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Zero-density estimate of L-functions attached to Maass forms

by

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1. Introduction. Zero-density theorems for L-functions to the right of the critical line play a significant role in analytic number theory. These results have been established in various research papers by many mathematicians for different L-functions. As a sample we quote a few of them below. Let $L(s)$ be any normalised L-function with the first coefficient being 1, which is absolutely convergent in $\Re s > 1$ and satisfies a functional equation of the Riemann zeta-type. We define, for $\sigma \geq 1/2$,

$$N_L(\sigma, T) := \#\{\varrho = \beta + i\gamma : L(\varrho) = 0,\ \beta \geq \sigma,\ |\gamma| \leq T\}. \tag{1.1}$$

For the Riemann zeta-function $\zeta(s)$, we know for example the familiar result of Ingham (for $\sigma \geq 1/2$) that

$$N_\zeta(\sigma, T) \ll T^{3(1-\sigma)/(2-\sigma)}(\log T)^5. \tag{1.2}$$

In the case of Dirichlet L-functions, there is an averaging result of Bombieri (see [1]) which states that when $T \leq Q$,

$$\sum_{q \leq Q} \sum_{\chi}{}^{*} N_\chi(\sigma, T) \ll TQ^{8(1-\sigma)/(3-2\sigma)}(\log Q)^{10}. \tag{1.3}$$

Here the superscript $*$ means that the sum is over primitive characters. It is also known (see [5] or [21]) that

$$N_\zeta(\sigma, T) \ll T^{12(1-\sigma)/5}(\log T)^{100}. \tag{1.4}$$

We also refer to [3] for sharp density results for the zeros of $\zeta(s)$ in certain ranges of σ. Some zero-density theorems for L-functions can be found in [5], [13], [14] and [21]. As a sample, we quote a result due to Montgomery (see [12]) which we state as

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Theorem A. *For* $T \geq 2$, *let*

$$M(T) = \max_{\substack{2 \leq t \leq T \\ \alpha \geq 1/2}} |\zeta(\alpha + it)|.$$

Then for $3/4 \leq \sigma \leq 1$, *we have*

$$N_\zeta(\sigma, T) \ll \{M(5T)(\log T)^6\}^{\frac{8(1-\sigma)(3\sigma-2)}{(4\sigma-3)(2\sigma-1)}} (\log T)^{11}.$$

The central idea is to study how frequently certain Dirichlet polynomials can be large. This main idea was developed and used first by Montgomery (see [12]) and later by many mathematicians (see [3], [8], and [16]). It should be mentioned that zero-density theorems have been established in various situations: for the Dedekind zeta-functions of a number field (see [2]), for the L-function attached to a holomorphic cusp form for the full modular group (see [6]), and for the symmetric square L-function attached to a holomorphic cusp form for the full modular group (see [20]).

Let $s = \sigma + it$ denote a complex variable. The parameter $T > 0$ will be chosen to be sufficiently large. The letters C, C' etc. denote positive constants which are not necessarily the same at each occurrence. Let f denote a normalised (i.e. the first Fourier coefficient is 1) Maass cusp form for $\mathrm{SL}(2, \mathbb{Z})$ which is an eigenfunction of all the Hecke operators $T(n)$ as well as the reflection operator $T_{-1} : z \mapsto -\overline{z}$. We have $T(n)f = \lambda(n)f$ for $n \in \mathbb{N}$, and $\lambda(1) = 1$. For $\sigma > 1$, we define the standard L-function of f as

$$L(s, f) := \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \prod_p (1 - \lambda(p)p^{-s} + p^{-2s})^{-1}. \tag{1.5}$$

We note that $L(s, f)$ extends as an entire function to the whole complex plane and it satisfies a Riemann zeta-type functional equation under $s \mapsto 1 - s$ (see [7]). We also note that $N_L(\sigma, T) \ll T \log T$ for $1/2 \leq \sigma \leq 1/2 + 1/\log T$. The zero-density estimates to the right of the critical line in the case of the standard L-function attached to a normalised Maass cusp form are of great interest and seem to be unavailable in the literature.

The main aim of this paper is to prove

Theorem 1. *For* $\sigma \geq 1/2 + 1/\log T$, *we have*

$$N_L(\sigma, T) \ll T^{4(1-\sigma)/(3-2\sigma)} (\log T)^{26},$$

where the implied constant depends on f.

As an application, we can extend Theorem 1 of [19] by Ramachandra and the first author. Precisely, we can prove the local theorem on the zeros of $L(s, f)$ in the neighbourhood of the critical line:

THEOREM 2. *If $L(s,f) \neq 0$ in the rectangle*

$$\left\{\frac{1}{2} + \frac{1}{10\log\log T} < \sigma \le 1,\, T - H \le t \le T + H\right\}$$

with $H = C\log\log\log T$, $T \ge 100$, then there is at least one zero of $L(s,f)$ in the disc of radius $C'(\log\log T)^{-1}$ with centre $1/2 + iT$. Here C, C' are effective positive constants depending on f.

REMARK. The zero-counting argument adapted in this paper is somewhat familiar (see for example [17] and [20]). However, the real difficulty in this situation lies in getting certain mean-value estimates of the zero-detector function $F_2(s)$ on certain lines in terms of precise log powers. For this, we need to first establish upper bounds on the discrete mean involving certain arithmetical functions. We prove these estimates in a sequence of lemmas in Section 3.

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2. Notation and preliminaries. The letters C, A and B (with or without subscripts) denote effective positive constants unless otherwise specified. They need not be the same at every occurrence. Throughout the paper we assume $T \ge T_0$ where T_0 is a large positive constant. We write $f(x) \ll g(x)$ to mean $|f(x)| < C_1 g(x)$ for $x \ge x_0$ where C_1 is some absolute positive constant (sometimes we denote this by the O notation also). Let $s = \sigma + it$ and $w = u + iv$. The implied constants are all effective but they will depend on the form f in question.

For $\sigma > 1$, let

$$\frac{1}{L(s,f)} = \sum_{n=1}^{\infty} \frac{\mu^*(n)}{n^s}. \tag{2.1}$$

Then $\mu^*(n)$ is a multiplicative function and its values on prime powers are as follows:

$$\mu^*(p^a) = \begin{cases} 1 & \text{if } a = 0, \\ -\lambda(p) & \text{if } a = 1, \\ 1 & \text{if } a = 2, \\ 0 & \text{if } a \ge 3. \end{cases} \tag{2.2}$$

We keep in mind that $\mu^*(n) = 0$ unless n is cube-free.

3. Some lemmas

LEMMA 3.1. *We have the estimate*

$$\sum_{n\le x} |\lambda(n)|^4 \ll x \log x.$$

Proof. Let $L_4(s) = \sum_{n=1}^{\infty} (\lambda(n))^4/n^s$. There is a dominant Dirichlet series $L_4^*(s)$ with positive coefficients $\lambda^*(n)$ which has the property that

$$\sum_{n\le x} |\lambda(n)|^4 \le \sum_{n\le x} \lambda^*(n) \tag{3.1}$$

(see for example [4]). In fact, $L_4^*(s)$ has a pole of order 2 at $s = 1$, is otherwise analytic and is given by

$$L_4^*(s) = L(s, f, \vee^4)(L(s, f, \vee^2))^3(\zeta(s))^2.$$

The two functions $L\left(s, f, \vee^4\right)$ and $L(s, f, \vee^2)$ on the right hand side are respectively the symmetric fourth and symmetric square L-series associated to f. As the analytic continuation and functional equations of these series are known (see for example [9], [10]), it follows from a standard Tauberian argument that

$$\sum_{n\le x} \lambda^*(n) \asymp Cx \log x$$

where the constant C depends on f. Now, the lemma follows from (3.1). ∎

LEMMA 3.2. *We have the estimate*

$$\sum_{l\le x} \frac{(\mu^*(l))^2}{l} \ll \log x.$$

Proof. We note that $\mu^*(l) = 0$ unless l is cube-free. So we write $l = d_1^2 d_2$ with $(d_1, d_2) = 1$ and d_1, d_2 square-free. Then

$$(\mu^*(l))^2 = (\mu^*(d_1^2))^2(\mu^*(d_2))^2 = (\lambda(d_2))^2. \tag{3.2}$$

Using (3.2), we have

$$\begin{aligned}\sum_{l\le x} \frac{(\mu^*(l))^2}{l} &= \sum_{\substack{d_1^2 d_2\le x\\ (d_1,d_2)=1\\ d_1,d_2 \text{ square-free}}} \frac{(\lambda(d_2))^2}{d_1^2 d_2}\\ &= \sum_{d_1^2\le x} \frac{1}{d_1^2} \sum_{d_2\le x d_1^{-2}} \frac{(\lambda(d_2))^2}{d_2} \ll (\log x)\Bigg(\sum_{d_1^2\le x} \frac{1}{d_1^2}\Bigg)\\ &\ll \log x,\end{aligned}$$

since $\sum_{m\le Y} (\lambda(m))^2 \ll Y$ (see [7]). ∎

LEMMA 3.3. *We have the estimate*

$$\sum_{l\le x}\frac{(\mu^*(l))^4}{l}\ll(\log x)^2.$$

Proof. From (3.2), we observe that

$$\begin{aligned}\sum_{l\le x}\frac{(\mu^*(l))^4}{l} &= \sum_{\substack{d_1^2d_2\le x\\(d_1,d_2)=1\\d_1,d_2\text{ square-free}}}\frac{(\mu^*(d_1^2d_2))^4}{d_1^2d_2}\\ &\le\sum_{d_1^2d_2\le x}\frac{(\lambda(d_2))^4}{d_1^2d_2}=\sum_{d_1^2\le x}\frac{1}{d_1^2}\sum_{d_2\le xd_1^{-2}}\frac{(\lambda(d_2))^4}{d_2}\\ &\ll(\log x)^2,\end{aligned}$$

on using the estimate in Lemma 3.1. ∎

LEMMA 3.4. *Let*

$$c(n)=\sum_{\substack{d|n\\d\le T}}\mu^*(d)\lambda(n/d).$$

Then

$$\sum_{n\le x}|c(n)|^2\ll x(\log x)^{17}.$$

Proof. For any $a_i\in\mathbb{R}$ and $m\in\mathbb{N}$, we have

$$\Big(\sum_{i=1}^m a_i\Big)^2\le m^2\sum_{i=1}^m a_i^2.$$

Therefore (with $\tau(n)$ being the number of positive divisors of n),

$$(c(n))^2\le(\tau(n))^2\sum_{\substack{d|n\\d\le T}}(\mu^*(d))^2(\lambda(n/d))^2.$$

Hence we have

$$\begin{aligned}S:=\sum_{n\le x}(c(n))^2 &\le\sum_{lm\le x}(\tau(lm))^2(\mu^*(l))^2(\lambda(m))^2\\ &\le\sum_{l\le x}\sum_{m\le xl^{-1}}(\tau(l))^2(\tau(m))^2(\mu^*(l))^2(\lambda(m))^2\\ &=\sum_{l\le x}(\tau(l))^2(\mu^*(l))^2\sum_{m\le xl^{-1}}(\tau(m))^2(\lambda(m))^2\\ &\le\sum_{l\le x}(\tau(l))^2(\mu^*(l))^2\Big\{\Big(\sum_{m\le xl^{-1}}(\tau(m))^4\Big)^{1/2}\Big(\sum_{m\le xl^{-1}}(\lambda(m))^4\Big)^{1/2}\Big\}.\end{aligned}$$

Since $(\tau(m))^4 \le \tau_{2^4}(m)$ (where $\tau_j(n)$ denotes the j-fold divisor function), we have

$$\sum_{m\le xl^{-1}} (\tau(m))^4 \ll \frac{x}{l}(\log x)^{15}.$$

Now, using Lemma 3.1, we note that the term within the curly bracket above is $\ll (x/l)(\log x)^8$. Thus, we obtain

$$\begin{aligned} S &\ll x(\log x)^8 \sum_{l\le x} \frac{(\tau(l))^2(\mu^*(l))^2}{l} \\ &\ll x(\log x)^8 \Big(\sum_{l\le x}\frac{(\tau(l))^4}{l}\Big)^{1/2}\Big(\sum_{l\le x}\frac{(\mu^*(l))^4}{l}\Big)^{1/2}. \end{aligned}$$

Now, using Lemma 3.3, we get

$$\sum_{n\le x}(c(n))^2 \ll x(\log x)^{17}. \ \blacksquare$$

Lemma 3.5 (Montgomery–Vaughan). *If h_n is an infinite sequence of complex numbers such that $\sum_{n=1}^{\infty} n|h_n|^2$ is convergent, then*

$$\int_T^{T+H} \Big|\sum_{n=1}^{\infty} h_n n^{-it}\Big|^2 dt = \sum_{n=1}^{\infty} |h_n|^2 (H + O(n)).$$

Proof. See for example Lemma 3.3 of [15], or [18]. ■

Lemma 3.6. *If $N_L(\sigma, T, T+1)$ denotes the number of zeros $\varrho = \beta + i\gamma$ of $L(s,f)$ with $\beta \ge \sigma$, $T \le \gamma < T+1$, then*

$$N_L(\sigma, T, T+1) \ll \log T.$$

Proof. We define

$$F_1(s) = \frac{L(s,f)}{\prod_\varrho \big(1 - \frac{s-s_0}{\varrho - s_0}\big)}$$

where ϱ in the product runs over the zeros $\varrho = \beta + i\gamma$ of $L(s,f)$ with $0 \le \beta \le 1$ and $T < \gamma < T+1$ and $s_0 = \sigma_0 + i\gamma$ with σ_0 sufficiently large. We note that

$$\begin{aligned} |F_1(s_0)| = |L(s_0,f)| &\ge 1 - \sum_{n=2}^{\infty} \frac{|\lambda(n)|}{n^{\sigma_0}} \\ &\ge 2 - \Big(\sum_{n=1}^{\infty}\frac{|\lambda(n)|^2}{n^{\sigma_0}}\Big)^{1/2} (\zeta(\sigma_0))^{1/2} \ge C \end{aligned}$$

for sufficiently large σ_0 which may depend upon f (note that both the series $\sum_{n=1}^{\infty} |\lambda(n)|^2 n^{-\sigma_0}$ and $\zeta(\sigma_0)$ approach 1 as $\sigma_0 \to \infty$). Here C is a certain positive constant. For $|s-s_0| = 3\sigma_0$, we have

$$\left|1 - \frac{s-s_0}{\varrho - s_0}\right| \geq \left|\frac{s-s_0}{\varrho-s_0}\right| - 1 \geq \frac{3\sigma_0}{\sigma_0 - \beta} - 1 \geq 2.$$

This implies that

$$C < |F_1(s_0)| < \max_{|s-s_0|=3\sigma_0} |F_1(s)| < \max_{|s-s_0|\leq 3\sigma_0} \frac{|L(s,f)|}{2^N} \ll \frac{T^C}{2^N}$$

and hence we obtain the lemma. ∎

LEMMA 3.7. *For $\sigma > 1$, define*

$$F_2(s) := L(s,f) \sum_{n\leq T} \frac{\mu^*(n)}{n^s} - 1 = L(s,f)M_T(s) - 1 =: \sum_{n=1}^{\infty} \frac{c(n)}{n^s}.$$

Then

$$c(n) = \sum_{\substack{d|n \\ d\leq T}} \mu^*(d)\lambda(n/d), \tag{3.3}$$

and for $\sigma > 1$,

$$F_2(s) = \sum_{n>T} c(n)/n^s.$$

Proof. First we observe that

$$\sum_{d|n} \mu^*(d)\lambda(n/d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

Now, we define

$$a(d) := \begin{cases} \mu^*(d) & \text{if } d \leq T, \\ 0 & \text{if } d > T. \end{cases} \tag{3.4}$$

From the definition of $F_2(s)$, we notice that

$$c(n) = \sum_{d|n} a(d)\lambda(n/d) - \sum_{d|n} \mu^*(d)\lambda(n/d). \tag{3.5}$$

If $n \leq T$, then $d \leq T$ (since d is a divisor of n) so that $a(d) = \mu^*(d)$. Therefore $c(n) = 0$ for $n \leq T$. For $n > T$, the second sum in (3.5) is zero and hence from (3.4), we get (3.3). ∎

4. Proof of the theorems

Proof of Theorem 1. By using dyadic partitions, it is enough to prove the theorem for $T \leq \gamma \leq 2T$. We divide the rectangle bounded by the lines with real parts $\sigma, 1$ and imaginary parts $T, 2T$ into abutting smaller rectangles of height $2(\log T)^2$. From Lemma 3.6, the multiplicity of any zero ϱ of $L(s,f)$ is $\ll \log T$. Therefore, without loss, we can assume that the zeros are simple in the counting process. We count the number of those smaller rectangles of height $2(\log T)^2$ which contain at least one zero and multiply by $C(\log T)^3$ to get a bound for $N_L(\sigma, T, 2T)$.

We define the zero-detector function

$$F_2(s) := L(s,f)\sum_{n\le T}\frac{\mu^*(n)}{n^s} - 1 = L(s,f)M_T(s) - 1 =: \sum_{n>T}\frac{c(n)}{n^s}.$$

From Lemma 3.7, we notice that

$$c(n) = \sum_{\substack{d|n\\ d\le T}} \mu^*(d)\lambda(n/d).$$

For any fixed zero $\varrho = \beta + i\gamma$, we let

$$G(s) = F_2(s)Y^{s-\varrho}e^{(s-\varrho)^2}$$

where Y is a parameter satisfying $T^{-A} \le Y \le T^A$. We select one zero ϱ_j in each of the rectangles (for $j = 1, 2, \ldots$)

$$\left\{\frac{1}{2} + \frac{1}{\log T} \le \sigma \le 1,\ T + 2(j-1)(\log T)^2 \le t \le T + 2j(\log T)^2\right\}.$$

We partition these rectangles into odd and even ones. Note that for any two zeros ϱ, ϱ' in two even (respectively odd) rectangles, we have $|\gamma - \gamma'| \ge 2(\log T)^2$. Let $\mathcal{A}$ and $\mathcal{B}$ denote the sets of the chosen zeros corresponding to the sets of odd and even rectangles respectively. Let $\varrho \in \mathcal{A}$ be any typical chosen zero. By Cauchy's residue theorem, we have

$$\left|\frac{1}{2\pi i}\int_{R(\varrho)}\frac{G(s)}{s-\varrho}\,ds\right| = 1$$

where the integral is taken over the rectangle $R(\varrho)$ defined by

$$R(\varrho) := \left\{\frac{1}{2} \le \sigma \le 1 + \frac{1}{\log T},\ |t-\gamma| \le B(\log T)^2\right\}.$$

Here $1/4 \le B \le 1$ is chosen such that the horizontal sides of $R(\varrho)$ are free from zeros of $L(s,f)$. If Y is chosen to satisfy $T^{-A} \le Y \ll T^A$, then the contributions from the horizontal sides of $R(\varrho)$ to the integral are $O(T^{-10})$ owing to the exponentially decaying factor $e^{(s-\varrho)^2}$. We denote the vertical sides of $R(\varrho)$ by V_1 and V_2 so that we have

$$(4.1)\qquad 1 = O\Big(\log T\Big(\int_{V_1}|F_2(s)|\,dt\Big)Y^{1/2-\beta} + \log T\Big(\int_{V_2}|F_2(s)|\,dt\Big)Y^{1+1/\log T-\beta}\Big)$$

$$= O\Big(\log T\Big(1+\int_{V_1}|F_2(s)|\,dt\Big)Y^{1/2-\beta} + \log T\Big(T^{-10}+\int_{V_2}|F_2(s)|\,dt\Big)Y^{1+1/\log T-\beta}\Big).$$

We choose Y such that

$$Y^{1/2-\beta}\Big(1+\int_{V_1}|F_2(s)|\,dt\Big) = Y^{1-\beta}\Big(T^{-10}+\int_{V_2}|F_2(s)|\,dt\Big).$$

Let

$$J_1(\varrho) = 1+\int_{V_1}|F_2(s)|\,dt,$$
$$J_2(\varrho) = T^{-10}+\int_{V_2}|F_2(s)|\,dt.$$

We notice that (from Lemmas 3.2 and 3.5)

$$\int_T^{2T}|M_T(1/2+it)|^2\,dt = \sum_{n\le T}\frac{|\mu^*(n)|^2}{n}(T+O(n)) \ll T\log T.$$

From (5.7) of [11], we have

$$\int_T^{2T}|L(1/2+it)|^2\,dt \ll T\log T.$$

Therefore by the Cauchy–Schwarz inequality, we find that

$$\int_T^{2T}|F_2(1/2+it)|\,dt \ll T\log T \tag{4.2}$$

and using Lemma 3.5 (Montgomery–Vaughan theorem) and the estimate in Lemma 3.4, we have

$$\begin{aligned}\int_T^{2T}|F_2(1+1/\log T+it)|^2\,dt &= \sum_{n>T}\frac{|c(n)|^2}{n^{2+2/\log T}}(T+O(n))\\ &\ll \sum_{n>T}\frac{|c(n)|^2}{n^{1+2/\log T}} = \int_T^{\infty}\frac{d(\sum_{n\le u}|c(n)|^2)}{u^{1+2/\log T}}\\ &\ll (\log T)^{19},\end{aligned} \tag{4.3}$$

on integrating by parts.

Note that (from (4.2) and (4.3))

$$Y = \Big(\frac{J_1}{J_2}\Big)^2 \ge \frac{1}{T^{-10}+T^C},\qquad Y \le \frac{T^C}{T^{-10}},$$

so that the condition on Y is satisfied. Hence we have

$$1 \le 2C(\log T)\left(\frac{J_1}{J_2}\right)^{2(1-\beta)} J_2 = 2C(\log T) J_1^{2(1-\beta)} J_2^{2\beta-1}.$$

It follows from the above that

$$\sum_{\varrho\in\mathcal{A}} J_1(\varrho) \ll T\log T \quad \text{and} \quad \sum_{\varrho\in\mathcal{A}} (J_2(\varrho))^2 \ll (\log T)^{21}.$$

The same argument is applicable to the zeros in the set $\mathcal{B}$. Thus we obtain

$$\sum_{\varrho\in\mathcal{A}} J_1(\varrho) + \sum_{\varrho\in\mathcal{B}} J_1(\varrho) = \sum_{\varrho\in\mathcal{A}\cup\mathcal{B}} J_1(\varrho) \ll T\log T$$

and similarly

$$\sum_{\varrho\in\mathcal{A}} (J_2(\varrho))^2 + \sum_{\varrho\in\mathcal{B}} (J_2(\varrho))^2 = \sum_{\varrho\in\mathcal{A}\cup\mathcal{B}} (J_2(\varrho))^2 \ll (\log T)^{21}$$

and so

$$\begin{aligned} \#\{\varrho : J_1(\varrho) \ge W_1\} &\le A\,\frac{T\log T}{W_1}, \\ \#\{\varrho : J_2(\varrho) \ge W_2\} &\le A\,\frac{(\log T)^{21}}{W_2^2}. \end{aligned} \tag{4.4}$$

Now we fix $W_1 = W_2^2 T$. Hence the total number of zeros coming from the two sets in (4.4) is at most

$$A(\log T)^{21}\left\{\frac{T}{W_1} + \frac{1}{W_2^2}\right\}.$$

From (4.1), for the remaining zeros, we have

$$J_1(\varrho) < W_1 \quad \text{and} \quad J_2(\varrho) < W_2$$

and also

$$\begin{aligned} 3/4 &\le 2C(\log T) W_1^{2(1-\beta)} W_2^{2\beta-1} \\ &= 2C(\log T) W_1^{2(1-\sigma)} W_1^{2(\sigma-\beta)} W_2^{2\sigma-1} W_2^{2(\beta-\sigma)} \\ &= 2C(\log T) W_1^{2(1-\sigma)} W_2^{2\sigma-1}\left(\frac{W_2}{W_1}\right)^{2(\beta-\sigma)} \\ &= 2C(\log T) W_1^{2(1-\sigma)} W_2^{2\sigma-1}\left(\frac{1}{W_2 T}\right)^{2(\beta-\sigma)}. \end{aligned}$$

Suppose that $W_2 > 1/T$ and so $(1/W_2T)^{2(\beta-\sigma)} \le 1$. Then we get

$$\begin{aligned} 3/4 &\le 2C(\log T) W_1^{2(1-\sigma)} W_2^{2\sigma-1} \\ &= 2C(\log T)(W_2^2 T)^{2(1-\sigma)} W_2^{2\sigma-1} = 2C(\log T) T^{2(1-\sigma)} W_2^{3-2\sigma}. \end{aligned} \tag{4.5}$$

We choose

$$W_2 = (4C \log T)^{-\frac{1}{3-2\sigma}} T^{-\frac{2(1-\sigma)}{3-2\sigma}}.$$

Clearly $W_2 > T^{-1}$. For this choice of W_2, (4.5) implies that $3/4 \le 1/2$, which is absurd; this means that we should count only those zeros which satisfy (4.4). Hence we get

$$N_L(\sigma, T, 2T) \ll \frac{(\log T)^{21}}{W_2^2} (\log T)^3 \ll T^{4(1-\sigma)/(3-2\sigma)} (\log T)^{26},$$

which proves the theorem.

Proof of Theorem 2. The proof is entirely similar to the proof of Theorem 1 of [19] and hence is omitted.

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