltjes integrals we deduce from (3.1) in a similar way to [12] that

\[(3.2) \quad |Z_F(c + it)| \ll 1 + \frac{c}{(c - 1)^4}\]

whenever \(c > 1\).

Next by the functional equation of \(Z_F^*(s)\) we get

\[|Z_F(1 - s)| = (2\pi)^{2-4\sigma} \left| \frac{\Gamma(s + k - 3/2)\Gamma(s + 1/2)}{\Gamma(-s + k - 1/2)\Gamma(3/2 - s)} \right| \cdot |Z_F(s)|.\]

Putting \(s = c + it\) and observing that \(|\Gamma(z)| = |\Gamma(\bar{z})|\), we in particular obtain

\[|Z_F(1 - c - it)| = (2\pi)^{2-4c} \left| \frac{\Gamma(k - 3/2 + c + it)\Gamma(c + 1/2 + it)}{\Gamma(k - 1/2 - c + it)\Gamma(3/2 - sc + it)} \right| \cdot |Z_F(c + it)|.\]

We estimate the quotients of \(\Gamma\)-factors in the same way as in [12] to deduce that

\[|Z_F(1 - c - it)| \ll |k - 1 + 2it|^{2c-1}|1 + it|^{2c-1}|Z_F(c + it)|,

hence

\[(3.3) \quad |Z_F(1 - c - it)| \ll k^{2c-1}|1 + it|^{4c-2}|Z_F(c + it)|.

Now put

\[c := 1 + \frac{1}{2\log k}.

Then from (3.2) we infer that

\[(3.4) \quad \left| Z_F \left(1 + \frac{1}{2\log k} + it\right) \right| \ll \log^4 k

and therefore combining with (3.3) it follows that

\[(3.5) \quad \left| Z_F \left(-\frac{1}{2\log k} + it\right) \right| \ll k\log^4 k \cdot |1 + it|^{2 + 2/\log k}.

Let us now recall the following “strong convexity” principle, due to Rademacher (cf. e.g. [12, Sect. 3]).

**Lemma 1.** Suppose that \(g(s)\) is continuous on the closed strip \(a \leq \sigma \leq b\) and holomorphic and of finite order on \(a < \sigma < b\). Furthermore suppose that

\[|g(a + it)| \leq E|P + a + it|^{\alpha}, \quad |g(b + it)| \leq F|P + b + it|^{\beta}.

Here \(E\) and \(F\) are positive constants and \(P\), \(\alpha\) and \(\beta\) are real constants that satisfy

\[P + a > 0, \quad \alpha \geq \beta.

Then for all \(a < \sigma < b\) we have

\[|g(s)| \leq (E|P + s|^{\alpha})^{(b - \sigma)/(b - a)}(F|P + s|^{\beta})^{(\sigma - a)/(b - a)}.

We apply Lemma 1 to \( Z_F(s) \) with
\[
a = -\frac{1}{2 \log k}, \quad b = P = 1 + \frac{1}{2 \log k}, \quad E = k \log^4 k, \quad F = \log^4 k, \\
\alpha = 2 \left(1 + \frac{1}{\log k}\right), \quad \beta = 0
\]
and \( s = \delta + it \) where \( 0 < \delta < 1/2 \). From (3.4) and (3.5) we then obtain easily

**Proposition 1.** Let \( F \in S_k(\Gamma_2) \) be a non-zero Hecke eigenform with normalized eigenvalues \( \lambda(n) \) \( (n \in \mathbb{N}) \). Assume that either \( k \) is odd, or \( k \) is even and \( F \notin S^*_k(\Gamma_2) \). Let \( 0 < \delta < 1/2 \). Then for all \( t \in \mathbb{R} \) one has
\[
|Z_F(\delta + it)| \ll k^{1-\delta} \log^4 k \cdot \left|1 + \frac{1}{2 \log k} + \delta + it\right|^{2+1/\log k-2\delta}.
\]

**4. An upper bound for sums of eigenvalues.** We shall prove

**Proposition 2.** Let \( F \in S_k(\Gamma_2) \) be a non-zero Hecke eigenform with normalized eigenvalues \( \lambda(n) \) \( (n \in \mathbb{N}) \). Assume that either \( k \) is odd, or \( k \) is even and \( F \notin S^*_k(\Gamma_2) \). Also suppose that (2.6) holds. Then
\[
\sum_{n \leq x} \lambda(n) \log^2 \left(\frac{x}{n}\right) \ll k \log^8 k \cdot x^{2/3 \log k}.
\]

**Proof.** By Perron’s formula and (2.2) we have
\[
\sum_{n \leq x} \lambda(n) \log^2 \left(\frac{x}{n}\right) = \frac{2}{2 \pi i} \int_{2-i \infty}^{2+i \infty} \frac{1}{\zeta(2s+1)} Z_F(s) \frac{x^s}{s^3} ds
\]
(cf. [12, Sect. 5]).

Let \( \frac{1}{2 \log k} < \delta < 1/2 \). We shift the line of integration to the line \( \sigma = \delta \) and recall the well-known estimate (say)
\[
\left|\frac{1}{\zeta(\sigma + it)}\right| \ll \beta(t)
\]
valid (uniformly) for \( \sigma > 1 \), where
\[
\beta(t) := \begin{cases} 
1 & \text{if } |t| \leq 10, \\
\log |t| & \text{if } |t| > 10.
\end{cases}
\]
Applying (3.6) we then obtain in a standard way
\[
\sum_{n \leq x} \lambda(n) \log^2 \left(\frac{x}{n}\right) \ll k^{1-\delta} \log^4 k \cdot \int_{-\infty}^{\infty} \beta(t) \frac{\left|1 + \frac{1}{2 \log k} + \delta + it\right|^{2+1/\log k-2\delta}}{|\delta + it|^3} dt \cdot x^{\delta}.
\]

**The first negative Hecke eigenvalue**

57
Note that the integral on the right-hand side of (4.1) is absolutely convergent since $2 + 1/\log k - 2\delta < 2$ by hypothesis.

We have to estimate this integral from above uniformly in $k$. Replacing $t$ by $-t$, it is sufficient to get an upper bound on

$$(4.2) \quad I_{k,\delta} := \int_0^\infty \beta(t) \left| \frac{1 + \frac{1}{2\log k} + \delta + it}{|\delta + it|^3} \right|^{2+1/\log k - 2\delta} dt.$$  

Note that for $0 < B < A$ one has

$$|A + it| \leq \frac{A}{B}|B + it| \quad (\forall t \in \mathbb{R}).$$

Applying this with

$$A := 1 + \frac{1}{2\log k} + \delta, \quad B := \delta$$

we see that the integrand in (4.2) is bounded from above by

$$C_{k,\delta} \beta(t)|\delta + it|^{-1+1/\log k - 2\delta}$$

where

$$C_{k,\delta} := \left(1 + \frac{1}{2\log k} \right)^{2+1/\log k - 2\delta}.$$

We split up $I_{k,\delta}$ into an integral from 0 to 10 and an integral from 10 to $\infty$. The first integral is clearly bounded by

$$\ll C_{k,\delta} \delta^{-1+1/\log k - 2\delta}.$$  

The second integral is bounded by

$$\ll C_{k,\delta} \int_{10}^\infty \log t \cdot t^{-1+1/\log k - 2\delta} dt \ll C_{k,\delta} \left( \frac{1}{2\delta - \frac{1}{\log k}} \right)^2,$$

where the last estimate follows by partial integration.

We now choose

$$\delta := \frac{2}{3\log k}.$$

We then obtain

$$I_{k,\delta} \ll \log^2 k \cdot (\log k + \log^2 k) \ll \log^4 k.$$  

Also $k^{1-\delta} \ll k$ (in fact $k^{1-\delta}$ is of the same order of magnitude as $k$).

Thus from (4.1) we obtain our assertion.

5. A lower bound for sums of eigenvalues

Proposition 3. Let $F \in S_k(\Gamma_2)$ be a non-zero Hecke eigenform and assume that either $k$ is odd, or $k$ is even and $F \notin S_k^*(\Gamma_2)$. Suppose that (2.6)
holds. Let \( \lambda(n) (n \in \mathbb{N}) \) be the normalized eigenvalues of \( F \) and suppose that \( \lambda(n) \geq 0 \) for \( 1 \leq n \leq x \). Then
\[
\sum_{n \leq x} \lambda(n) \log^2 \left( \frac{x}{n} \right) \gg \frac{\sqrt{x}}{\log^2 x} \quad (x > 1).
\]

Proof. Clearly
\[
\sum_{n \leq x} \lambda(n) \log^2 \left( \frac{x}{n} \right) \gg \sum_{n \leq x/2} \lambda(n),
\]
hence it suffices to show that
\[
(5.1) \quad \sum_{n \leq x} \lambda(n) \gg \frac{\sqrt{x}}{\log^2 x} \quad (x > 1).
\]

By [1], for each prime \( p \) the local spinor polynomial \( Z_{F,p}(X) \) given by (2.4) is equal to
\[
Z_{F,p}(X) = 1 - \lambda(p)X + (\lambda(p^2) - \lambda(p^2) - 1/p)X^2 - \lambda(p)X^3 + X^4,
\]
hence by (2.2) we have
\[
(5.2) \quad \frac{1 - \frac{1}{p}X^2}{Z_{F,p}(X)} = \sum_{n \geq 0} \lambda(p^n)X^n.
\]

Clearly (5.2) is equivalent to saying that
\[
(5.3) \quad \lambda(p^n) = \lambda(p)\lambda(p^{n-1}) - (\lambda(p^2) - \lambda(p^2) - 1/p)\lambda(p^{n-2})
\]
\[
+ \lambda(p)\lambda(p^{n-3}) - \lambda(p^{n-4})
\]
for all \( n \geq 0 \), with the convention that \( \lambda(p^n) = 0 \) for \( n < 0 \).

Note that (2.4), (2.6), (2.7) and (5.2) imply that
\[
(5.4) \quad |\lambda(p)|, |\lambda(p^2)|, |\lambda(p^3)| \ll 1.
\]

To prove (5.1), bearing in mind that \( \lambda(n) \geq 0 \) for \( n \leq x \), let us write
\[
(5.5) \quad \sum_{n \leq x} \lambda(n) \geq \sum_{p,q \leq \sqrt{x}} \lambda(p^2q^2) + \sum_{p,q \leq \sqrt{x}} \lambda(p^2q) + \sum_{p,q \leq \sqrt{x}} \lambda(pq)
\]
where on the right-hand side \( p \) and \( q \) run over primes.

Taking \( n = 4 \) in (5.3) we obtain
\[
(5.6) \quad \lambda(p^4) = \lambda(p^2)^2 + \lambda(p)\lambda(p^3) + \lambda(p^2)(-\lambda(p^2) + 1/p) + \lambda(p)^2 - 1.
\]

Similarly, for \( n = 3 \) we find that
\[
(5.7) \quad \lambda(p^3) = \lambda(p)(2\lambda(p^2) + 1 + 1/p - \lambda(p)^2).
\]

From (5.6), observing (5.4) we see that
\[
\lambda(p^4) \gg \lambda(p^2)^2 - c_1
\]
where $c_1 > 0$ is an absolute constant. Thus
\begin{equation}
\sum_{p,q \leq \sqrt[3]{x}} \lambda(p^2 q^2) \gg \left( \sum_{p \leq \sqrt[3]{x}} \lambda(p^2) \right)^2 - c_1 \pi(\sqrt[3]{x})
\end{equation}
where as usual $\pi(x) (x > 1)$ denotes the number of primes $p \leq x$.

Next, from (5.7) taking into account (5.4) we see that
\[ \lambda(p^3) \gg \lambda(p) \lambda(p^2) - c_2 \]
where $c_2 > 0$ is an absolute constant. Hence
\begin{equation}
\sum_{p,q \leq \sqrt[3]{x}} \lambda(p^2 q) \gg \left( \sum_{p \leq \sqrt[3]{x}} \lambda(p^2) \right) \left( \sum_{p \leq \sqrt[3]{x}} \lambda(p) \right) - c_2 \pi(\sqrt[3]{x}).
\end{equation}

We finally look at the sum
\[ \sum_{p,q \leq \sqrt[3]{x}} \lambda(pq) \]
in (5.5). For $p \leq \sqrt[3]{x}$ the quantities $\lambda(p^3), \lambda(p^2)$ and $\lambda(p)$ are non-negative, hence we deduce from (5.7) for such $p$ that
\[ \lambda(p^2) \gg \lambda(p)^2 - c_3 \]
where $c_3 > 0$ is an absolute constant. Therefore as before
\begin{equation}
\sum_{p,q \leq \sqrt[3]{x}} \lambda(pq) \gg \left( \sum_{p \leq \sqrt[3]{x}} \lambda(p) \right)^2 - c_3 \pi(\sqrt[3]{x}).
\end{equation}

Combining (5.8), (5.9) and (5.10) we infer from (5.5) that
\begin{equation}
\sum_{n \leq x} \lambda(n) \gg \left( \sum_{p \leq \sqrt[3]{x}} \lambda(p^2) + \sum_{p \leq \sqrt[3]{x}} \lambda(p) \right)^2 - c \pi(\sqrt[3]{x})
\end{equation}
where $c > 0$ is an absolute constant.

We now claim that $\lambda(p^2)$ and $\lambda(p)$ cannot be simultaneously small for $p \leq \sqrt[3]{x}$. Indeed, otherwise $\lambda(p^3)$ would also be small, by (5.7), and then (5.6) would give a contradiction since $\lambda(p^4) \geq 0$ by hypothesis. Thus there exists an absolute constant $\alpha > 0$ such that
\[ \lambda(p^2) + \lambda(p) \geq \alpha \quad (p \leq \sqrt[3]{x}). \]

From (5.11) we now conclude using the prime number theorem that
\[ \sum_{n \leq x} \lambda(n) \gg \frac{\sqrt{x}}{\log^2 x} \]
as claimed.
6. Proof of Theorem. Assuming that \( \lambda(n) \geq 0 \) for \( n \leq x \), we infer from Propositions 2 and 3 that

\[
\frac{\sqrt{x}}{\log^2 x} \ll k \log^8 k \cdot x^{2/3 \log k} \quad (x > 1).
\]

Clearly for \( x \) large this is a contradiction.

To get an explicit bound, quoting the more general Lemma 4 in [4] we see that (6.1) implies that

\[
x \ll \left( \frac{A}{\delta^2} \right)^{1/\delta} \log^{2/\delta} \left( \frac{A}{\delta^2} \right)
\]

where

\[
A := k \log^8 k, \quad \delta := \frac{1}{2} - \frac{2}{3 \log k}.
\]

We have

\[
\frac{1}{\delta} = 2 + \frac{8}{3 \log k - 4}.
\]

Hence

\[
A^{1/\delta} = (k \log^8 k)^{2+8/(3 \log k-4)} \ll k^2 \log^{16} k
\]

and

\[
\log^{2/\delta} \left( \frac{A}{\delta^2} \right) \ll \log^{2/\delta} A \ll \log^4 k.
\]

Thus (6.2) implies that

\[
x \ll k^2 \log^{20} k.
\]

Therefore we obtain the assertion of the Theorem.

References


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