

A mean-square bound for the lattice discrepancy of bodies of rotation with flat points on the boundary

by

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Dedicated to Professor Edmund Hlawka on his 90th birthday

1. Introduction. Let \mathcal{B} denote a compact convex body in s -dimensional Euclidean space, $s \geq 2$, which contains the origin as an inner point and whose boundary $\partial\mathcal{B}$ is sufficiently smooth. The central question of the classical *lattice point theory of large domains* is to estimate the *lattice point discrepancy* of a linearly dilated copy $t\mathcal{B}$, i.e.,

$$(1.1) \quad P_{\mathcal{B}}(t) := \#(t\mathcal{B} \cap \mathbb{Z}^s) - \text{vol}(\mathcal{B})t^s,$$

where t is a large real parameter. For enlightening accounts on this topic, the reader is referred to E. Krätzel's monographs [14] and [15], to a recent survey article by A. Ivić, E. Krätzel, M. Kühleitner, and W. G. Nowak [10], and to M. Huxley's book [7] where he presented his breakthrough in the planar case ("discrete Hardy–Littlewood method").

If $\partial\mathcal{B}$ is of bounded nonzero Gaussian curvature throughout, the usual and plausible conjecture is that

$$(1.2) \quad P_{\mathcal{B}}(t) \ll t^{\Theta_s + \varepsilon}$$

for every $\varepsilon > 0$, where $\Theta_2 := 1/2$ and $\Theta_s := s - 2$ for $s \geq 3$. It is well-known that for every dimension, Θ_s is the minimal possible value with this property, and that (1.2) is actually true for spheres of dimension $s \geq 4$, even with $\varepsilon = 0$ if $s \geq 5$: see, e.g., E. Krätzel [15, p. 227]. Quite recently, V. Bentkus and F. Götze [1] and F. Götze [3] established (1.2) for arbitrary ellipsoids of dimension $s \geq 5$.

However, for $s = 2$ and 3, and for general bodies of higher dimensions, the proof or disproof of (1.2) remains an open problem. The sharpest known

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estimates are due to W. Müller [24]. Researchers subsequently dealt with the task of verifying (1.2) “on average”, i.e., showing that

$$(1.3) \quad \int_0^T (P_{\mathcal{B}}(t))^2 dt \ll T^{2\Theta_s+1+\varepsilon}.$$

In fact, (1.3) was established for planar domains by D. G. Kendall [13] and the author [25], ultimately in the form of an asymptotics [26]. For dimensions $s \geq 4$, (1.3) was proved by W. Müller [23] who however had to leave open the case $s = 3$. This gap was filled by A. Iosevich, E. Sawyer, and A. Seeger [8] who showed that

$$(1.4) \quad \int_0^T (P_{\mathcal{B}}(t))^2 dt \ll \begin{cases} T^{2\Theta_s+1} & \text{for } s \geq 4, \\ T^3 (\log T)^2 & \text{for } s = 3. \end{cases}$$

The very last estimate comes rather close to the asymptotic formula known for the three-dimensional sphere \mathcal{B}_0 , namely

$$\int_0^T (P_{\mathcal{B}_0}(t))^2 dt = CT^3 \log T + O(T^3 (\log T)^{1/2}).$$

Cf. V. Jarník [12], and also Y.-K. Lau [22] who improved the error term to $O(T^3)$.

2. Recent developments and statement of the present result.

The topic of this note will combine two recent trends in lattice point theory: On the one hand, increased interest arose in \mathbb{R}^3 -bodies of rotation (with respect to one of the coordinate axes), denoted by \mathcal{R} in what follows. For the case of nonzero curvature, F. Chamizo [2] obtained the upper bound

$$(2.1) \quad P_{\mathcal{R}}(t) \ll t^{11/8+\varepsilon},$$

while the papers by M. Kühleitner [20] and M. Kühleitner and W. G. Nowak [21] provided Ω -results. A recent article of E. Krätzel and W. G. Nowak [19] gives a version of (2.1) with numerical constants, for the special case of an ellipsoid.

On the other hand, a number of papers investigated the influence of boundary points with curvature zero on the lattice discrepancy. While Krätzel’s monograph [15] provides an enlightening survey of the planar case (which is comparatively well-understood), results for dimension 3 and higher can be found in the works of K. Haberland [4], E. Krätzel [16]–[18], and M. Peter [27]. These are all (pointwise) O -estimates, partially providing a precise evaluation of the contribution of an isolated flat point on $\partial\mathcal{B}$ to $P_{\mathcal{B}}(t)$, with a remainder of smaller order.

In the present paper we shall take up both of these matters, in the aspect of a mean-square estimate in the sense of (1.3). We will consider an \mathbb{R}^3 -body of rotation \mathcal{R} (with respect to one of the coordinate axes), with smooth boundary $\partial\mathcal{R}$ of nonzero Gaussian curvature κ throughout, except for the points of intersection of $\partial\mathcal{R}$ with the axis of rotation, where κ may vanish. It will turn out that the contribution of these flat points to the lattice point discrepancy can be evaluated quite accurately, leaving a remainder term which is in mean-square “as small as it should be”, in the sense of formula (1.3).

We remark parenthetically that if κ vanished anywhere else on $\partial\mathcal{B}$, it would do so on a whole circle. This would presumably have a more dramatic effect on the lattice discrepancy. It seems much more difficult to obtain a sharp result in this general case.

Precise formulation of the present assumptions. Let $\varrho : [0, \pi] \rightarrow \mathbb{R}_{>0}$ be a function of class C^4 , with $\varrho'(0) = \varrho'(\pi) = 0$ and ⁽¹⁾

$$(2.2) \quad \varrho\varrho'' - 2\varrho'^2 - \varrho^2 \neq 0$$

throughout $]0, \pi[$. Suppose that ϱ is analytic at π and 0. At these two values, the left-hand side of (2.2) may vanish, of orders (exactly) $N_1, N_2 \geq 0$, as a function of θ , the case that $\min(N_1, N_2) = 0$ not being excluded. It will be convenient to write for short

$$\alpha_i := \frac{1}{N_i + 2} \quad (i = 1, 2).$$

Now

$$\mathcal{C} = \{(x, y) = (\varrho(|\theta|) \cos \theta, \varrho(|\theta|) \sin \theta) : \theta \in [-\pi, \pi]\}$$

defines a smooth curve in the (x, y) -plane, symmetric with respect to the x -axis. Rotating \mathcal{C} around the latter, we obtain a smooth surface in (x, y, z) -space, which we call $\partial\mathcal{R}$, where \mathcal{R} is the compact convex body bounded by $\partial\mathcal{R}$. We denote by a_1, a_2 the minimal, resp., maximal x -coordinate on $\partial\mathcal{R}$. Obviously, the Gaussian curvature of $\partial\mathcal{R}$ vanishes at most at the points of intersection with the x -axis.

THEOREM. *Suppose that the conditions stated above are satisfied, in particular, the Gaussian curvature of $\partial\mathcal{R}$ vanishes at most at the two points of intersection with the axis of rotation. Then for the number $A_{\mathcal{R}}(t)$ of lattice points in the linearly dilated body $t\mathcal{R}$ we have the asymptotic formula*

$$A_{\mathcal{R}}(t) = \text{vol}(\mathcal{R})t^3 + \sum_{i=1}^2 \sum_{j=2}^{N_i+1} d_{i,j}^* \lambda_{i,j}(t) t^{2-j\alpha_i} + \Delta_{\mathcal{R}}(t),$$

⁽¹⁾ Recall that the curvature of a curve whose equation in polar coordinates is $\varrho = \varrho(\theta)$ is given, in absolute value, by $|\varrho(\theta)\varrho''(\theta) - 2\varrho'^2(\theta) - \varrho^2(\theta)|/(\varrho^2(\theta) + \varrho'^2(\theta))^{3/2}$.

where

$$\lambda_{i,j}(t) := (2\pi)^{-j\alpha_i} \Gamma(j\alpha_i) \sum_{k=1}^{\infty} k^{-1-j\alpha_i} \sin((-1)^i \pi j \alpha_i / 2 - 2\pi k \alpha_i t) \tag{i = 1, 2},$$

and the remainder satisfies the mean-square estimate

$$\int_0^T (\Delta_{\mathcal{R}}(t))^2 dt = O(T^{3+\varepsilon})$$

for each $\varepsilon > 0$. The coefficients $d_{1,j}^*$, $d_{2,j}^*$ are computable, from the formulas (3.1)–(3.5) below. In particular, $d_{1,1}^* > 0$, $d_{2,1}^* < 0$.

REMARKS. 1. It is easy to see that the error term satisfies in fact the pointwise upper estimate

$$(2.3) \quad \Delta_{\mathcal{R}}(t) = O(t^{3/2+\varepsilon}).$$

This is a straightforward consequence of the works of Krätzel [15]–[18], but also follows as a simple by-product from the argument in this paper: see the concluding remark at the end.

2. There is a crucial difference in the treatment of the problem, depending on whether there are boundary points of curvature zero or not. For κ nonzero, the analysis leading to the results (1.3), (1.4) is based on the asymptotic expansion of the Fourier transform of the indicator function of the body \mathcal{B} , which is due to E. Hlawka [5], [6]. In the case that $\inf \kappa = 0$ the latter is not at our disposal. Thus we have to employ a quite different approach which uses a truncated Hardy identity (Lemma 1) and a transformation of trigonometric sums.

3. Some auxiliary results

LEMMA 1. For integers $k \geq 0$, let as usual $r(k)$ denote the number of pairs $(m_1, m_2) \in \mathbb{Z}^2$ with $m_1^2 + m_2^2 = k$. Then for large real parameters X, Y , and any $\varepsilon > 0$,

$$\begin{aligned} P(X) &:= \sum_{0 \leq k \leq X} r(k) - \pi X \\ &= \frac{1}{\pi} X^{1/4} \sum_{1 \leq n \leq Y} \frac{r(n)}{n^{3/4}} \cos(2\pi\sqrt{nX} - 3\pi/4) \\ &\quad + O(X^{1/2+\varepsilon} Y^{-1/2}) + O(Y^\varepsilon). \end{aligned}$$

Proof. This is contained in formula (1.9) of A. Ivić [9].

LEMMA 2. Let $F \in C^4[A, B]$, $G \in C^2[A, B]$, and suppose that, for positive parameters X, Y, Z , we have $1 \leq B - A \ll X$ and

$$\begin{aligned} F^{(j)} &\ll X^{2-j} Y^{-1} && \text{for } j = 2, 3, 4, \quad |F''| \geq c_0 Y^{-1}, \\ G^{(j)} &\ll X^{-j} Z && \text{for } j = 0, 1, 2, \end{aligned}$$

throughout the interval $[A, B]$, with some constant $c_0 > 0$. Let \mathcal{J}' denote the image of $[A, B]$ under F' , and F^* the inverse function of F' . Then

$$\begin{aligned} &\sum_{A < m \leq B} G(m) e(F(m)) \\ &= e\left(\frac{\operatorname{sgn}(F'')}{8}\right) \sum_{k \in \mathcal{J}'} \frac{G(F^*(k))}{\sqrt{|F''(F^*(k))|}} e(F(F^*(k)) - kF^*(k)) \\ &\quad + O(Z(\sqrt{Y} + \log(2 + \text{length}(\mathcal{J}')))). \end{aligned}$$

Proof. Transformation formulas of this kind are quite common, though often with worse error terms. This very sharp version can be found as Theorem 8.16, together with formula (8.47), in the recent monograph [11] of H. Iwaniec and E. Kowalski. ■

For our argument it will be essential to have at hand a close analysis of the situation near the points where the Gaussian curvature (possibly) vanishes. To this end, let \mathcal{C}_+ denote the upper half of \mathcal{C} , and set

$$\mathcal{C}_+ = \{(\varrho(\theta) \cos \theta, \varrho(\theta) \sin \theta) : \theta \in [0, \pi]\} = \{(x, y) : a_1 \leq x \leq a_2, y = f(x)\},$$

with $a_1 := -\varrho(\pi) < 0 < a_2 := \varrho(0)$. This defines $f : [a_1, a_2] \rightarrow \mathbb{R}_{\geq 0}$ as a strictly positive C^4 -function on $]a_1, a_2[$, with $f(a_1) = f(a_2) = 0$, and f'' strictly negative throughout. By our assumptions, for each of the a_i 's, and $(x, y) \in \mathcal{C}_+$ in a suitable neighborhood of $(a_i, 0)$,

$$(3.1) \quad x = a_i + c_i y^{N_i+2} + \sum_{m=1}^{\infty} c_{i,m} y^{N_i+2+m} \quad (c_i \neq 0).$$

Consequently,

$$(3.2) \quad y = f(x) = \sum_{j=1}^{\infty} d_{i,j} |x - a_i|^{\alpha_i j}, \quad d_{i,1} = |c_i|^{-\alpha_i} \neq 0,$$

and the other $d_{i,j}$'s can be computed recursively from the $c_{i,m}$'s. It thus follows that, for $r = 0, 1, 2, \dots$,

$$(3.3) \quad f^{(r)}(x) \asymp |x - a_i|^{\alpha_i - r}$$

for x close to a_i . Similarly, we deduce that

$$(3.4) \quad \frac{d^r}{dx^r}(\sqrt{f(x)}) \asymp |x - a_i|^{\alpha_i/2 - r}$$

for $r = 0, 1, 2, \dots$, and x near a_i . Furthermore,

$$(3.5) \quad \frac{d}{dx}(f^2(x)) = \sum_{j=2}^{\infty} d_{i,j}^* |x - a_i|^{\alpha_i j - 1},$$

with $d_{i,2}^* = 2\alpha_i d_{i,1}^2 (-1)^{i+1}$, again in an appropriate neighborhood of a_i .

For our proof we will also need some knowledge about the *tac-function* of \mathcal{R} ,

$$H(u, v, w) := \max_{(x,y,z) \in \mathcal{R}} (ux + vy + wz)$$

and the *polar body* \mathcal{R}^* , given by $H(u, v, w) \leq 1$. The connection between the smoothness and curvature of $\partial\mathcal{R}$ and of $\partial\mathcal{R}^*$ has been neatly worked out in W. Müller [23, Lemma 1]. It is clear that H and thus \mathcal{R}^* is again invariant under rotations about the first coordinate axis. Let \mathcal{C}_+^* denote the intersection of $\partial\mathcal{R}^*$ with the closed upper half of the (u, v) -plane, i.e., \mathcal{C}_+^* is the polar curve of \mathcal{C}_+ . Then

$$\mathcal{C}_+^* = \{(u, v) : 1/a_1 \leq u \leq 1/a_2, v = h(u)\},$$

where $h :]1/a_1, 1/a_2[\rightarrow \mathbb{R}_{\geq 0}$ is a strictly positive C^3 -function on $]1/a_1, 1/a_2[$, with $h(1/a_1) = h(1/a_2) = 0$ (cf. W. Müller [23, Lemma 1]).

LEMMA 3. *With the conditions and definitions stated,*

$$\sup_{1/a_1 < u < 1/a_2} |h(u)h'(u)| < \infty.$$

Further, as $u \rightarrow 1/a_i$ ($i = 1, 2$), $|(d/du)(h(u)h'(u))|$ either is bounded or tends to ∞ .

Proof. Let the real numbers $x \in]a_1, a_2[$ and $u \in]1/a_1, 1/a_2[$ be such that the points $(x, f(x))$ and $(u, h(u))$ are polar reciprocal to each other, i.e.,

$$1 = xu + f(x)h(u) = \max_{(\xi,\eta) \in \mathcal{C}_+} (\xi u + \eta h(u)) = \max_{(\mu,\nu) \in \mathcal{C}_+^*} (x\mu + f(x)\nu).$$

Plainly, $(x, f(x)) \rightarrow (a_i, 0)$ if $(u, h(u)) \rightarrow (1/a_i, 0)$, and vice versa. We may assume that u is close to $1/a_i$ ($i = 1$ or 2), since elsewhere the boundedness of hh' is obvious. Then x belongs to a suitable neighborhood of a_i , and the inverse function $x = x(y)$ is well-defined and given in an explicit form by (3.1). By elementary calculus, for $y = f(x) \Leftrightarrow x = x(y)$, we have

$$(3.6) \quad 0 = u + f'(x)h(u) = x'(y)u + h(u) = x + f(x)h'(u) = x(y) + yh'(u).$$

Consequently, $h(u) = -x'(y)u$ and $h'(u) = -x(y)/y$, thus

$$(3.7) \quad h(u)h'(u) = ux(y)x'(y)/y \ll 1,$$

since $x'(y)/y$ remains bounded as $y \rightarrow 0$, in view of (3.1). This verifies the first part of Lemma 3.

Moreover, eliminating $h(u)$ from the pair of equations

$$1 = x(y)u + yh(u), \quad 0 = x'(y)u + h(u),$$

we see that $u = (x(y) - x'(y)y)^{-1}$, hence

$$\frac{du}{dy} = \frac{x''(y)y}{(x(y) - x'(y)y)^2}.$$

Therefore, using again (3.7), we obtain

$$\begin{aligned} (3.8) \quad \frac{d}{du}(h(u)h'(u)) &= \frac{d}{dy} \left(\frac{xx'}{xy - x'y^2} \right) \left(\frac{du}{dy} \right)^{-1} \\ &= \frac{-x^2x' + 2xx'^2y - x'^3y^2 + x^2x''y}{x''y^3}, \end{aligned}$$

where we have suppressed the argument y of x, x', x'' for short. By (3.1), for small y this is equal to a Laurent series, resp., Taylor series in y . ■

LEMMA 4. For a large real parameter X and the tac-function H defined above, we have the asymptotics

$$(3.9) \quad N(X) := \#\{(m, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} : H(m, \sqrt{n}, 0) \leq X\} = CX^3 + O(X)$$

with a constant $C > 0$. Furthermore, for $0 < \delta < 1$,

$$(3.10) \quad \begin{aligned} N_\delta(X) &:= \#\{(m, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} : H(m, \sqrt{n}, 0) \leq X, \sqrt{n} \leq \delta|m|\} \\ &= C_\delta X^3 + O(X), \end{aligned}$$

with a positive $C_\delta \ll \delta^2$ and the O -constant independent of δ . As a consequence, for large X and $0 < \omega, \delta < 1$,

$$(3.11) \quad \begin{aligned} &\#\{(m, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} : |H(m, \sqrt{n}, 0) - X| < \omega\} \ll X^2\omega + X, \\ &\#\{(m, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} : |H(m, \sqrt{n}, 0) - X| < \omega, \sqrt{n} \leq \delta|m|\} \\ &\ll X^2\omega\delta^2 + X. \end{aligned}$$

Proof. Let \mathcal{D}_+^* denote the compact planar domain bounded by the curve \mathcal{C}_+^* and the u -axis. Obviously,

$$\begin{aligned} N(X) &= \#\{(m, n) \in \mathbb{Z} \times \mathbb{Z}_{\geq 0} : (m, \sqrt{n}) \in X\mathcal{D}_+^*\} \\ &= \sum_{(1/a_1)X \leq m \leq (1/a_2)X} (1 + [X^2h^2(m/X)]) \\ &= X^2 \sum_{(1/a_1)X \leq m \leq (1/a_2)X} h^2(m/X) + O(X) \\ &= X^2 \int_{(1/a_1)X}^{(1/a_2)X} h^2(u/X) du + 2X \int_{(1/a_1)X}^{(1/a_2)X} \psi(u)h(u/X)h'(u/X) du + O(X), \end{aligned}$$

by the Euler–MacLaurin formula (see E. Krätzel [14, p. 20]), with $\psi(u) := u - [u] - 1/2$. Here the first integral equals CX with $C = \int_{1/a_1}^{1/a_2} h^2(\xi) d\xi$, which yields the main term of (3.9). Further, for any interval $[\beta_1, \beta_2] \subset]1/a_1, 1/a_2[$, an integration by parts gives

$$\begin{aligned}
 (3.12) \quad 2X \int_{\beta_1 X}^{\beta_2 X} \psi(u)h(u/X)h'(u/X) du &= 2X(h(\beta_2)h'(\beta_2)\psi_1(\beta_2 X) - h(\beta_1)h'(\beta_1)\psi_1(\beta_1 X)) \\
 &\quad - 2 \int_{\beta_1 X}^{\beta_2 X} (h(u/X)h''(u/X) + h'^2(u/X))\psi_1(u) du,
 \end{aligned}$$

where

$$\psi_1(u) := \int_0^u \psi(v) dv \ll 1.$$

If $hh'' + h'^2 = (hh')'$ is bounded on $]1/a_1, 1/a_2[$, we simply let $\beta_1 \rightarrow 1/a_1$ and $\beta_2 \rightarrow 1/a_2$ to obtain the desired bound $O(X)$ for the remainder. In case $(hh')'$ is unbounded near $1/a_1$ (say), we choose $\beta_1 > 1/a_1$ such that $(hh')'$ has no sign change on $]1/a_1, \beta_1]$. By the second mean-value theorem and Lemma 3,

$$\int_{(1/a_1)X}^{\beta_1 X} \psi(u)h(u/X)h'(u/X) du \ll \sup_{]1/a_1, \beta_1]} |hh'| \ll 1.$$

A similar reasoning holds near $1/a_2$ if necessary. On the remaining interval $[\beta_1 X, \beta_2 X]$, (3.12) readily yields the bound $O(X)$ and thus completes the proof of (3.9). Quite similarly,

$$\begin{aligned}
 (3.13) \quad N_\delta(X) &= \sum_{(1/a_1)X \leq m \leq (1/a_2)X} \min(X^2 h^2(m/X), \delta^2 m^2) + O(X) \\
 &= \int_{(1/a_1)X}^{(1/a_2)X} \min(X^2 h^2(u/X), \delta^2 u^2) du \\
 &\quad + \int_{(1/a_1)X}^{(1/a_2)X} \psi(u) \frac{d}{du} (\min(X^2 h^2(u/X), \delta^2 u^2)) du + O(X).
 \end{aligned}$$

Here it is important to notice that the equation $X^2 h^2(u/X) = \delta^2 u^2$ can have at most two solutions in $]X/a_1, X/a_2[$. To see this, the change of variable $u/X \mapsto u$ gives $h(u) = \pm \delta u$, $1/a_1 < u < 1/a_2$. Recalling the first equation of (3.6) leads to $f'(x) = \mp 1/\delta$, with $a_1 < x < a_2$. By convexity of the arc \mathcal{C}_+ , this has at most two solutions.

The first integral in (3.13) equals $C_\delta X^3$ with

$$C_\delta := \int_{1/a_1}^{1/a_2} \min(h^2(\xi), \delta^2 \xi^2) d\xi \ll \delta^2.$$

The remainder integral can be treated as before, with the bound $O(X)$, since for any interval $I \subseteq]X/a_1, X/a_2[$,

$$\int_I \psi(u) \delta^2 u du \ll X.$$

The deduction of (3.11) from (3.9), (3.10) is trivial. ■

LEMMA 5. *For an arbitrary finite index set \mathcal{M} , let $(\gamma_m)_{m \in \mathcal{M}}$ be a complex sequence and let $(\xi_m)_{m \in \mathcal{M}}$ be reals. Then, for any real T_0 and $T > 0$,*

$$\int_{T_0}^{T_0+T} \left| \sum_{m \in \mathcal{M}} \gamma_m e(\xi_m t) \right|^2 dt \leq 3T \sum_{\substack{m_1, m_2 \in \mathcal{M} \\ |\xi_{m_1} - \xi_{m_2}| < 1/T}} |\gamma_{m_1} \gamma_{m_2}|.$$

Proof. This is essentially Lemma 7.1 in H. Iwaniec and E. Kowalski [11].

4. Asymptotic evaluation of the main terms. For a large parameter t it follows, with the definitions of Section 3, that

$$\begin{aligned} (4.1) \quad A_{\mathcal{R}}(t) &= \sum_{a_1 t \leq m \leq a_2 t} \left(\sum_{0 \leq k \leq t^2 f^2(m/t)} r(k) \right) \\ &= \pi t^2 \sum_{a_1 t \leq m \leq a_2 t} f^2(m/t) + \sum_{a_1 t \leq m \leq a_2 t} P(t^2 f^2(m/t)). \end{aligned}$$

We proceed to evaluate the first sum on the right-hand side, postponing the mean-square estimation of the last one to the next section. By the Euler–MacLaurin formula,

$$\begin{aligned} (4.2) \quad \pi t^2 \sum_{a_1 t \leq m \leq a_2 t} f^2(m/t) &= \pi t^2 \int_{a_1 t}^{a_2 t} f^2(\tau/t) d\tau + \pi t^2 \int_{a_1 t}^{a_2 t} \psi(\tau) \frac{d}{d\tau}(f^2(\tau/t)) d\tau \\ &= \text{vol}(\mathcal{R})t^3 + \pi t^2 \int_{a_1}^{a_2} \psi(tx) \frac{d}{dx}(f^2(x)) dx. \end{aligned}$$

By (3.5),

$$(4.3) \quad \frac{d}{dx}(f^2(x)) = \sum_{j=2}^{2N_1+3} d_{1,j}^* (x - a_1)^{\alpha_1 j - 1} + \sum_{j=2}^{2N_2+3} d_{2,j}^* (a_2 - x)^{\alpha_2 j - 1} + \Phi(x),$$

with $\Phi \in C^1[a_1, a_2]$. Integrating by parts and using again $\psi_1(u) \ll 1$, we obtain

$$(4.4) \quad t^2 \int_{a_1}^{a_2} \psi(tx)\Phi(x) dx = O(t).$$

The same argument works for $|x - a_i|^{\alpha_i j - 1}$ instead of $\Phi(x)$, with $N_i + 2 \leq j \leq 2N_i + 3$, $i \in \{1, 2\}$. In the subsequent analysis we may thus replace the upper summation limits in the sums from (4.3) by $N_1 + 1$, resp., $N_2 + 1$. To deal with the first of these remaining sums, we use the Fourier series

$$\psi(z) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin(2\pi kz) \quad (z \notin \mathbb{Z})$$

and an obvious shift of variable. For $j = 2, \dots, N_1 + 1$, we conclude that

$$\int_{a_1}^{a_2} \psi(tx)(x - a_1)^{\alpha_1 j - 1} dx = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \int_0^{a_2 - a_1} x^{\alpha_1 j - 1} \sin(2\pi kt(a_1 + x)) dx.$$

An integration by parts shows that

$$\int_{a_2 - a_1}^{\infty} x^{\alpha_1 j - 1} \sin(2\pi kt(a_1 + x)) dx = O((kt)^{-1}).$$

Further,

$$\begin{aligned} \int_0^{\infty} x^{\alpha_1 j - 1} \sin(2\pi kt(a_1 + x)) dx &= \Im \left(e(a_1 kt) (kt)^{-\alpha_1 j} \int_0^{\infty} \tau^{\alpha_1 j - 1} e(\tau) d\tau \right) \\ &= \Im (e(a_1 kt) (2\pi kt)^{-\alpha_1 j} \Gamma(\alpha_1 j) e(\alpha_1 j / 4)) \\ &= \Gamma(\alpha_1 j) (2\pi kt)^{-\alpha_1 j} \sin(2\pi a_1 kt + \pi \alpha_1 j / 2), \end{aligned}$$

using well-known formulas for the last integral (cf., e.g., H. Rademacher [28, p. 82]). Collecting the results, we get

$$\pi t^2 \int_{a_1}^{a_2} \psi(tx) \left(\sum_{j=2}^{N_1+1} d_{1,j}^* (x - a_1)^{\alpha_1 j - 1} \right) dx = \sum_{j=2}^{N_1+1} d_{1,j}^* \lambda_{1,j}(t) t^{2-j\alpha_1} + O(t)$$

where $\lambda_{i,j}(t)$ has been defined in our Theorem. Dealing quite similarly with the second sum coming from (4.3), and recalling (4.2), (4.4), we obtain altogether

$$(4.5) \quad \pi t^2 \sum_{a_1 t \leq m \leq a_2 t} f^2(m/t) = \text{vol}(\mathcal{R}) t^3 + \sum_{i=1}^2 \sum_{j=2}^{N_i+1} d_{i,j}^* \lambda_{i,j}(t) t^{2-j\alpha_i} + O(t).$$

5. Estimating the remainder in mean-square. It remains to deal with the last sum in (4.1), i.e., to show that

$$\int_T^{2T} \left(\sum_{a_1 t \leq m \leq a_2 t} P(t^2 f^2(m/t)) \right)^2 dt \ll T^{3+\varepsilon}.$$

For given large T we divide $]a_1, (a_1 + a_2)/2]$ and $](a_1 + a_2)/2, a_2]$ into dyadic subintervals $\mathcal{J}^{(1,r)} =]u^{(1,r+1)}, u^{(1,r)}]$, $\mathcal{J}^{(2,r)} =]u^{(2,r)}, u^{(2,r+1)}]$, $0 \leq r \leq R$, where $u^{(i,r)} := a_i - (-1)^i 2^{-r-1}(a_2 - a_1)$, and R is chosen such that the shortest of these intervals are of length $\asymp T^{-1}$. Dropping the superscripts for short, we write \mathcal{J} for any of these subintervals, whose number is obviously $O(\log T)$. Let $\mathcal{K} := [a_1, a_2] \setminus \bigcup \mathcal{J}$; then $|\mathcal{K}| \asymp T^{-1}$, and the trivial bound $P(z) \ll \sqrt{z}$ readily implies

$$\int_T^{2T} \left(\sum_{m \in t\mathcal{K}} P(t^2 f^2(m/t)) \right)^2 dt \ll T^3.$$

Thus it suffices to prove that, for each \mathcal{J} and $\varepsilon > 0$,

$$(5.1) \quad \int_T^{2T} \left(\sum_{m \in t\mathcal{J}} P(t^2 f^2(m/t)) \right)^2 dt \ll T^{3+\varepsilon}.$$

For every $t \in [T, 2T]$ and $m \in t\mathcal{J}$, we apply Lemma 1, with $X = t^2 f^2(m/t)$ and $Y = T^2$. We obtain

$$\begin{aligned} (5.2) \quad & \sum_{m \in t\mathcal{J}} P(t^2 f^2(m/t)) \\ &= \sum_{m \in t\mathcal{J}} \left(\frac{\sqrt{t}}{\pi} \sqrt{f(m/t)} \sum_{1 \leq n \leq T^2} \frac{r(n)}{n^{3/4}} \cos(2\pi \sqrt{n} t f(m/t) - 3\pi/4) + O(T^\varepsilon) \right) \\ &= \frac{\sqrt{t}}{\pi} \sum_{1 \leq n \leq T^2} \frac{r(n)}{n^{3/4}} \left\{ \sum_{m \in t\mathcal{J}} \sqrt{f(m/t)} \cos(2\pi \sqrt{n} t f(m/t) - 3\pi/4) \right\} + O(T^{1+\varepsilon}). \end{aligned}$$

We shall transform the inner sum here by means of Lemma 2, with

$$G(\tau) := \sqrt{f(\tau/t)}, \quad F(\tau) := \sqrt{n} t f(\tau/t).$$

To do so we put $L := T \text{length}(\mathcal{J})$, and observe that $|\tau - at| \asymp L$ for all $\tau \in t\mathcal{J}$, where a is the one of a_1, a_2 which is nearer to \mathcal{J} . Hence, in view of (3.3),

$$F''(\tau) = \sqrt{n} t^{-1} f''(\tau/t) \asymp \sqrt{n} t^{-1} |\tau/t - a|^{\alpha-2} \asymp \sqrt{n} t^{1-\alpha} L^{\alpha-2}$$

for all $\tau \in t\mathcal{J}$ (α the appropriate α_i), and similarly

$$F^{(j)}(\tau) \ll \sqrt{n} t^{1-\alpha} L^{\alpha-j} \quad \text{for } j = 3, 4.$$

Furthermore, by (3.4),

$$G^{(j)}(\tau) = \frac{d^j}{d\tau^j}(\sqrt{f(\tau/t)}) \ll t^{-j}|\tau/t - a|^{\alpha/2-j} \ll t^{-\alpha/2}L^{\alpha/2-j}$$

for $\tau \in t\mathcal{J}$, $j = 0, 1, 2$. We may thus apply Lemma 2 with the parameters

$$X := L, \quad Y := n^{-1/2}t^{\alpha-1}L^{2-\alpha}, \quad Z := (L/t)^{\alpha/2}.$$

To simplify the notation in what follows, put

$$\beta(z) := \sqrt{\frac{f(f^*(z))}{|f''(f^*(z))|}}.$$

After a short computation, Lemma 2 yields

$$\begin{aligned} (5.3) \quad & \sum_{m \in t\mathcal{J}} \sqrt{f(m/t)} \cos(2\pi\sqrt{n}tf(m/t) - 3\pi/4) \\ &= \frac{\sqrt{t}}{n^{1/4}} \Re\left(- \sum_{k \in \sqrt{n}\mathcal{J}^*} \beta(k/\sqrt{n})e(t(\sqrt{n}f(f^*(k/\sqrt{n})) - kf^*(k/\sqrt{n})))\right) \\ &+ O(n^{-1/4}t^{(\alpha-1)/2}L^{1-\alpha/2}) + O(\log t), \end{aligned}$$

where f^* denotes the inverse function of f' and \mathcal{J}^* the image of the closure $\bar{\mathcal{J}}$ of \mathcal{J} under f' . The contribution of the error terms here to the whole of (5.2) is

$$\ll t \sum_{1 \leq n \leq T^2} \frac{r(n)}{n} + \sqrt{t} \log t \sum_{1 \leq n \leq T^2} \frac{r(n)}{n^{3/4}} \ll T \log T,$$

hence small enough. Now $k \in \sqrt{n}\mathcal{J}^*$ implies that $f^*(k/\sqrt{n}) \in \bar{\mathcal{J}}$. Hence, by (3.3) and the fact that f'' is bounded away from zero,

$$(5.4) \quad \beta(k/\sqrt{n}) \ll L/T \quad \text{for } k \in \sqrt{n}\mathcal{J}^*.$$

Furthermore, by the definition of the tac-function H , for all $k \in \mathbb{Z}, n \in \mathbb{Z}^+$,

$$\begin{aligned} H(-k, \sqrt{n}, 0) &= \max_{(x,y,0) \in \partial\mathcal{R}} (-kx + \sqrt{n}y) = \max_{a_1 \leq x \leq a_2} (-kx + \sqrt{n}f(x)) \\ &= \sqrt{n}f(f^*(k/\sqrt{n})) - kf^*(k/\sqrt{n}). \end{aligned}$$

Hence it will suffice to show that

$$(5.5) \quad I(\mathcal{J}, T) := \int_T^{2T} t^2 |S(\mathcal{J}, t, T)|^2 dt \ll T^{3+\varepsilon},$$

where

$$S(\mathcal{J}, t, T) := \sum_{1 \leq n \leq T^2} \frac{r(n)}{n} \sum_{k \in \sqrt{n}\mathcal{J}^*} \beta(k/\sqrt{n})e(tH(-k, \sqrt{n}, 0)).$$

Dividing into dyadic subintervals, it is thus enough to show that

$$(5.6) \quad \int_T^{2T} \left| \sum_{n \asymp N} \sum_{k \asymp K} \gamma_{n,k} e^{i(tH(-k, \sqrt{n}, 0))} \right|^2 dt \ll T^{1+\varepsilon}$$

with

$$(5.7) \quad \begin{aligned} N &\ll T^2, & K &\ll \sqrt{N}(L/T)^{\alpha-1}, \\ \gamma_{n,k} &:= \beta(k/\sqrt{n}) \frac{r(n)}{n} &&\ll LT^{-1+\varepsilon/2}N^{-1}. \end{aligned}$$

Here we used (5.4) and the fact that, in view of (3.3),

$$\max_{\xi \in \mathcal{J}^*} |\xi| = \max_{x \in \mathcal{J}} |f'(x)| \ll (L/T)^{\alpha-1}.$$

By Lemma 5, the left-hand side of (5.6) is

$$\begin{aligned} &\ll T^{1+\varepsilon} \frac{L^2}{T^2 N^2} NK \\ &\quad \times \max_{H_0 \ll K + \sqrt{N}} \#\{n \asymp N, k \asymp K : |H(-k, \sqrt{n}, 0) - H_0| < 1/T\}. \end{aligned}$$

According to Lemma 4, this cardinality is

$$\ll N/T + \sqrt{N} + K \ll \sqrt{N} + K.$$

To see this, we use the first inequality of (3.11) if $K \ll \sqrt{N}$ and the second one otherwise, with δ a constant multiple of \sqrt{N}/K . Therefore, the left-hand side of (5.6) is overall

$$\ll T^{1+\varepsilon} (L/T)^2 \frac{K}{\sqrt{N}} \left(1 + \frac{K}{\sqrt{N}} \right) \ll T^{1+\varepsilon} ((L/T)^{1+\alpha} + (L/T)^{2\alpha}) \ll T^{1+\varepsilon},$$

with an appeal to (5.7). This completes the proof of (5.6) and thereby, in view of (4.1) and (4.5), that of our Theorem.

6. Concluding remark. We indicate briefly how the pointwise bound (2.3) follows as a by-product from the above analysis. Estimating the right-hand side of (5.3) trivially, we get, for $t = T$,

$$\begin{aligned} \sum_{m \in t\mathcal{J}} \sqrt{f(m/t)} \cos(2\pi\sqrt{n}tf(m/t) - 3\pi/4) \\ \ll n^{1/4}t^{1/2}|\mathcal{J}^*| + n^{-1/4}t^{(\alpha-1)/2}L^{1-\alpha/2} + \log t \\ \ll n^{1/4}t^{3/2-\alpha}L^{\alpha-1} + n^{-1/4}t^{(\alpha-1)/2}L^{1-\alpha/2} + \log t, \end{aligned}$$

in view of (5.7). Using Lemma 1 with

$$X = t^2 f^2(m/t) \ll t^2, \quad Y = L^{2-\alpha} t^{\alpha-1} \ll t,$$

we obtain, as an obvious variant of (5.2),

$$\begin{aligned}
 & \sum_{m \in t\mathcal{J}} P(t^2 f^2(m/t)) \\
 & \ll \sqrt{t} \sum_{1 \leq n \leq Y} \frac{r(n)}{n^{3/4}} (n^{1/4} t^{3/2-\alpha} L^{\alpha-1} + n^{-1/4} t^{(\alpha-1)/2} L^{1-\alpha/2} + \log t) \\
 & \quad + Lt^{1+\varepsilon}/Y^{1/2} \\
 & \ll t^{2-\alpha} L^{\alpha-1} Y^{1/2} + t^{\alpha/2} L^{1-\alpha/2} \log t + t^{1/2} Y^{1/4} \log t + Lt^{1+\varepsilon}/Y^{1/2} \\
 & \ll L^{\alpha/2} t^{3/2-\alpha/2+\varepsilon} + L^{1-\alpha/2} t^{\alpha/2} \log t + L^{1/2-\alpha/4} t^{\alpha/4+1/4} \log t \\
 & \ll t^{3/2+\varepsilon}. \blacksquare
 \end{aligned}$$

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