Zero-density estimate of $L$-functions attached to Maass forms

by

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1. Introduction. Zero-density theorems for $L$-functions to the right of the critical line play a significant role in analytic number theory. These results have been established in various research papers by many mathematicians for different $L$-functions. As a sample we quote a few of them below. Let $L(s)$ be any normalised $L$-function with the first coefficient being 1, which is absolutely convergent in $\Re s > 1$ and satisfies a functional equation of the Riemann zeta-type. We define, for $\sigma \geq 1/2$,

\begin{equation}
N_L(\sigma, T) := \#\{\varrho = \beta + i\gamma : L(\varrho) = 0, \beta \geq \sigma, |\gamma| \leq T\}.
\end{equation}

For the Riemann zeta-function $\zeta(s)$, we know for example the familiar result of Ingham (for $\sigma \geq 1/2$) that

\begin{equation}
N_\zeta(\sigma, T) \ll T^{3(1-\sigma)/(2-\sigma)}(\log T)^5.
\end{equation}

In the case of Dirichlet $L$-functions, there is an averaging result of Bombieri (see [1]) which states that when $T \leq Q$,

\begin{equation}
\sum_{q \leq Q} \sum_{\chi} N_{\chi}(\sigma, T) \ll TQ^{8(1-\sigma)/(3-2\sigma)}(\log Q)^{10}.
\end{equation}

Here the superscript $\ast$ means that the sum is over primitive characters. It is also known (see [5] or [21]) that

\begin{equation}
N_\zeta(\sigma, T) \ll T^{12(1-\sigma)/5}(\log T)^{100}.
\end{equation}

We also refer to [3] for sharp density results for the zeros of $\zeta(s)$ in certain ranges of $\sigma$. Some zero-density theorems for $L$-functions can be found in [5], [13], [14] and [21]. As a sample, we quote a result due to Montgomery (see [12]) which we state as
THEOREM A. For $T \geq 2$, let

$$M(T) = \max_{2 \leq t \leq T} |\zeta(\alpha + it)|.$$ 

Then for $3/4 \leq \sigma \leq 1$, we have

$$N_\zeta(\sigma, T) \ll \left\{ M(5T)(\log T)^6 \right\}^{8(1-\sigma)(3\sigma-2)/(4\sigma-3)(2\sigma-3)}(\log T)^{11}.$$ 

The central idea is to study how frequently certain Dirichlet polynomials can be large. This main idea was developed and used first by Montgomery (see [12]) and later by many mathematicians (see [3], [8], and [16]). It should be mentioned that zero-density theorems have been established in various situations: for the Dedekind zeta-functions of a number field (see [2]), for the $L$-function attached to a holomorphic cusp form for the full modular group (see [6]), and for the symmetric square $L$-function attached to a holomorphic cusp form for the full modular group (see [20]).

Let $s = \sigma + it$ denote a complex variable. The parameter $T > 0$ will be chosen to be sufficiently large. The letters $C, C'$ etc. denote positive constants which are not necessarily the same at each occurrence. Let $f$ denote a normalised (i.e. the first Fourier coefficient is 1) Maass cusp form for $SL(2, \mathbb{Z})$ which is an eigenfunction of all the Hecke operators $T(n)$ as well as the reflection operator $T_{-1}: z \mapsto -\bar{z}$. We have $T(n)f = \lambda(n)f$ for $n \in \mathbb{N}$, and $\lambda(1) = 1$. For $\sigma > 1$, we define the standard $L$-function of $f$ as

$$(1.5) \quad L(s, f) := \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p (1 - \lambda(p)p^{-s} + p^{-2s})^{-1}.$$ 

We note that $L(s, f)$ extends as an entire function to the whole complex plane and it satisfies a Riemann zeta-type functional equation under $s \mapsto 1 - s$ (see [7]). We also note that $N_L(\sigma, T) \ll T \log T$ for $1/2 \leq \sigma \leq 1/2 + 1/\log T$. The zero-density estimates to the right of the critical line in the case of the standard $L$-function attached to a normalised Maass cusp form are of great interest and seem to be unavailable in the literature.

The main aim of this paper is to prove

THEOREM 1. For $\sigma \geq 1/2 + 1/\log T$, we have

$$N_L(\sigma, T) \ll T^{4(1-\sigma)/(3-2\sigma)}(\log T)^{26},$$

where the implied constant depends on $f$.

As an application, we can extend Theorem 1 of [19] by Ramachandra and the first author. Precisely, we can prove the local theorem on the zeros of $L(s, f)$ in the neighbourhood of the critical line:
Theorem 2. If \( L(s, f) \neq 0 \) in the rectangle
\[
\left\{ \frac{1}{2} + \frac{1}{10 \log \log T} < \sigma \leq 1, \ T - H \leq t \leq T + H \right\}
\]
with \( H = C \log \log \log T, \ T \geq 100, \) then there is at least one zero of \( L(s, f) \) in the disc of radius \( C'(\log \log T)^{-1} \) with centre \( 1/2 + iT \). Here \( C, C' \) are effective positive constants depending on \( f \).

Remark. The zero-counting argument adapted in this paper is somewhat familiar (see for example [17] and [20]). However, the real difficulty in this situation lies in getting certain mean-value estimates of the zero-detector function \( F_2(s) \) on certain lines in terms of precise log powers. For this, we need to first establish upper bounds on the discrete mean involving certain arithmetical functions. We prove these estimates in a sequence of lemmas in Section 3.

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2. Notation and preliminaries. The letters \( C, A \) and \( B \) (with or without subscripts) denote effective positive constants unless otherwise specified. They need not be the same at every occurrence. Throughout the paper we assume \( T \geq T_0 \) where \( T_0 \) is a large positive constant. We write \( f(x) \ll g(x) \) to mean \( |f(x)| < C_1 g(x) \) for \( x \geq x_0 \) where \( C_1 \) is some absolute positive constant (sometimes we denote this by the \( O \) notation also). Let \( s = \sigma + it \) and \( w = u + iv \). The implied constants are all effective but they will depend on the form \( f \) in question.

For \( \sigma > 1 \), let
\[
\frac{1}{L(s, f)} = \sum_{n=1}^{\infty} \mu^*(n) n^{-s}.
\]
Then \( \mu^*(n) \) is a multiplicative function and its values on prime powers are as follows:
\[
\mu^*(p^a) = \begin{cases} 
1 & \text{if } a = 0, \\
-\lambda(p) & \text{if } a = 1, \\
1 & \text{if } a = 2, \\
0 & \text{if } a \geq 3.
\end{cases}
\]
We keep in mind that \( \mu^*(n) = 0 \) unless \( n \) is cube-free.
3. Some lemmas

**Lemma 3.1.** We have the estimate
\[ \sum_{n \leq x} |\lambda(n)|^4 \ll x \log x. \]

**Proof.** Let \( L_4(s) = \sum_{n=1}^{\infty} (\lambda(n))^4/n^s \). There is a dominant Dirichlet series \( L^*_4(s) \) with positive coefficients \( \lambda^*(n) \) which has the property that
\[ \sum_{n \leq x} |\lambda(n)|^4 \leq \sum_{n \leq x} \lambda^*(n) \]
(see for example [4]). In fact, \( L^*_4(s) \) has a pole of order 2 at \( s = 1 \), is otherwise analytic and is given by
\[ L^*_4(s) = L(s, f, \sqrt[4]{1})(L(s, f, \sqrt{2}))^3(\zeta(s))^2. \]
The two functions \( L(s, f, \sqrt[4]{1}) \) and \( L(s, f, \sqrt{2}) \) on the right hand side are respectively the symmetric fourth and symmetric square \( L \)-series associated to \( f \). As the analytic continuation and functional equations of these series are known (see for example [9], [10]), it follows from a standard Tauberian argument that
\[ \sum_{n \leq x} \lambda^*(n) \asymp Cx \log x \]
where the constant \( C \) depends on \( f \). Now, the lemma follows from (3.1). \( \blacksquare \)

**Lemma 3.2.** We have the estimate
\[ \sum_{l \leq x} \frac{(\mu^*(l))^2}{l} \ll \log x. \]

**Proof.** We note that \( \mu^*(l) = 0 \) unless \( l \) is cube-free. So we write \( l = d_1^2d_2 \) with \( (d_1, d_2) = 1 \) and \( d_1, d_2 \) square-free. Then
\[ (\mu^*(l))^2 = (\mu^*(d_1^2))^2(\mu^*(d_2))^2 = (\lambda(d_2))^2. \]
Using (3.2), we have
\[ \sum_{l \leq x} \frac{(\mu^*(l))^2}{l} = \sum_{d_1^2d_2 \leq x \atop (d_1, d_2) = 1, d_1, d_2 \text{ square-free}} \frac{(\lambda(d_2))^2}{d_1^2d_2} \]
\[ = \sum_{d_1^2 \leq x} \frac{1}{d_1^2} \sum_{d_2 \leq \frac{x}{d_1^2}} (\lambda(d_2))^2 \frac{d_2}{d_2} \ll (\log x) \left( \sum_{d_1^2 \leq x} \frac{1}{d_1^2} \right) \]
\[ \ll \log x, \]
since \( \sum_{m \leq Y} (\lambda(m))^2 \ll Y \) (see [7]). \( \blacksquare \)
Lemma 3.3. We have the estimate
\[ \sum_{l \leq x} \frac{(\mu^* (l))^4}{l} \ll (\log x)^2. \]

Proof. From (3.2), we observe that
\[
\sum_{l \leq x} \frac{(\mu^* (l))^4}{l} = \sum_{d_1^2 d_2 \leq x} \frac{(\mu^* (d_1^2 d_2))^4}{d_1^2 d_2} \\
\leq \sum_{d_1^2 d_2 \leq x} \frac{(\lambda (d_2))^4}{d_1^4 d_2} \leq \sum_{d_1^2 \leq x} \frac{1}{d_1^4} \sum_{d_2 \leq x d_1^{-2}} \frac{(\lambda (d_2))^4}{d_2} \\
\ll (\log x)^2,
\]
on using the estimate in Lemma 3.1. ■

Lemma 3.4. Let
\[ c(n) = \sum_{\substack{d|n \\quad d \leq T}} \mu^* (d) \lambda (n/d). \]
Then
\[ \sum_{n \leq x} |c(n)|^2 \ll x (\log x)^{17}. \]

Proof. For any \( a_i \in \mathbb{R} \) and \( m \in \mathbb{N} \), we have
\[ \left( \sum_{i=1}^{m} a_i \right)^2 \leq m^2 \sum_{i=1}^{m} a_i^2. \]
Therefore (with \( \tau(n) \) being the number of positive divisors of \( n \)),
\[ (c(n))^2 \leq (\tau(n))^2 \sum_{\substack{d|n \\quad d \leq T}} (\mu^* (d))^2 (\lambda (n/d))^2. \]
Hence we have
\[
S := \sum_{n \leq x} (c(n))^2 \leq \sum_{lm \leq x} (\tau(lm))^2 (\mu^* (l))^2 (\lambda (m))^2 \\
\leq \sum_{l \leq x} \sum_{m \leq x l^{-1}} (\tau(l))^2 (\tau(m))^2 (\mu^* (l))^2 (\lambda (m))^2 \\
= \sum_{l \leq x} (\tau(l))^2 (\mu^* (l))^2 \sum_{m \leq x l^{-1}} (\tau(m))^2 (\lambda (m))^2 \\
\leq \sum_{l \leq x} (\tau(l))^2 (\mu^* (l))^2 \left\{ \left( \sum_{m \leq x l^{-1}} (\tau(m))^4 \right)^{1/2} \left( \sum_{m \leq x l^{-1}} (\lambda(m))^4 \right)^{1/2} \right\}.
\]
Since \((\tau(m))^4 \leq \tau_{24}(m)\) (where \(\tau_j(n)\) denotes the \(j\)-fold divisor function), we have
\[
\sum_{m \leq x^{1-\epsilon}} (\tau(m))^4 \ll \frac{x}{\log x}^{15}.
\]
Now, using Lemma 3.1, we note that the term within the curly bracket above is \(\ll (x/l)(\log x)^8\). Thus, we obtain
\[
S \ll x(\log x)^8 \sum_{l \leq x} \left(\frac{\tau(l)^2(\mu^*(l))^2}{l}\right)
\ll x(\log x)^8 \left(\sum_{l \leq x} \left(\frac{\tau(l)^4}{l}\right)\right)^{1/2} \left(\sum_{l \leq x} \left(\frac{\mu^*(l)^4}{l}\right)\right)^{1/2}.
\]
Now, using Lemma 3.3, we get
\[
\sum_{n \leq x} (c(n))^2 \ll x(\log x)^{17}.
\]

**Lemma 3.5** (Montgomery–Vaughan). If \(h_n\) is an infinite sequence of complex numbers such that \(\sum_{n=1}^{\infty} n|h_n|^2\) is convergent, then
\[
\left| \sum_{n=1}^{\infty} h_n n^{-it} \right|^2 dt = \sum_{n=1}^{\infty} |h_n|^2 (H + O(n)).
\]

*Proof.* See for example Lemma 3.3 of [15], or [18].

**Lemma 3.6.** If \(N_L(\sigma, T, T+1)\) denotes the number of zeros \(\varrho = \beta + i\gamma\) of \(L(s, f)\) with \(\beta \geq \sigma, T \leq \gamma < T+1\), then
\[
N_L(\sigma, T, T+1) \ll \log T.
\]

*Proof.* We define
\[
F_1(s) = \frac{L(s, f)}{\prod_{\varrho} (1 - \frac{s-s_0}{\varrho-s_0})}
\]
where \(\varrho\) in the product runs over the zeros \(\varrho = \beta + i\gamma\) of \(L(s, f)\) with \(0 \leq \beta \leq 1\) and \(T < \gamma < T+1\) and \(s_0 = \sigma_0 + i\gamma\) with \(\sigma_0\) sufficiently large. We note that
\[
|F_1(s_0)| = |L(s_0, f)| \geq 1 - \sum_{n=2}^{\infty} \frac{|\lambda(n)|}{n^{\sigma_0}}
\geq 2 - \left(\sum_{n=1}^{\infty} \frac{|\lambda(n)|^2}{n^{\sigma_0}}\right)^{1/2} (\zeta(\sigma_0))^{1/2} \geq C
\]
for sufficiently large \(\sigma_0\) which may depend upon \(f\) (note that both the series \(\sum_{n=1}^{\infty} |\lambda(n)|^2 n^{-\sigma_0}\) and \(\zeta(\sigma_0)\) approach 1 as \(\sigma_0 \to \infty\)). Here \(C\) is a certain
positive constant. For $|s - s_0| = 3\sigma_0$, we have
\[
1 - \frac{s - s_0}{\rho - s_0} \geq \frac{|s - s_0|}{|\rho - s_0|} - 1 \geq \frac{3\sigma_0}{\sigma_0 - \beta} - 1 \geq 2.
\]
This implies that
\[
C < |F_1(s_0)| < \max_{|s - s_0| = 3\sigma_0} |F_1(s)| < \max_{|s - s_0| \leq 3\sigma_0} \frac{|L(s, f)|}{2^N} \ll \frac{T^C}{2^N}
\]
and hence we obtain the lemma.  

**Lemma 3.7.** For $\sigma > 1$, define
\[
F_2(s) := L(s, f) \sum_{n \leq T} \frac{\mu^*(n)}{n^s} - 1 = L(s, f) M_T(s) - 1 =: \sum_{n=1}^{\infty} \frac{c(n)}{n^s}.
\]
Then
\[
(3.3) \quad c(n) = \sum_{\substack{d|n \\ \mu^*(d)}} \lambda(n/d),
\]
and for $\sigma > 1$,
\[
F_2(s) = \sum_{n > T} c(n)/n^s.
\]

**Proof.** First we observe that
\[
\sum_{d | n} \mu^*(d) \lambda(n/d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}
\]
Now, we define
\[
(3.4) \quad a(d) := \begin{cases} \mu^*(d) & \text{if } d \leq T, \\ 0 & \text{if } d > T. \end{cases}
\]
From the definition of $F_2(s)$, we notice that
\[
(3.5) \quad c(n) = \sum_{d | n} a(d) \lambda(n/d) - \sum_{d | n} \mu^*(d) \lambda(n/d).
\]
If $n \leq T$, then $d \leq T$ (since $d$ is a divisor of $n$) so that $a(d) = \mu^*(d)$. Therefore $c(n) = 0$ for $n \leq T$. For $n > T$, the second sum in (3.5) is zero and hence from (3.4), we get (3.3).  

**4. Proof of the theorems**

**Proof of Theorem 1.** By using dyadic partitions, it is enough to prove the theorem for $T \leq \gamma \leq 2T$. We divide the rectangle bounded by the lines with real parts $\sigma, 1$ and imaginary parts $T, 2T$ into abutting smaller rectangles of height $2(\log T)^2$. From Lemma 3.6, the multiplicity of any zero $\rho$ of $L(s, f)$ is $\ll \log T$. Therefore, without loss, we can assume that the zeros are simple.
in the counting process. We count the number of those smaller rectangles of height \(2(\log T)^2\) which contain at least one zero and multiply by \(C(\log T)^3\) to get a bound for \(N_L(\sigma, T, 2T)\).

We define the zero-detector function

\[
F_2(s) := \frac{\mu^*(n)}{n^s} - 1 = \frac{L(s, f) M_T(s) - 1}{\sum_{n>T} \frac{c(n)}{n^s}}.
\]

From Lemma 3.7, we notice that

\[
c(n) = \sum_{\frac{d|n}{d \leq T}} \mu^*(d) \lambda(n/d).
\]

For any fixed zero \(\varrho = \beta + i\gamma\), we let

\[
G(s) = F_2(s) Y^{s-\varrho} e^{(s-\varrho)^2}
\]

where \(Y\) is a parameter satisfying \(T^{-A} \leq Y \leq T^A\). We select one zero \(\varrho_j\) in each of the rectangles (for \(j = 1, 2, \ldots\))

\[
\left\{ \frac{1}{2} + \frac{1}{\log T} \leq \sigma \leq 1, T + 2(j - 1)(\log T)^2 \leq t \leq T + 2j(\log T)^2 \right\}
\]

We partition these rectangles into odd and even ones. Note that for any two zeros \(\varrho, \varrho'\) in two even (respectively odd) rectangles, we have \(|\gamma - \gamma'| \geq 2(\log T)^2\). Let \(\mathcal{A}\) and \(\mathcal{B}\) denote the sets of the chosen zeros corresponding to the sets of odd and even rectangles respectively. Let \(\varrho \in \mathcal{A}\) be any typical chosen zero. By Cauchy’s residue theorem, we have

\[
\left| \frac{1}{2\pi i} \int_{R(\varrho)} \frac{G(s)}{s - \varrho} ds \right| = 1
\]

where the integral is taken over the rectangle \(R(\varrho)\) defined by

\[
R(\varrho) := \left\{ \frac{1}{2} \leq \sigma \leq 1 + \frac{1}{\log T}, |t - \gamma| \leq B(\log T)^2 \right\}.
\]

Here \(1/4 \leq B \leq 1\) is chosen such that the horizontal sides of \(R(\varrho)\) are free from zeros of \(L(s, f)\). If \(Y\) is chosen to satisfy \(T^{-A} \leq Y \ll T^A\), then the contributions from the horizontal sides of \(R(\varrho)\) to the integral are \(O(T^{-10})\) owing to the exponentially decaying factor \(e^{(s-\varrho)^2}\). We denote the vertical sides of \(R(\varrho)\) by \(V_1\) and \(V_2\) so that we have

\[
1 = O \left( \log T \left( \int_{V_1} |F_2(s)| \, dt \right) Y^{1/2 - \beta} \right.
\]

\[
+ \log T \left( \int_{V_2} |F_2(s)| \, dt \right) Y^{1 + 1/\log T - \beta} \right).
\]
\[= O\left( \log T \left( 1 + \int_{V_1} |F_2(s)| \, dt \right) Y^{1/2 - \beta} \right.
\]
\[+ \log T \left( T^{-10} + \int_{V_2} |F_2(s)| \, dt \right) Y^{1+1/\log T - \beta} \).\]

We choose \( Y \) such that
\[Y^{1/2 - \beta} \left( 1 + \int_{V_1} |F_2(s)| \, dt \right) = Y^{-\beta} \left( T^{-10} + \int_{V_2} |F_2(s)| \, dt \right).\]

Let
\[J_1(\varrho) = 1 + \int_{V_1} |F_2(s)| \, dt,\]
\[J_2(\varrho) = T^{-10} + \int_{V_2} |F_2(s)| \, dt.\]

We notice that (from Lemmas 3.2 and 3.5)
\[2T \int_{T} |M_T(1/2 + it)|^2 \, dt = \sum_{n \leq T} \frac{|\mu^*(n)|^2}{n} (T + O(n)) \ll T \log T.\]

From (5.7) of [11], we have
\[2T \int_{T} |L(1/2 + it)|^2 \, dt \ll T \log T.\]

Therefore by the Cauchy–Schwarz inequality, we find that
\[(4.2) \int_{T} |F_2(1/2 + it)| \, dt \ll T \log T\]
and using Lemma 3.5 (Montgomery–Vaughan theorem) and the estimate in Lemma 3.4, we have
\[(4.3) \int_{T} |F_2(1 + 1/\log T + it)|^2 \, dt = \sum_{n > T} \frac{|c(n)|^2}{n^{2+2/\log T}} (T + O(n))\]
\[\ll \sum_{n > T} \frac{|c(n)|^2}{n^{1+2/\log T}} = \int_{T} \frac{d(\sum_{n \leq u} |c(n)|^2)}{u^{1+2/\log T}} \ll (\log T)^{19},\]

on integrating by parts.

Note that (from (4.2) and (4.3))
\[Y = \left( \frac{J_1}{J_2} \right)^2 \geq \frac{1}{T^{-10} + T^C}, \quad Y \leq \frac{T^C}{T^{-16}},\]
so that the condition on $Y$ is satisfied. Hence we have

$$1 \leq 2C(\log T) \left( \frac{J_1}{J_2} \right)^{2(1-\beta)} J_2 = 2C(\log T)J_1^{2(1-\beta)} J_2^{2\beta-1}.$$ 

It follows from the above that

$$\sum_{\varrho \in A} J_1(\varrho) \ll T \log T \quad \text{and} \quad \sum_{\varrho \in A} (J_2(\varrho))^2 \ll (\log T)^{21}.$$ 

The same argument is applicable to the zeros in the set $B$. Thus we obtain

$$\sum_{\varrho \in A} J_1(\varrho) + \sum_{\varrho \in B} J_1(\varrho) = \sum_{\varrho \in A \cup B} J_1(\varrho) \ll T \log T$$

and similarly

$$\sum_{\varrho \in A} (J_2(\varrho))^2 + \sum_{\varrho \in B} (J_2(\varrho))^2 = \sum_{\varrho \in A \cup B} (J_2(\varrho))^2 \ll (\log T)^{21}$$

and so

$$\#\{\varrho : J_1(\varrho) \geq W_1\} \leq A \frac{T \log T}{W_1},$$

(4.4)

$$\#\{\varrho : J_2(\varrho) \geq W_2\} \leq A \frac{(\log T)^{21}}{W_2^2}.$$ 

Now we fix $W_1 = W_2^2 T$. Hence the total number of zeros coming from the two sets in (4.4) is at most

$$A(\log T)^{21} \left\{ \frac{T}{W_1} + \frac{1}{W_2^2} \right\}.$$ 

From (4.1), for the remaining zeros, we have

$$J_1(\varrho) < W_1 \quad \text{and} \quad J_2(\varrho) < W_2$$

and also

$$\frac{3}{4} \leq 2C(\log T)W_1^{2(1-\sigma)}W_2^{2\beta-1}$$

$$= 2C(\log T)W_1^{2(1-\sigma)}W_2^{2(\sigma-\beta)}W_2^{2\sigma-1}W_2^{2(\beta-\sigma)}$$

$$= 2C(\log T)W_1^{2(1-\sigma)}W_2^{2\sigma-1} \left( \frac{W_2}{W_1} \right)^{2(\beta-\sigma)}$$

$$= 2C(\log T)W_1^{2(1-\sigma)}W_2^{2\sigma-1} \left( \frac{1}{W_2T} \right)^{2(\beta-\sigma)}.$$ 

Suppose that $W_2 > 1/T$ and so $(1/W_2T)^{2(\beta-\sigma)} \leq 1$. Then we get

(4.5) $$\frac{3}{4} \leq 2C(\log T)W_1^{2(1-\sigma)}W_2^{2\sigma-1}$$

$$= 2C(\log T)(W_2^2 T)^{2(1-\sigma)}W_2^{2\sigma-1} = 2C(\log T)T^{2(1-\sigma)}W_2^{3-2\sigma}.$$
We choose
\[ W_2 = (4C \log T)^{-\frac{1}{3-2\sigma}} T^{-\frac{2(1-\sigma)}{3-2\sigma}}. \]
Clearly \( W_2 > T^{-1} \). For this choice of \( W_2 \), (4.5) implies that \( 3/4 \leq 1/2 \), which is absurd; this means that we should count only those zeros which satisfy (4.4). Hence we get
\[ N_L(\sigma, T, 2T) \ll \frac{(\log T)^{21}}{W_2^2} (\log T)^3 \ll T^{4(1-\sigma)/(3-2\sigma)} (\log T)^{26}, \]
which proves the theorem.

Proof of Theorem 2. The proof is entirely similar to the proof of Theorem 1 of [19] and hence is omitted.

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