ACTA ARITHMETICA
129.1 (2007)

Density of rational points on elliptic fibrations

by

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1. Introduction. Let V be a Fano variety defined over $\mathbb{Q}$, and let H denote the anticanonical height function defined on V. For any open set $U \subset V$, we define the counting function

$$N_{U,H}(B) := \#\{x \in U(\mathbb{Q}) : H(x) \leq B\}.$$

Manin *et al.* have put forward a conjecture about the asymptotic behavior of $N_{U,H}$ for suitable open sets U. For simplicity, we state a weak form of the conjecture for two-dimensional Fano varieties, or del Pezzo surfaces.

CONJECTURE (Manin *et al.*). *Let V be a del Pezzo surface with at most rational double points over $\mathbb{Q}$. Let H be the anticanonical height function defined on V. Then for the open set U obtained by deleting the exceptional divisors from V, and for any positive ε, we have*

$$N_{U,H}(B) = O(B^{1+\varepsilon}).$$

Geometrically, the smooth del Pezzo surfaces are projective surfaces isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ or the blow up of $\mathbb{P}^2$ in up to eight points in general position. In the latter case the del Pezzo surface has degree equal to 9 minus the number of points blown up. The arithmetic of del Pezzo surfaces over number fields has drawn a huge amount of research in recent times. It is well known that the arithmetic complexity of the surface increases as the degree falls. The above conjecture has been proved for the surfaces of degree 6 and higher. For the surfaces of degree 5 and degree 4, the conjecture has been established in a few instances. For lower degree the situation is rather less satisfactory.

The smooth cubic surfaces are del Pezzo surfaces of degree 3. If such a surface contains three coplanar lines defined over $\mathbb{Q}$, then Heath-Brown [4] has proved that $N_{U,H}(B) \ll_\varepsilon B^{4/3+\varepsilon}$. In particular, this bound holds for the Fermat cubic surface $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$. Better estimates are available

2000 *Mathematics Subject Classification*: 11D45, 14G05.
Key words and phrases: elliptic pencil, del Pezzo surfaces, rational points.

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for singular cubics. (See the survey article by T. D. Browning [2] on the subject, and the references listed therein.)

For the del Pezzo surfaces of degree 2, even the basic bound $N_{U,H}(B) \ll_\varepsilon B^{2+\varepsilon}$ is not yet known. In this case, Broberg [1] has established the weaker bound $N_{U,H}(B) \ll_\varepsilon B^{9/4+\varepsilon}$. However, for the del Pezzo surfaces of degree 2 given by the equation

$$Y^2 = ax_0^4 + bx_1^4 + c^2x_2^4,$$

where a, b and c are non-zero integers, one may do better. For fixed x_0 and x_1, using the standard estimate for the divisor function, we see that there are $O(B^\varepsilon)$ pairs of integers (Y, x_2), satisfying the equation

$$(Y - cx_2^2)(Y + cx_2^2) = ax_0^4 + bx_1^4.$$

Summing over all possible x_0 and x_1, we get $N_{U,H}(B) \ll_\varepsilon B^{2+\varepsilon}$. (The author wishes to thank the referee for this observation.)

In this article we focus on the del Pezzo surfaces of degree 1. These surfaces are intimately linked to elliptic surfaces. Recall that an *elliptic surface* is a morphism $\pi : \mathcal{S} \to \mathcal{C}$, where $\mathcal{S}$ is a projective surface and $\mathcal{C}$ is a smooth projective curve, and such that for all but finitely many points $t \in \mathcal{C}$, the fiber $\pi^{-1}(t)$ is an elliptic curve (with a choice of 0-section). Now, in a del Pezzo surface of degree 1 the anticanonical system consists of irreducible cubic curves and has a fixed point. Suppose the fixed point is rational. Then blowing up that point yields an elliptic surface with all its fibers irreducible. Of course, since the del Pezzo surface is rational the associated elliptic surface is also rational. Conversely, starting from a rational elliptic surface we can produce a del Pezzo surface by contracting the 0-section.

A rational elliptic surface over a projective line has a Weierstrass model given by

$$Y^2 = X^3 + g_4(T)X + g_6(T),$$

where g_4 is a polynomial of degree 4 and g_6 is a polynomial of degree 6 (see [5]). Our next task is to define an appropriate height function on such elliptic fibrations. Perhaps the easiest way is to embed the surfaces in the weighted projective space $\mathbb{P}(1,1,2,3)$. To do so, we homogenize g_4 and g_6 by introducing a new variable S. Then assigning weight 1 to both S and T, weight 2 to X and weight 3 to Y, we see that the above equation is homogeneous of degree 6. Hence it defines a variety in the weighted projective space $\mathbb{P}(1,1,2,3)$. Then the anticanonical height of a point $P = (S,T,X,Y)$ with reduced (weighted) homogeneous coordinates is given by

$$H(P) = \max\{|S|, |T|, |X|^{1/2}, |Y|^{1/3}\}.$$

Our problem is now reduced to counting the number of integers S, T, X and Y, reduced with respect to the above weights, with $|S| < B$, $|T| < B$,

$|X| < B^2$ and $|Y| < B^3$, and satisfying the Weierstrass equation. Moreover, they need to satisfy certain inequalities so that the corresponding point lies outside the lines.

In the next section we apply the theory of elliptic curves to establish the following basic bound for the density of rational points on certain del Pezzo surfaces of degree 1.

THEOREM A. *For surfaces given by*

(i) $Y^2 = X^3 + f_4(T,S)X$,
(ii) $Y^2 = X^3 + bf_2(T,S)^3$, *or*
(iii) $Y^2 = X^3 + bf_3(T,S)^2$,

where $f_k(T,S)$ *denotes a form defined over* $\mathbb{Z}$ *of degree* k, *and* $b \in \mathbb{Z}$, *we have* $N_{U,H}(B) \ll_\varepsilon B^{2+\varepsilon}$.

In the last section, we study the surface given by $Y^2 = X^3 + Q(S,T)^3$, where Q is a positive-definite quadratic form defined over $\mathbb{Z}$. Using a uniform bound of Heath-Brown's on the density of rational points on plane conics, we establish a much refined bound.

THEOREM B. *For a positive-definite quadratic form* Q *defined over* $\mathbb{Z}$, *let* $\mathcal{S}$ *denote the surface given by* $Y^2 = X^3 + Q(S,T)^3$. *Let* $N(B)$ *denote the cardinality of the set*

$$\{(S,T,X,Y) \in \mathcal{S} : 0 < |X| < B^2,\ |S| < B,\ |T| < B,\ 0 < |Y| < B^3\}.$$

Then $N(B) \ll_\varepsilon B^{4/3+\varepsilon}$.

REMARK. In the above theorem, we have put the inequality $Y \neq 0$ to avoid the points on the curve $X = -Q(S,T)$.

The results appearing in this article are taken from the last chapter of the author's Ph.D. thesis [6]. The author wishes to thank his advisor Prof. Andrew Wiles for many enlightening discussions. The author also thanks the referee for a thorough reading of the manuscript and many helpful suggestions.

2. Proof of Theorem A. The proof involves some well known results from the theory of elliptic curves. We start with the theory of descent. (For details see Silverman [8].) Suppose $E/\mathbb{Q}$ is an elliptic curve with an m-torsion point, P say. For all purposes (here), we may assume that $E(\mathbb{Q})[m] \cong \mu_m$ or $\mu_m \times \mu_m$ (depending on the choice of basis). Then we define the elliptic curve $E'/\mathbb{Q} = E/\langle P\rangle$ and an isogeny

$$\phi :\ E \to E'.$$

Then $\ker(\phi) = \langle P\rangle$. Let

$$S = \{\infty\} \cup \{p : E \text{ has bad reduction at } p\} \cup \{p : p \mid m\}.$$

We have the fundamental exact sequence

$$0 \to E'(\mathbb{Q})/\phi(E(\mathbb{Q})) \to S^{(\phi)}(E/\mathbb{Q}) \to \mathbf{ST}(E/\mathbb{Q})[\phi] \to 0$$

with the Selmer group $S^{(\phi)}(E/\mathbb{Q}) \subset H^1(G_{\overline{K}/K}, E[\phi]; S)$, and the Tate–Shafarevich group $\mathbf{ST}$. Now a choice of a basis of $E(\mathbb{Q})[m]$ gives an isomorphism

$$H^1(G_{\overline{K}/K}, E[\phi]; S) \cong \mathbb{Q}(S, m)^i,$$

where $i = 1$ or 2, depending on the dimension of $E(\mathbb{Q})[m]$. Hence

$$\#E'(\mathbb{Q})/\phi(E(\mathbb{Q})) \le \#S^{(\phi)}(E/\mathbb{Q}) \le m^{2\#S}.$$

Let $\widetilde{\phi}$ be the dual isogeny. Then we also have the inequality

$$\#E(\mathbb{Q})/\widetilde{\phi}(E'(\mathbb{Q})) \le m^{2\#S}.$$

Then using the exact sequence

$$0 \to \frac{E'(\mathbb{Q})[\widetilde{\phi}]}{\phi(E(\mathbb{Q})[m])} \to \frac{E'(\mathbb{Q})}{\phi(E(\mathbb{Q}))} \to \frac{E(\mathbb{Q})}{mE(\mathbb{Q})} \to \frac{E(\mathbb{Q})}{\widetilde{\phi}(E'(\mathbb{Q}))} \to 0,$$

we get

$$\#\frac{E(\mathbb{Q})}{mE(\mathbb{Q})} \ll m^{4\#S}.$$

Hence

$$\operatorname{rank}_{\mathbb{Z}} E(\mathbb{Q}) \le \dim_{\mathbb{Z}/m\mathbb{Z}} \frac{E(\mathbb{Q})}{mE(\mathbb{Q})} \ll \#S,$$

where the implied constant is absolute. Using this we get the following.

Lemma 1. *Let $E/\mathbb{Q}$ be an elliptic curve given by any of the following Weierstrass equations*:

(1) $Y^2 = X^3 + DX$,
(2) $Y^2 = X^3 + bD^2$, *or*
(3) $Y^2 = X^3 + bD^3$,

where D and b are integers, and b is considered to be fixed. Then

$$\operatorname{rank} E(\mathbb{Q}) \ll 1 + \nu(D),$$

where $\nu(D)$ denotes the number of prime factors of D. Here the implied constant is 2 *in case* (1), *and depends only on b in cases* (2) *and* (3).

The above lemma and the following result of Silverman (see [7]) are the crucial ingredients in our argument.

Lemma 2 (Silverman). *For the elliptic curves $Y^2 = X^3 + D$ and $Y^2 = X^3 + DX$, the number of integral points is bounded by $c_1 2^{c_2 \times \operatorname{rank} E}$ for some absolute constants c_1 and c_2.*

Coupling the above two results together, we find that the number of integral points on $E_D : Y^2 = X^3 + DX$ is bounded by $O(2^{c_3 \times \nu(D)}) = O(\mathbf{d}(D)^{c_3})$. Here $\mathbf{d}$ denotes the divisor function, and c_3 is an absolute constant. Hence

$$\#E_D(\mathbb{Z}) \ll_\varepsilon D^\varepsilon.$$

Now consider the surface

$$V : \quad Y^2 = X^3 + f_4(T,S)X.$$

Specializing $S = s$ and $T = t$, with s and $t \in \mathbb{Z}$, $|s| < B$ and $|t| < B$, we get an elliptic curve

$$E_{f_4(t,s)} : \quad Y^2 = X^3 + f_4(t,s)X$$

of the above type (ignoring any fourth power that might appear). Then each of these curves has at most $O(B^\varepsilon)$ integral points. Adding up the contribution of all the B^2 many elliptic fibers we conclude that $N_{U,H}(B) \ll_\varepsilon B^{2+\varepsilon}$, for any open subset U of V.

Similarly we conclude the same bound for the surfaces given by

$$Y^2 = X^3 + bf_2(T,S)^3, \qquad Y^2 = X^3 + bf_3(T,S)^2.$$

This proves Theorem A. ∎

3. Proof of Theorem B. Let $K = \mathbb{Q}(\sqrt{-3})$, and $\omega = (-1 + \sqrt{-3})/2$ be a cube root of unity in K. Then the ring of integers in K is given by $O_K = \mathbb{Z}[\omega]$, and the norm form is given by $\mathbf{N}(a + \omega b) = a^2 - ab + b^2$.

LEMMA 3. *Let d_1 and d_2 be a pair of coprime square-free integers, and let $d = d_1 d_2$. Then the number of integer solutions of the pair of equations*

$$d_1 y_1^2 = x + z, \qquad d_2 y_2^2 = x^2 - xz + z^2,$$

with the restriction $0 < |x| < B$, $0 < |z| < B$, $y_i \neq 0$, and $(x,z) = 1$, is bounded by $O((1 + \sqrt{B/d})(Bd)^\varepsilon)$.

Proof. Since x and z are coprime, it follows that $x + \omega z$ and $x + \omega^2 z$ are coprime except for a possible common factor $1 - \omega$. Hence from the second equation we conclude that

$$\begin{aligned} \text{either} \quad & x + \omega z = (a + \omega b)\eta^2, \\ \text{or} \quad & x + \omega z = (a + \omega b)(1 - \omega)\eta^2, \end{aligned}$$

for some $\eta \in O_K$ and $a, b \in \mathbb{Z}$ satisfying $\mathbf{N}(a + \omega b) = d_2$. We note that there are at most $O(d^\varepsilon)$ choices for the pair (a, b).

Now write $\eta = M + \omega N$; then $\eta^2 = (M^2 - N^2) + \omega(2MN - N^2)$. We will only consider the first equation. The other one can be dealt with similarly.

We get

$$x = a(M^2 - N^2) - b(2MN - N^2),$$
$$z = a(2MN - N^2) + b(M^2 - N^2) - b(2MN - N^2).$$

Also,

$$M^2 + N^2 + (M - N)^2 = 2(M^2 - MN + N^2) = 2\mathbf{N}(\eta) \ll B/\sqrt{d_2}.$$

Hence $|M|, |N| \ll \sqrt{B}/d_2^{1/4}$. Putting the values of x and z in the equation $d_1 y_1^2 = x + z$, we get

$$d_1 y_1^2 = a(M^2 + 2MN - 2N^2) + b(M^2 - 4MN + N^2),$$

that is,

$$d_1 y_1^2 = (a + b)M^2 + 2(a - 2b)MN - (2a - b)N^2.$$

Let $\alpha = a + b$, $\beta = a - 2b$, and $\gamma = 2a - b = \alpha + \beta$. Then from above we get

$$d_1 y_1^2 = \alpha M^2 + 2\beta MN - \gamma N^2.$$

Now we want to use the following result of Heath-Brown [3] to count the number of solutions of the ternary quadratic equations. Before that we observe that $M \neq 0$, $N \neq 0$, and if $\delta = (M, N, y_1)$, then $\delta \mid y_2$. But y_1 and y_2 are coprime, so we are only concerned with primitive solutions of the above ternary form.

Lemma 4 (Heath-Brown). *Let q be a ternary quadratic form with matrix $\mathcal{M}$. Let $\Delta = |\det \mathcal{M}|$, and assume that $\Delta \neq 0$. Write Δ_0 for the highest common factor of the 2×2 minors of $\mathcal{M}$. Then the number of primitive integer solutions of $q(x) = 0$ in the box $|x_i| < R_i$ is*

$$\ll_\varepsilon \left\{1 + \left(\frac{R_1 R_2 R_3 \Delta_0^2}{\Delta}\right)^{1/3+\varepsilon}\right\}(R_1 R_2 R_3)^\varepsilon$$

for any $\varepsilon > 0$.

We apply the result to the ternary quadratic form

$$d_1 y_1^2 - \alpha M^2 - 2\beta MN + \gamma N^2.$$

In this case the determinant is

$$\Delta = |d_1(\alpha\gamma + \beta^2)| = d_1(\alpha^2 + \alpha\beta + \beta^2) = 3d_1 d_2 = 3d.$$

Also, the gcd of all 2×2 minors is $(\alpha d_1, \beta d_1, \gamma d_1, \alpha\gamma + \beta^2)$, which is either 1 or 3. The restrictions on y_1, M and N are given by: $0 < |y_1| < \sqrt{B}/\sqrt{d_1}$ and $|M|, |N| < \sqrt{B}/d_2^{1/4}$. Hence in Heath-Brown's notation

$$R_1 R_2 R_3 = \frac{\sqrt{B}}{\sqrt{d_1}}\left(\frac{\sqrt{B}}{d_2^{1/4}}\right)^2 = \frac{B^{3/2}}{d^{1/2}}.$$

So, Heath-Brown's estimate implies that the number of possible primitive solutions is given by

$$\ll_\varepsilon \left\{1+\sqrt{\frac{B}{d}}\right\}(Bd)^\varepsilon.$$

The lemma follows. ∎

LEMMA 5. *The number of integer solutions to the equation* $Y^2 = X^3+Z^3$ *with the restrictions* $0 < |X| < B^2$, $0 < |Z| < B^2$ *and* $Y > 0$ *is bounded by* $O(B^{4/3+\varepsilon})$.

Proof. Suppose (X,Y,Z) is a solution of the equation $Y^2 = X^3 + Z^3$ satisfying the size restrictions. Let $g = \gcd(X,Z)$. Then $g < B^2$. Suppose $g = d\tau^2$, where d is square-free. Then $\tau^6 \mid (X^3+Z^3) = Y^2$. But for primes $p \mid d$, $v_p(g)$ is odd, say $2k+1$. Then $p^{6k+3} \mid Y^2$, and hence $p^{6k+4} \mid Y^2$. So we conclude that $d^2\tau^3 \mid Y$.

Defining $x = X/g$, $z = Z/g$, and $y = Y/d^2\tau^3$, we see that (x,y,z) satisfies the equation

$$dy^2 = x^3 + z^3 \quad \text{with } (x,z) = 1.$$

Now since $d^2\tau^3 \mid Y$ and $Y < B^3$, we get $d < (B/\tau)^{3/2}$. Also, $0 < |x| < B^2/d\tau^2$, $0 < |z| < B^2/d\tau^2$ and $0 < y < 2B^3/d^2\tau^3$. As the form of the equation does not depend on τ, we do the calculation writing B in place of B/τ, and then replace τ back at the end and sum over the range $1 \le \tau < B$. However, from the form of the estimate it will be clear that the sum over τ only increases the bound by a constant, which will not concern us.

We factorize the right hand side of the above equation, and get

$$dy^2 = (x+z)(x^2 - xz + z^2) \quad \text{with } (x,z) = 1.$$

Let κ denote the largest common square-free factor of $x+z$ and x^2-xz+z^2. Then we conclude that

$$d_1\kappa y_1^2 = x+z, \qquad d_2\kappa y_2^2 = x^2 - xz + z^2,$$

where $d_1\kappa$ and $d_2\kappa$ are the square-free parts, $d = d_1d_2$ and $y = \kappa y_1 y_2$. Also, since $(x,z) = 1$, it follows that κ is either 1 or 3. So, the problem is reduced to counting the number of solutions of the pair of equations

$$d_1y_1^2 = x+z, \qquad d_2y_2^2 = x^2 - xz + z^2,$$

with the restriction $0 < |x| < B^2/d$, $0 < |z| < B^2/d$, and $y_i \neq 0$. (The case of $\kappa = 3$ is just similar.)

For $d \le B^{4/3}$, we use the bound obtained in Lemma 3. The number of solutions to the above pair of equations, for any given d, is bounded by $O((1+B/d)B^\varepsilon)$. Then adding the contribution of all the $O(B^{4/3})$ possible values for d, we deduce that the number of solutions is bounded above by $O(B^{4/3+\varepsilon})$.

For $d > B^{4/3}$, we fix x and z and observe that the trivial estimate for the divisor function implies that the total number of d_1, d_2, y_1 and y_2 satisfying the above pair of equations is bounded by $O(B^{\varepsilon})$. Then adding up the contributions of all x and z, we conclude that number of solutions to the equation $Y^2 = X^3 + Z^3$ with $(X, Z) > B^{4/3}$ is bounded by $O((B^2/B^{4/3})^2 B^{\varepsilon}) = O(B^{4/3+\varepsilon})$. ∎

Theorem B follows directly from the last lemma, by setting $Z = Q(S, T)$, and observing that there are at most $O(B^{\varepsilon})$ pairs (S, T) for each fixed Z. ∎

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Received on 6.11.2006
and in revised form on 5.4.2007 (5312)