Restricted sums of subsets of $\mathbb{Z}$

by

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1. Introduction. Let

\begin{equation}
\{A_i\}_{i=1}^n
\end{equation}

be a finite sequence of sets. If $a_1 \in A_1, \ldots, a_n \in A_n$, and $a_1, \ldots, a_n$ are pairwise different, then we call \(\{a_i\}_{i=1}^n\) a \textit{system of distinct representatives} (abbreviated to SDR) of (1.1). Apparently (1.1) has an SDR provided that

\begin{equation}
|A_i| \geq i \quad \text{for all } i = 1, \ldots, n.
\end{equation}

If $A_1, \ldots, A_n$ are contained in a finite set $\{x_1, \ldots, x_k\}$ with cardinality $k$, then (1.1) has as many SDR’s as \(\{A_i^*\}_{i=1}^n\) does where $A_i^* = \{1 \leq j \leq k : x_j \in A_i\} \subseteq \{1, \ldots, k\}$.

Let $A_1, \ldots, A_n$ be finite subsets of an additive abelian group $G$. Their sumset is given by

\begin{equation}
A_1 + \ldots + A_n = \{a_1 + \ldots + a_n : a_1 \in A_1, \ldots, a_n \in A_n\}.
\end{equation}

If we require the summands to be distinct, then we are led to the restricted sumset

\begin{equation}
S(\{A_i\}_{i=1}^n) = S(A_1, \ldots, A_n)
\end{equation}

\begin{equation}
= \left\{ \sum_{i=1}^n a_i : \{a_i\}_{i=1}^n \text{ forms an SDR of } \{A_i\}_{i=1}^n \right\}.
\end{equation}

Of course there are many other kinds of restricted sumsets. An interesting problem is to provide a nontrivial lower bound for the cardinality of a restricted sumset of $A_1, \ldots, A_n$. In the light of the fundamental theorem on finitely generated abelian groups, it suffices to work within the ring $\mathbb{Z}$ of integers instead of a torsionfree abelian group $G$.

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For a finite subset $A$ of $\mathbb{Z}$, in 1995 M. B. Nathanson [N1] obtained the inequality
\begin{equation}
|n^\cap A| \geq n|A| - n^2 + 1
\end{equation}
and determined when equality holds. (By $n^\cap A$ we mean $S(\{A_i\}_{i=1}^n)$ with $A_1 = \ldots = A_n = A$.) Soon after this, Y. Bilu [B] gave the same result independently. Let $p$ be a prime. In 1994 J. A. Dias da Silva and Y. O. Hamidoune [DH] proved the following generalization of a conjecture of P. Erdős and H. Heilbronn (cf. [EH] and [G]):
\begin{equation}
|n^\cap A| \geq \min\{p, n|A| - n^2 + 1\} \quad \text{for any } A \subseteq \mathbb{Z}/p\mathbb{Z}.
\end{equation}
By the so-called polynomial method, in 1996 N. Alon, M. B. Nathanson and I. Z. Ruzsa [ANR] got the following result: Let $F$ be any field of characteristic $p$ and $A_1, \ldots, A_n$ its subsets with $0 < |A_1| < \ldots < |A_n| < \infty$, then
\begin{equation}
\left|S(\{A_i\}_{i=1}^n)\right| \geq \min\left\{p, \sum_{i=1}^n |A_i| - \frac{n(n+1)}{2} + 1\right\}.
\end{equation}
Their method does not allow one to determine when the bound can be attained. Provided that $A_1, \ldots, A_n$ are finite subsets of $\mathbb{Z}$ with $0 < |A_1| < \ldots < |A_n|$, we have
\begin{equation}
\left|S(\{A_i\}_{i=1}^n)\right| \geq 1 + \sum_{i=1}^n (|A_i| - i).
\end{equation}
A purely combinatorial proof of this inequality was given by Hui-Qin Cao and Zhi-Wei Sun [CS], where the authors obtained some necessary conditions for the equality case.

Now we introduce our basic notations in this paper. For $A \subseteq \mathbb{Z}$ we put $-A = \{-x : x \in A\}$ and $a + A = A + a = \{a + x : x \in A\}$ for $a \in \mathbb{Z}$. An arithmetic progression $A$ is a set of the form $\{a, a + d, \ldots, a + kd\}$ where $a$ and $d, k > 0$ are integers; we use $d(A)$ to denote the (common) difference $d$ of $A$. (A set having a single element is not considered as an arithmetic progression.) For the sake of convenience, AP will denote the class of all arithmetic progressions. For $a, b \in \mathbb{Z}$ we put
\begin{align*}
(a, b) &= \{x \in \mathbb{Z} : a < x < b\}, \\
[a, b) &= \{x \in \mathbb{Z} : a \leq x < b\}, \\
[a, b] &= \{x \in \mathbb{Z} : a \leq x \leq b\}, \\
(a, b] &= \{x \in \mathbb{Z} : a < x \leq b\}.
\end{align*}
In this paper we study lower bounds for cardinalities of various restricted sumsets of subsets of $\mathbb{Z}$. We use the powerful techniques developed in [CS]. In the next section we will prove the following general result on linearly restricted sums of subsets of $\mathbb{Z}$. 
Theorem 1.1. Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{Z}$, and $V$ a set of tuples $(s, t, \mu, \nu, w)$ where $1 \leq s, t \leq n$, $s \neq t$, $\mu, \nu \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$ and $w \in \mathbb{Z}$. Set

$$C = \{a_1 + \ldots + a_n : a_i \in A_i, \text{ and } \mu a_i + \nu a_j \neq w \text{ if } (i, j, \mu, \nu, w) \in V\}.$$  

If each $V_i = \{(s, t, \mu, \nu, w) \in V : i \in \{s, t\}\}$ has cardinality less than $|A_i|$, then

$$|C| \geq \sum_{i=1}^{n} |A_i| - 2|V| - n + 1 = 1 + \sum_{i=1}^{n} (|A_i| - |V_i| - 1) > 0. \quad (1.10)$$

Remark 1.1. If we replace $a_1 + \ldots + a_n$ by $\lambda_1 a_1 + \ldots + \lambda_n a_n$ in the definition (1.9) of $C$ where $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}^*$, then Theorem 1.1 remains valid. For, when $(i, j, \mu, \nu, w) \in V$, $a_i \in A_i$ and $a_j \in A_j$, we have

$$\mu a_i + \nu a_j = w \Leftrightarrow \lambda_i \lambda_j (\mu a_i + \nu a_j) = \lambda_i \lambda_j w \Leftrightarrow \mu' (\lambda_i a_i) + \nu' (\lambda_j a_j) = w'$$

where $\mu' = \lambda_j \mu$, $\nu' = \lambda_i \nu$ and $w' = \lambda_i \lambda_j w$.

Now we give several consequences of Theorem 1.1.

Corollary 1.1. Let $A_1, \ldots, A_n$ be subsets of $\mathbb{Z}$ which are nonempty and finite. Then

$$|A_1 + \ldots + A_n| \geq |A_1| + \ldots + |A_n| - n + 1. \quad (1.11)$$

Proof. Just apply Theorem 1.1 with $V = \emptyset$. ■

Remark 1.2. Corollary 1.1 is a known result. Equality in (1.11) holds if and only if all those $A_i$ with $|A_i| \geq 2$ are arithmetic progressions with the same difference. See Theorems 1.4 and 1.5 of [N2].

Corollary 1.2. Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{Z}$ such that $|A_i| \geq |J_i|$ for all $i = 1, \ldots, n$ where $J_i = \{1 \leq j \leq n : A_i \cap A_j \neq \emptyset\}$. Then

$$|S(\{A_i\}_{i=1}^{n})| \geq 1 + \sum_{i=1}^{n} (|A_i| - |J_i|). \quad (1.12)$$

Proof. Put $V = \{(i, j, 1, -1, 0) : 1 \leq i < j \leq n \text{ and } A_i \cap A_j \neq \emptyset\}$. Then

$$|V_i| = |\{1 \leq j \leq n : j \neq i \text{ and } A_i \cap A_j \neq \emptyset\}| = |J_i \setminus \{i\}| < |A_i| \quad \text{for } i \in [1, n].$$

Applying Theorem 1.1 we immediately get the desired inequality. ■

Corollary 1.3. Let $A, A_1, \ldots, A_n$ be finite subsets of $\mathbb{Z}$ such that

$$|A_i| > \sum_{j \neq i} |(A_i + A_j) \cap A| \quad \text{for all } i = 1, \ldots, n.$$

Let $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}^*$ and

$$L = \{\lambda_1 a_1 + \ldots + \lambda_n a_n : a_1 \in A_1, \ldots, a_n \in A_n, \ a_i + a_j \notin A \text{ if } i \neq j\}.$$
Then
\[
\sum_{i=1}^n |A_i| - |L| \leq 2 \sum_{1 \leq i < j \leq n} |(A_i + A_j) \cap \Lambda| + n - 1 \leq (n|\Lambda| + 1)(n - 1).
\]

**Proof.** Set
\[
V = \{(i, j, 1, 1, \lambda) : 1 \leq i < j \leq n \& \lambda \in (A_i + A_j) \cap \Lambda\}.
\]
Then
\[
|V| = \sum_{1 \leq i < j \leq n} |(A_i + A_j) \cap \Lambda| \leq \binom{n}{2}|\Lambda|,
\]
and \(|V_i| = \sum_{j \neq i} |(A_i + A_j) \cap \Lambda|\) for \(i = 1, \ldots, n\). Thus the required result follows from Theorem 1.1 and Remark 1.1. ■

**Corollary 1.4.** Let \(A_1, \ldots, A_n\) be finite subsets of \(\mathbb{Z}\), and
\[
S = \{a_1 + \cdots + a_n : a_1 \in A_1, \ldots, a_n \in A_n, \ a_i \neq \mu_{ij}a_j + \nu_{ij} \text{ if } i \neq j\},
\]
where \(\mu_{ij} \in \mathbb{Z}^*\) and \(\nu_{ij} \in \mathbb{Z}\). If \(|A_i| \geq 2n - 1\) for all \(i = 1, \ldots, n\), then
\[
|S| \geq \sum_{i=1}^n |A_i| - 2n^2 + n + 1.
\]

**Proof.** Let \(V = \{(i, j, 1, -\mu_{ij}, \nu_{ij}) : 1 \leq i < j \leq n \& i \neq j\}\). If \(1 \leq i \leq n\) then \(|V_i| = n - 1 + (n - 1) = 2n - 2\). Clearly \(2|V| + n - 1 = 2(n^2 - n) + n - 1 = 2n^2 - n - 1\). So it suffices to apply Theorem 1.1. ■

**Remark 1.3.** For \(1 \leq i < j \leq n\) let \(\mu_{ij} = 1\), \(\mu_{ji} = -1\) and \(\nu_{ij} = \nu_{ji} = 0\). Then the set \(S\) given in Corollary 1.4 becomes \(\{\sum_{i=1}^n a_i : a_i \in A_i\} \text{ and all the } a_i^2 \text{ are distinct}\).

**Corollary 1.5.** For each \(i = 1, \ldots, n\) let \(A_i \subseteq \mathbb{Z}\) and \(3 \leq |A_i| < \infty\). Then the set
\[
\{a_1 + \cdots + a_n : a_i \in A_i, \ a_i \neq a_{i+1} \text{ if } i < n, \text{ and } a_n \neq a_1\}
\]
has cardinality at least \(\sum_{i=1}^n |A_i| - 3n + 1\).

**Proof.** Let \(V = \{(i, i + 1, 1, -1, 0) : i \in [1, n]\} \cup \{(n, 1, 1, -1, 0)\}\). Then \(|V| = n\), and \(|V_i| = 2 < |A_i|\) for all \(i \in [1, n]\). So the desired result follows immediately from Theorem 1.1. ■

Let \(F\) be a field of characteristic \(p\) where \(p\) is a prime, and \(A_1, \ldots, A_n\) its finite subsets satisfying (1.2). Then Theorem 3.2 of [ANR] essentially asserts that
\[
|S(A_i\}_{i=1})| \geq \min\left\{p, 1 + \sum_{i=1}^n \min_{1 \leq j \leq n} (|A_j| - j)\right\}.
\]

In the last section we will show the following general result by our combinatorial method.
THEOREM 1.2. Let \( A_1, \ldots, A_n \) be finite subsets of \( \mathbb{Z} \) with (1.2) and \( |A_1| \leq \ldots \leq |A_n| \). Then

\[
|S\{A_i\}_{i=1}^n| \geq 1 + \sum_{i=1}^n \min_{i \leq j \leq n} (|A_j| - j).
\]

In the equality case, \( \bigcup_{i=1}^n A_i = A_m \) if \( m \) lies in

\[
M = \{1 \leq i \leq n : |A_i| - i < |A_j| - j \text{ for all } j \in (i, n)\},
\]

and providing \( |A_i| > i \) for all \( i \in [1, n] \) the set \( \bigcup_{i=1}^n A_i = A_n \) lies in \( \text{AP} \) with the only exceptions as follows:

(i) \( n = 1 \) or \( |A_n| = n + 1 \);
(ii) \( n = 2 \), \( |A_1| \in \{3, 4\} \) and \( A_2 \) has the form

\[
\{x_1, x_2, x_3, x_4\} \quad \text{with} \quad x_1 < x_2 < x_3 < x_4 \text{ and } x_4 - x_3 = x_2 - x_1;
\]
(iii) \( n > 1 \), \( |A_{n-1}| = n \), \( A_{n-1} \) and \( A_n \setminus A_{n-1} \) belong to \( \text{AP} \), and \( d(A_{n-1}) = d(A_n \setminus A_{n-1}) \).

REMARK 1.4. Let \( A_1, \ldots, A_n \) be finite subsets of \( \mathbb{Z} \) with \( k_i = |A_i| \geq i \) for all \( i \in [1, n] \). Providing \( k_s > k_{s+1} \) for some \( s \in [1, n] \), we still have inequality (1.13). To see this, we exchange \( A_s \) and \( A_{s+1} \), i.e. we arrange \( A_1, \ldots, A_n \) in the order

\[
A^*_1 = A_1, \quad \ldots, \quad A^*_{s-1} = A_{s-1}, \quad A^*_s = A_{s+1},
\]
\[
A^*_{s+1} = A_s, \quad A^*_{s+2} = A_{s+2}, \quad \ldots, \quad A^*_n = A_n.
\]

Clearly

\[
|A^*_{s+1}| - (s + 1) = k_s - s - 1 > k_{s+1} - (s + 1)
\]

and

\[
\min\{|A^*_s| - s, |A^*_{s+1}| - (s + 1)| = \min\{k_{s+1} - s, k_s - s - 1\}
\]
\[
= k_{s+1} - s > k_{s+1} - (s + 1)
\]
\[
\geq \min\{k_s - s, k_{s+1} - (s + 1)\},
\]

thus

\[
\min_{i \leq j \leq n} (|A^*_j| - j) \geq \min_{i \leq j \leq n} (k_j - j) \quad \text{for all } i = 1, \ldots, n.
\]

The following example shows that in Theorem 1.2 the lower bound (in terms of cardinalities \( |A_1|, \ldots, |A_n| \)) is best possible.

EXAMPLE 1.1. Let \( k_1, \ldots, k_n \) be integers for which \( k_1 \leq \ldots \leq k_n \) and \( k_i \geq i \) for all \( i = 1, \ldots, n \). Let \( d_i = \min_{i \leq j \leq n} (k_j - j) \) for each \( i = 1, \ldots, n \). Apparently \( d_1 \leq \ldots \leq d_n \). Put \( A_1 = [0, k_1 - 1], \ldots, A_n = [0, k_n - 1] \). Observe
that $S(\{A_i\}_{i=1}^n)$ contains the following sets:

\[
0 + 1 + 2 + \ldots + (n-3) + (n-2) + [n-1, n-1 + d_n],
\]
\[
0 + 1 + 2 + \ldots + (n-3) + [n-2, n-2 + d_{n-1}] + (n-1 + d_n),
\]
\[
\vdots
\]
\[
0 + [1, 1 + d_2] + (2 + d_3) + \ldots + (n-2 + d_{n-1}) + (n-1 + d_n),
\]
\[
[0, d_1] + (1 + d_2) + (2 + d_3) + \ldots + (n-2 + d_{n-1}) + (n-1 + d_n).
\]

Therefore

\[
S(\{A_i\}_{i=1}^n) \supseteq [0 + 1 + \ldots + (n-1), d_1 + (1 + d_2) + \ldots + (n-1 + d_n)]
\]
\[
= \frac{n(n-1)}{2} + \left[0, \sum_{i=1}^n d_i \right].
\]

Suppose that $\max S(\{A_i\}_{i=1}^n) = \sum_{i=1}^n x_i$ where $x_1 < \ldots < x_n$ and these $n$ integers can be rearranged to form an SDR of $\{A_i\}_{i=1}^n$. Choose a permutation $\sigma$ on $\{1, \ldots, n\}$ such that $x_{\sigma(i)} \in A_i$. When $1 \leq i \leq n$, there exists a $j \in [i, n]$ such that $\sigma^{-1}(j) \notin (i, n]$ and hence $x_j \in A_{\sigma^{-1}(j)} \subseteq A_i$. So $x_i \in A_i$ for every $i = 1, \ldots, n$. If $x_n < k_n - 1$, then by substituting $k_n - 1$ for $x_n$ we would obtain an SDR of $\{A_i\}_{i=1}^n$ with the corresponding sum larger than $\sum_{i=1}^n x_i$. Thus $x_n = k_n - 1 = n-1+d_n$. Let $1 \leq i < n$ and assume that $x_j = j-1+d_j$ for all $j \in (i, n]$. When $i < j \leq n$, we have $x_j = j-1+d_j \geq i + d_i$. If $x_i < i - 1 + d_i$ then by substituting $i - 1 + d_i \in A_i$ for $x_i$ we would obtain a sum larger than $x_1 + \ldots + x_n$, thus $x_i = i - 1 + d_i$. By the above,

\[
\max S(\{A_i\}_{i=1}^n) = \sum_{i=1}^n x_i = \sum_{i=1}^n (i-1 + d_i) = \frac{n(n-1)}{2} + \sum_{i=1}^n d_i.
\]

Obviously

\[
\min S(\{A_i\}_{i=1}^n) = 0 + 1 + \ldots + (n-1) = \frac{n(n-1)}{2}.
\]

So we also have

\[
S(\{A_i\}_{i=1}^n) \subseteq \frac{n(n-1)}{2} + \left[0, \sum_{i=1}^n d_i \right].
\]

Therefore

\[
S(\{A_i\}_{i=1}^n) = \left[\frac{n(n-1)}{2}, \frac{n(n-1)}{2} + \sum_{i=1}^n d_i \right]
\]

and hence $|S(\{A_i\}_{i=1}^n)| = 1 + \sum_{i=1}^n d_i$.

**Remark 1.5.** Example 1.1 was realized by Alon, Nathanson and Ruzsa [ANR], but they did not go into details. Let $k_1, \ldots, k_n$ and $A_1, \ldots, A_n$ be as in Example 1.1. For $i = 1, \ldots, n$ put $A_i^* = \{a + jd : j \in [0, k_i)\}$ where
a ∈ Z and d ∈ Z*. By Example 1.1,

\[ |S(\{A_i^n\}_{i=1}^n)| = |S(\{A_i^n\}_{i=1}^n)| = 1 + \sum_{i=1}^n \min_{i \leq j \leq n} (|A_i^n| - j). \]

As for the exceptions (i) and (ii), here we give

**Example 1.2.** Let A be a finite subset of Z with |A| ≥ n ≥ 1, and A_1, \ldots, A_n subsets of Z with \( \bigcup_{i=1}^n A_i = A_n = A \). Suppose that |A_i| - i ≥ |A_n| - n for all i = 1, \ldots, n (i.e. the set M defined by (1.14) only contains n). If \{a_i\}_{i=1}^n is an SDR of \{A_i^n\}_{i=1}^n, then \{a_1, \ldots, a_n\} is a subset of A with cardinality n. If S ⊆ A and |S| = n, then for each i ∈ [1, n] we have

\[ |S \cap A_i| ≥ |S| - |A \setminus A_i| = n - (|A_n| - |A_i|) ≥ i, \]

therefore \{S \cap A_i\}_{i=1}^n has an SDR \{a_i\}_{i=1}^n and hence S = \{a_1, \ldots, a_n\}. Thus S(\{A_i^n\}_{i=1}^n) = n^A, (1.13) is equivalent to (1.5), and the equality case of (1.13) is the same as that of (1.5). A result of Nathanson says that for \( n^A = n|A| - n^2 + 1 \) if and only if n ∈ \{1, |A| - 1, |A|\}, or A ∈ AP, or n = 2 and A can be written in the form (1.15). (See Section 3 of [N1] and Section 1.3 of [N2].) Thus, if n = 1 or |A| = n + 1, whether A ∈ AP or not, the two sides of (1.13) are always equal; this corresponds to the exception (i). In the case n = 2, if A_2 = A is of the form (1.15), then |A_1| ∈ \{|A_2| - 1, |A_2|\} = \{3, 4\} and

\[ |S(\{A_i^n\}_{i=1}^2)| = |2^A| = 2|A| - 2^2 + 1 = 5 \]

\[ = 1 + \min\{|A_1| - 1, |A_2| - 2\} + |A_2| - 2 \]

though we may not have A_2 = A ∈ AP.

For the equality case of (1.13), Example 1.2 shows that the necessary conditions given by Theorem 1.2 are also sufficient in the case M = \{n\}.

From Theorem 1.2 we have

**Corollary 1.6.** Let A_1, \ldots, A_n be finite subsets of Z with |A_1| ≤ \ldots ≤ |A_n| and \( \min_{1 \leq i \leq n} (|A_i| - i) = 0 \). Put \( m = \max\{1 \leq i \leq n : |A_i| = i\} \). Suppose that the two sides of (1.13) are equal. Then A_n \ A_m ∈ AP unless we have one of the following:

(i') \( m \in \{n - 1, n\} \) or |A_n| = n + 1;

(ii') \( m = n - 2 \), |A_{n-1}| \in \{n + 1, n + 2\} and A_n \ A_{n-2} is of the form (1.15);

(iii') \( m < n - 1 \), |A_{n-1}| = n, A_{n-1} \ A_m and A_n \ A_{n-1} lie in AP, and d(A_{n-1} \ A_m) = d(A_n \ A_{n-1}).

**Proof.** Write M = \{m_1, \ldots, m_l\} where m_0 = 0 < m_1 < \ldots < m_l = n. Clearly m_1 = m. For any j ∈ [1, l] set A_i^j = A_{m_j} for all i ∈ (m_{j-1}, m_j]. By Theorem 1.2, A_i ⊆ A_{m_j} for all i = 1, \ldots, m_j. In the light of Example 1.2,
any $m_j - m_{j-1}$ distinct elements of $A_{m_j}$ can be arranged to form an SDR of $\{A_i\}_{m_j-1 < i \leq m_j}$. So

$$S(\{A_i\}_{i=1}^n) = S(\{A_i^*\}_{i=1}^n) = \left\{ \sum_{x \in A_m} x + \sum_{m < i \leq n} a_i : a_i \in A_i^* \setminus A_m, \text{ all the } a_i \text{ are distinct} \right\}$$

$$= \sum_{x \in A_m} x + S(\{A_i^* \setminus A_m\}_{i \in (m, n]})$$

where we regard $S(\emptyset)$ as $\{0\}$. Observe that

$$\sum_{i=1}^n \min_{i \leq j \leq n} (|A_j| - j) = \sum_{j=1}^i \sum_{m_{j-1} < i \leq m_j} (|A_m| - m_j)$$

$$= \sum_{m < i \leq n} \min_{i \leq j \leq n} (|A_i^*| - j)$$

$$= \sum_{m < i \leq n} \min_{i \leq j \leq n} (|A_i^* \setminus A_m| - (j - m)).$$

Thus

$$|S(\{A_i^* \setminus A_m\}_{i \in (m, n]})| = |S(\{A_i\}_{i=1}^n)|$$

$$= 1 + \sum_{m < i \leq n} \min_{i \leq j \leq n} (|A_i^* \setminus A_m| - (j - m)).$$

If $i \in (m, n]$, then $|A_i| - i > |A_m| - m = 0$ and hence $|A_i^* \setminus A_m| = |A_i^*| - m > i - m$.

Below we assume that $m \neq n$. Let us apply Theorem 1.2 to the sets $A_{m+1}^* \setminus A_m, \ldots, A_n^* \setminus A_m$. If $A_n^* \setminus A_m = A_n \setminus A_m \not\in \text{AP}$, then we are led to the exceptions corresponding to (i)–(iii) in Theorem 1.2. Obviously

$$|(m, n)| = 1 \iff m = n - 1 \quad \text{and} \quad |A_n^* \setminus A_m| = (n - m) + 1 \iff |A_n| = n + 1.$$ 

In the case $n - m = 2$, $A_n^* \setminus A_m = A_n \setminus A_{n-2}$ and

$$|A_{n-1}^* \setminus A_m| \in |A_n^* \setminus A_m| + \{0, -1\} \iff |A_{n-1}^*| \in |A_n| + \{0, -1\} \iff n - 1 \not\in M,$$

if $|A_n^* \setminus A_m| = |A_n \setminus A_{n-2}| = 4$ then $|A_n| = |A_{n-2}| + 4 = n + 2$ and

$$|A_{n-1}^* \setminus A_m| \in \{3, 4\} \iff |A_{n-1}| \in |A_n| + \{0, -1\} = \{n + 1, n + 2\}.$$ 

When $n - m > 1$, we have

$$|A_{n-1}^* \setminus A_m| = n - m \& (A_n^* \setminus A_m) \setminus (A_{n-1}^* \setminus A_m) \in \text{AP}$$

$$\iff |A_{n-1}^*| = n, \ A_{n-1}^* \neq A_n^* = A_n \& A_n^* \setminus A_{n-1}^* \in \text{AP}$$

$$\iff n - 1 \in M, \ |A_{n-1}| = n \& A_n \setminus A_{n-1} \in \text{AP}$$

$$\iff |A_{n-1}| = n \& A_n \setminus A_{n-1} \in \text{AP}.$$
In view of this, we have (i') or (ii') or (iii') if \( A_n \setminus A_m \not\subseteq \text{AP} \). ■

**Remark 1.6.** Clearly (i), (ii) and (iii) correspond to (i'), (ii') and (iii') with \( m = 0 \) and \( A_0 = \emptyset \). The proof of Corollary 1.6 shows that in the equality case of (1.13) those \( A_m \) with \( m \in M \) are vital.

Let \( A_1, \ldots, A_n \) be finite subsets of \( \mathbb{Z} \) satisfying (1.2). Theorem 1.2, together with Example 1.1, Remark 1.5 and Corollary 1.6, shows that we have completely determined the set \( \bigcup_{i=1}^{n} A_i = A_n \) in the equality case of (1.13).

**Corollary 1.7.** Let \( A_1, \ldots, A_n \) be finite subsets of \( \mathbb{Z} \) with (1.2) and \( |A_1| \leq \ldots \leq |A_n| \). Then

\[
|S(\{A_i\}_{i=1}^{n})| \geq 1 + \sum_{i=1}^{n} (|A_i| + h_i - n)
\]

where

\[
h_i = |\{ |A_j| : 1 \leq j \leq n & |A_j| > |A_i| \}|.
\]

Furthermore, when the lower bound in (1.16) is reached, \( A_i \subseteq A_n \) for all \( i = 1, \ldots, m \) if \( |A_m| < |A_{m+1}| - 1 \) or \( m = n \); also \( |A_l| < \ldots < |A_n| \) where \( l \) is the least index with \( |A_l| < |A_{l+1}| - 1 \) or \( l = n \); and providing \( \min\{n, |A_1| - 1, \ldots, |A_n| - n\} \geq 2 \) we have \( A_n \in \text{AP} \) unless \( A_n \) is of the form (1.15).

**Proof.** Let \( k_i = |A_i| \) for \( i \in [1, n] \). When \( i \in [1, n] \), if \( k_i = k_{i+1} \) then \( h_i = h_{i+1} \), if \( k_i \leq k_{i+1} - 1 \) then \( h_i = h_{i+1} + 1 \); thus \( k_i + h_i \leq k_{i+1} + h_{i+1} \), and \( k_i + h_i < k_{i+1} + h_{i+1} \) if and only if \( k_i < k_{i+1} - 1 \). For \( i \in [1, n] \), if \( j \in [i, n] \) then \( k_i + h_i - n \leq k_j + h_j - n \leq k_j - j \), so \( k_i + h_i - n \leq d_i = \min_{i \leq j \leq n} (k_j - j) \). Thus (1.16) holds by Theorem 1.2.

Clearly \( k_1 + h_1 = \ldots = k_l + h_l \) by the above, and \( d_1 = \ldots = d_l \) since \( k_1 - 1 \geq \ldots \geq k_l - l \). When \( k_i + h_i - n = d_i \) for all \( i = 1, \ldots, n \), for each \( m \in [1, n] \) we have

\[
m \in M \iff d_m < d_{m+1} \iff k_m + h_m < k_{m+1} + h_{m+1} \iff k_m < k_{m+1} - 1,
\]

so \( l \in M \) and \( k_l + h_l - n = d_l = k_l - l \), therefore \( h_l = n - l \) and \( |A_l| < \ldots < |A_n| \). Conversely, if \( |A_l| < \ldots < |A_n| \), then \( k_l - l \leq \ldots \leq k_n - n \) and hence \( d_i = k_i - i = k_i + h_i - n \) for all \( i \in [l, n] \). So \( k_i + h_i - n = d_i \) for all \( i \in [1, n] \) if and only if \( k_l < \ldots < k_n \).

Suppose that the two sides of (1.16) are equal. Then the two sides of (1.13) are equal, and \( k_l < \ldots < k_n \) by the above. In view of Theorem 1.2, \( \bigcup_{i=1}^{m} A_i = A_m \) provided that \( k_m < k_{m+1} - 1 \) or \( m = n \). If \( n \geq 2 \) and \( d_1 = \min_{1 \leq i \leq n} (k_i - i) \geq 2 \), then either \( A_n \in \text{AP} \), or \( n = 2 \) and \( A_2 \) can be written in the form (1.15). ■

**Remark 1.7.** In the case \( A_1 = \ldots = A_n = A \), we have \( h_1 = \ldots = h_n = 0 \) and Corollary 1.7 reduces to Theorem 2 of Nathanson [N1]. When
\[|A_1| < \ldots < |A_n|, \text{ Corollary 1.7 is a slight improvement on the main theorem of Cao and Sun [CS].} \]

**Corollary 1.8.** Let \( A_1, \ldots, A_n \) be finite subsets of \( \mathbb{Z} \) with (1.2). Then

\[
|S(\{A_i\}_{i=1}^n)| \geq \sum_{i=1}^n |A_i| - n^2 + 1.
\]

Providing \( 2 \leq n \leq |A_n| - 2 \) and \( |A_n| \neq 4 \), the two sides are equal if and only if \( A_1 = \ldots = A_n \in \text{AP} \).

**Proof.** If we rearrange the order of \( A_1, \ldots, A_n \), both sides of (1.16) keep unchanged. Suppose that \( |A_{\sigma(1)}| \leq \ldots \leq |A_{\sigma(n)}| \) where \( \sigma \) is a permutation on \( \{1, \ldots, n\} \). If \( |A_{\sigma(i)}| < i \), then

\[
[i, n] \subseteq \{1 \leq j \leq n : |A_j| \geq i\} \subseteq \{\sigma(j) : j \in (i, n)\},
\]

which is impossible. So \( |A_{\sigma(i)}| \geq i \) for all \( i \in [1, n] \). By Corollary 1.7, (1.16) holds and hence (1.18) follows. If both sides of (1.18) are equal, then \( h_i = 0 \) for all \( i = 1, \ldots, n \) and hence \( |A_1| = \ldots = |A_n| \), as \( \bigcup_{i=1}^n A_i = A_n \) by Corollary 1.7 we must have \( A_1 = \ldots = A_n \). Now it suffices to apply the Nathanson result.  

For the equality case of (1.13), let us look at one more example.

**Example 1.3.** Let \( k \) and \( n \) be integers with \( k > n > 1 \). Let \( A_1, \ldots, A_{n-1} \) be subsets of \( A_n = [0, k-1] \) with \( A_1 = [0, k-n] \backslash \{k-n-1\} \) and \( |A_i+1| = |A_i| \in \{0, 1\} \) for all \( i \in (1, n) \). We assert that

\[
S = S(\{A_i\}_{i=1}^n) = \left[ \frac{n(n-1)}{2}, kn - \frac{n(n+1)}{2} \right] \backslash \left\{ kn - \frac{n(n+1)}{2} - 1 \right\}
\]

and hence

\[
|S| = kn - n^2 = 1 + (|A_1| - 1) + (n - 1)(k - n) = 1 + \sum_{i=1}^n \min_{1 \leq j \leq n} (|A_j| - j).
\]

Since \( M = \{1, n\} \), by the arguments in the proof of Corollary 1.6, we may assume \( A_2 = \ldots = A_n \) without any loss of generality.

In the case \( k = n + 1 \), clearly \( A_1 = \{1\} \) and \( A_i = [0, n] \) for \( i \in (1, n) \); setting \( A = [0, n] \backslash \{1\} \) we then have

\[
S = 1 + (n-1)^\wedge A = 1 + \left\{ \sum_{x \in A} x - a : a \in A \right\} = \sum_{i=1}^n i - A
\]

\[
= \frac{n(n+1)}{2} - ([0, n] \backslash \{1\}) = \left[ \frac{n(n-1)}{2}, \frac{n(n+1)}{2} \right] \backslash \left\{ \frac{n(n+1)}{2} - 1 \right\}
\]

\[
= \left[ \frac{n(n-1)}{2}, kn - \frac{n(n+1)}{2} \right] \backslash \left\{ kn - \frac{n(n+1)}{2} - 1 \right\}.
\]
Below we verify the assertion on the condition $k > n + 1$. By Example 1.1,
\[
S \subseteq S([0, k - n], A_2, \ldots, A_n) = \frac{n(n - 1)}{2} + \left[0, \sum_{i=1}^{n} (k-n)\right]
\]
\[
= \left[\frac{n(n-1)}{2}, kn - \frac{n(n+1)}{2}\right]
\]
and $S$ contains
\[
S([0, k - n - 2], A_2, \ldots, A_n) = \frac{n(n-1)}{2} + \left[0, k - n - 2 + (n-1)(k-n)\right]
\]
\[
= \left[\frac{n(n-1)}{2}, kn - \frac{n(n+1)}{2} - 2\right].
\]
Observe that
\[
\text{max } S = k - n + (k - n - 1) + \cdots + (k - 1) = kn - \frac{n(n+1)}{2}.
\]
Now it suffices to show that $kn - (n+1)/2 - 1 \not\in S$. On the contrary, we can write
\[
kn - \frac{n(n+1)}{2} - 1 = k - n + (k - i_1) + \cdots + (k - i_{n-1})
\]
where $1 \leq i_1 < \cdots < i_{n-1} \leq k$ and $n \not\in \{i_1, \ldots, i_{n-1}\}$. Apparently
\[
i_1 + \cdots + i_{n-1} = \frac{n(n+1)}{2} + 1 - n, \quad \text{i.e. } \sum_{j=1}^{n-1} (i_j - j) = 1.
\]
So $i_t - t = 1$ for some $t \in [1, n)$, and $i_j = j$ for all $j \in [1, n) \setminus \{t\}$. As $i_{n-1} \neq n$, we have $t < n - 1$ and hence $i_t = t + 1 = i_{t+1}$. This contradicts $i_t < i_{t+1}$.

Let $A_1, \ldots, A_{n-1}$ be subsets of $A_n = [0, k_n - 1]$ with the two sides of (1.13) equal. Set $A_i' = \{k_n - 1 - x : x \in A_i\}$ for $i = 1, \ldots, n$. Then
\[
|S(\{A_i'\}_{i=1}^{n})| = |S(\{A_i\}_{i=1}^{n})| = 1 + \min_{i \leq j \leq n} (|A_i'| - j).
\]
If $\min A_1 + \max A_1 \geq k_n$, then $\min A_1' + \max A_1' = 2(k_n - 1) - \min A_1 - \max A_1 < k_n$. So, to discuss the equality case of (1.13) with $A_n \in \text{AP}$, we may simply take $A_n = [0, k_n - 1]$ and assume that $\min A_1 + \max A_1 < k_n$.

Now we pose a conjecture which essentially determines the equality case of (1.13).

**Conjecture 1.1.** Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{Z}$ with $|A_1| \leq \cdots \leq |A_n|$, $k_i = |A_i| > i$ for $i \in [1, n]$, and $\bigcup_{i=1}^{n} A_i = A_m$ for all $m \in M$. Suppose that $A_n = [0, k_n - 1]$ and $\min A_1 + \max A_1 < k_n$. If the two sides of (1.13) are equal, then $A_m = [0, k_m - 1]$ for all $m \in M$, unless
\[
M = \{1, n\}, \quad k_n - k_1 = n \quad \text{and} \quad A_1 = [0, k_1] \setminus \{k_1 - 1\}.
\]
Though we are unable to solve this conjecture, we have found evidence to support it through computer calculations.

2. Proof of Theorem 1.1. We use induction on \( n \). In the case \( n = 1 \), the inequality is obvious since \( C = A_1 \) and \( V_1 = V = \emptyset \). So we proceed to the induction step.

Let \( n > 1 \) and assume the assertion holds for smaller values of \( n \). Set

\[
V' = \{(s, t, \mu, \nu, w) \in V : 1 \leq s, t \leq n - 1\}.
\]

For each \( i = 1, \ldots, n - 1 \) let \( A'_i \) consist of those \( a_i \in A_i \) for which \( \mu a_i + \nu a \neq w \) if \( (i, n, \mu, \nu, w) \in V \), and \( \mu a + \nu a_i \neq w \) if \( (n, i, \mu, \nu, w) \in V \). Apparently

\[
|A'_i| \geq |A_i| - |\{(s, t, \mu, \nu, w) \in V : \{s, t\} = \{i, n\}\}|
\]

and thus

\[
V'_i = \{(s, t, \mu, \nu, w) \in V' : i \in \{s, t\}\}
= V_i \setminus \{(s, t, \mu, \nu, w) \in V : \{s, t\} = \{i, n\}\}
\]

has cardinality not greater than \( |V_i| + |A'_i| - |A_i| < |A'_i| \). Let

\[
C' = \{a_1 + \ldots + a_{n-1} : a_i \in A'_i, \text{ and } \mu a_i + \nu a_j \neq w \text{ if } (i, j, \mu, \nu, w) \in V'\}.
\]

By the induction hypothesis,

\[
|C'| \geq 1 + \sum_{i=1}^{n-1} (|A'_i| - |V'_i| - 1) \geq 1 + \sum_{i=1}^{n-1} (|A_i| - |V_i| - 1) > 0.
\]

Write \( \max C' = \sum_{i=1}^{n-1} a'_i \) where \( a'_1 \in A'_1, \ldots, a'_{n-1} \in A'_{n-1}, \) and \( \mu a'_i + \nu a'_j \neq w \) if \( (i, j, \mu, \nu, w) \in V' \). Let \( A'_n \) consist of those \( a_n \in A_n \) for which \( \mu a'_i + \nu a_n \neq w \) if \( (i, n, \mu, \nu, w) \in V \), and \( \mu a_n + \nu a'_i \neq w \) if \( (n, i, \mu, \nu, w) \in V \). Note that \( a \in A'_n \) and \( |A'_n| \geq |A_n| - |V_n| > 0 \). Clearly

\[
(C' + a) \cup (a'_1 + \ldots + a'_{n-1} + A'_n) \subseteq C
\]

and

\[
\max(C' + a) = a'_1 + \ldots + a'_{n-1} + a = \min(a'_1 + \ldots + a'_{n-1} + A'_n).
\]

Therefore

\[
|C| \geq |C' + a| + |a'_1 + \ldots + a'_{n-1} + A'_n| - 1 = |C'| + |A'_n| - 1
\]

\[
\geq 1 + \sum_{i=1}^{n-1} (|A_i| - |V_i| - 1) + |A_n| - |V_n| - 1 = 1 + \sum_{i=1}^{n} (|A_i| - |V_i| - 1).
\]

Since \( \sum_{i=1}^{n} |V_i| = 2|V| \), we are done.
3. Several lemmas. We first check the exception (iii) given in Theorem 1.2.

**Lemma 3.1.** Let $A_1, \ldots, A_n$ ($n > 1$) be finite subsets of $\mathbb{Z}$ such that $|A_i| > i$ for all $i \in [1, n]$, $|A_{n-1}| = n < |A_n| - 1$ and $\bigcup_{i=1}^{n-1} A_i = A_{n-1} \subseteq A_n$. Then the two sides of (1.13) are equal if and only if $A_{n-1}, A_n \setminus A_{n-1} \in \text{AP}$ and $d(A_{n-1}) = d(A_n \setminus A_{n-1})$.

**Proof.** Let $S = S(\{A_i\}_{i=1}^n)$ and $k_i = |A_i|$ for all $i = 1, \ldots, n$. Write $A_{n-1} = \{x_1, \ldots, x_n\}$ and $A_n \setminus A_{n-1} = \{y_1, \ldots, y_{k_n-k_{n-1}}\}$ where $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_{k_n-k_{n-1}}$. Since $k_i - i \geq 1 = k_{n-1} - (n-1)$ for all $i \in [1, n-1]$, $S(\{A_i\}_{i=1}^{n-1}) = (n-1)^\text{AP} A_{n-1}$ as pointed out in Example 1.2. Thus

$$S = \bigcup_{i=1}^{n} \{x_1 + \ldots + x_n - x_i + y : y \in \{x_i, y_1, \ldots, y_{k_n-k_{n-1}}\}\}$$

and hence $|S| = 1 + |(A_n \setminus A_{n-1}) - A_{n-1}|$ where we let $A - B = A + (-B) = \{a - b : a \in A, b \in B\}$ for $A, B \subseteq \mathbb{Z}$. By a known result (cf. Lemma 1.3 and Theorem 1.5 of [N2]), for any finite subsets $A$ and $B$ of $\mathbb{Z}$ with $|A| \geq 2$ and $|B| \geq 2$, $|A + B| = |A| + |B| - 1$ if and only if $A, B \in \text{AP}$ and $d(A) = d(B)$. So

$$|S| = 1 + \sum_{i=1}^{n} \min_{i \leq j \leq n} (k_j - j) = 1 + (n-1)(k_{n-1} - (n-1)) + k_n - n - n = k_n$$

\[\iff |(A_n \setminus A_{n-1}) - A_{n-1}| = k_n - 1 = |A_n \setminus A_{n-1}| + |A_{n-1}| = 1\]

\[\iff x_{i+1} - x_i = y_{j+1} - y_j \text{ for all } i \in [1, n] \text{ and } j \in [1, k_n - k_{n-1}]\].

The following lemma is an improvement on Lemma 2 of [CS].

**Lemma 3.2.** Let $A_1$ and $A_2$ be finite subsets of $\mathbb{Z}$ with $|A_1| \geq 3, A_1 \subset A_2$, $\min A_1 = \min A_2$, $\max A_1 \neq \max A_2$ and $|S(A_1, A_2)| = |A_1| + |A_2| - 2$. Then $A_2 \in \text{AP}$ unless $|A_1| = 3$ and $A_2$ can be written in the form (1.15).

**Proof.** Let $A_1 = \{a_1, \ldots, a_k\}$ and $A_2 = \{b_1, \ldots, b_l\}$ where $a_1 < \ldots < a_k$ and $b_1 < \ldots < b_l$. By the proof of Lemma 2 of [CS], $a_i \in \{b_i, b_{i+1}\}$ for all $i \in [1, k]$,

$$S(A_1, A_2) = \{a_1 + b_2, \ldots, a_1 + b_{l-1}, a_1 + b_l, \ldots, a_k + b_l\},$$

and $A_2 \in \text{AP}$ if $a_3 < b_{l-1}$.

Suppose that $a_3 = b_{l-1}$. Then $k = 3$ since $a_3 \leq a_k < b_l$. As $a_1 + b_{l-1} < a_2 + b_{l-1} < a_2 + b_l$, we must have $a_2 + b_{l-1} = a_1 + b_l$, i.e. $b_l - b_{l-1} = a_2 - a_1$. If $a_3 = b_3$, then $l = 4, a_2 = b_2$ and hence $b_4 - b_3 = b_2 - b_1$, so $A_2$ is of the form (1.15). Below we let $a_3 = b_4$. Then $l = 5$ and $b_5 - b_4 = a_2 - a_1$. As $a_1 + b_4 < a_3 + b_2 = b_4 + b_2 \leq a_2 + b_4 = a_1 + b_5$, we must have $a_2 = b_2 < b_3$. Observe that
\[a_1 + b_3 < a_2 + b_3 < a_2 + b_4 = a_1 + b_5 < a_3 + b_3 < a_3 + b_5.\]  
So \(a_2 + b_3 = a_1 + b_4\) and \(a_3 + b_3 = a_2 + b_5\), therefore \(A_2 \in \text{AP} \). ■

We now present a lemma reflecting some symmetry.

**Lemma 3.3.** Let \(A_1, \ldots, A_n\) be finite subsets of \(\mathbb{Z}\) with \(A_1 = \ldots = A_m \subseteq A_{m+1} = \ldots = A_n\) and \(0 < |A_m| - m \leq |A_n| - n\) where \(m \in [1, n]\). Define the dual sequence \(\{B_j\}_{j=1}^{A_n - n}\) of \(\{A_i\}_{i=1}^n\) as follows:

\[B_i = A_n \setminus A_m\]  
for each \(i \in [1, |A_n| - n - (|A_m| - m)]\)

and

\[B_j = A_n\]  
for all \(j \in [|A_n| - n - (|A_m| - m), |A_n| - n]\).

Then \(|S(\{A_i\}_{i=1}^n)| = |S(\{B_i\}_{i=1}^{A_n - n})|\) and

\[\sum_{i=1}^{n} \min_{i \leq j \leq n} (|A_j| - j) = \sum_{i=1}^{n} \min_{i \leq j \leq n} (|B_j| - j).\]

**Proof.** Let \(k_m = |A_m|\) and \(k_n = |A_n|\). Suppose that \(A_m = \{x_1, \ldots, x_{k_m}\}\) and \(A_n \setminus A_m = \{y_1, \ldots, y_{k_n-k_m}\}\). Then \(S(\{A_i\}_{i=1}^n)\) consists of integers of the form \(\sum_{i \in I} x_i + \sum_{j \in J} y_j\) where \(I \subseteq [1, k_m]\), \(J \subseteq [1, k_n-k_m]\), \(|I| + |J| = n\) and \(|I| \geq m\), in other words the elements of \(S(\{A_i\}_{i=1}^n)\) are integers of the form

\[\sum_{i=1}^{k_m} x_i - \sum_{i \in I} x_i + \sum_{j=1}^{k_n-k_m} y_j - \sum_{j \in J} y_j\]

where \(I \subseteq [1, k_m]\), \(J \subseteq [1, k_n-k_m]\), \(|I| + |J| = k_m + (k_n-k_m) - n = k_n - n\) and \(|J| \geq k_n - k_m - (n - m) = k_n - n - (k_m - m)\). Thus

\[S(\{A_i\}_{i=1}^n) = \sum_{x \in A_n} x - S(\{B_i\}_{i=1}^{k_n-n})\]

and so

\[|S(\{A_i\}_{i=1}^n)| = |S(\{B_i\}_{i=1}^{k_n-n})|.\]

Clearly

\[\sum_{i=1}^{n} \min_{i \leq j \leq n} (|A_j| - j) = m(k_m - m) + (n - m)(k_n - n).\]

Also,

\[\sum_{i=1}^{k_n-n} \min_{i \leq j \leq n} (|B_j| - j) - (k_m - m)(|A_n| - (k_n - n))\]

\[= (k_n - n - (k_m - m))(|A_n \setminus A_m| - (k_n - n - (k_m - m)))\]

\[= (n - m)(k_n - n) + (m - n)(k_m - m).\]

This concludes the proof. ■
Let $A_1 \subseteq A_2 \subseteq \mathbb{Z}$, $|A_1| = 3$ and $|A_2| = 4$. Then the dual sequence of $\{A_i\}_{i=1}^n$ is the sequence $A_2, A_2$. Thus the example (given by Nathanson) with $|2^\wedge A_2| = 2|A_2| - 2^2 + 1$ and $A_2 \notin AP$, induces the exception (ii) in Theorem 1.2.

4. Reduction of Theorem 1.2. Let $A_1, \ldots, A_n$ be finite subsets of $\mathbb{Z}$ with (1.2) and $|A_1| \leq \ldots \leq |A_n|$. Put $d_i = \min_{i \leq j \leq n}(|A_j| - j)$ and $k'_i = d_i + i$ for $i = 1, \ldots, n$. Clearly $k''_i = |A_n|$ and $k'_i < k'_{i+1}$ for all $i \in [1, n)$. As $k'_i \leq |A_i|$, we can choose a subset $A'_i$ of $A_i$ with $|A'_i| = k'_i$. Obviously $A'_n = A_n$ and $\sum_{i=1}^n |A'_i| \leq \sum_{i=1}^n |A_i|$. By the Theorem of Cao and Sun [CS], we have

$$|S(\{A_i\}_{i=1}^n)| \geq |S(\{A'_i\}_{i=1}^n)| \geq 1 + \sum_{i=1}^n (k'_i - i) = 1 + \sum_{i=1}^n d_i.$$ 

So (1.13) holds. If equality is valid in (1.13), then

$$|S(\{A'_i\}_{i=1}^n)| = 1 + \sum_{i=1}^n (k'_i - i),$$

hence by the Theorem of [CS] we have $\bigcup_{i=1}^m A'_i = A'_m \subseteq A_m$ for any $m$ in the set

$$M = \{1 \leq i \leq n : k'_i < k'_{i+1} - 1\} \cup \{n\} = \{1 \leq i \leq n : d_i < d_{i+1}\} \cup \{n\} = \{1 \leq i \leq n : |A_i| - i < |A_j| - j \text{ for all } j \in (i, n]\}.$$ 

For any $i = 1, \ldots, n$, if $a_i \in A_i$ then we can select $A'_i \subseteq A_i$ so that $a_i \in A'_i$ and $|A'_i| = k'_i$. Thus, in the equality case of (1.13) we have $\bigcup_{i=1}^m A_i \subseteq A_m$ for all $m \in M$.

Let $1 \leq i \leq n$. Then

$$k'_i > i \iff d_i > 0 \iff |A_j| > j \text{ for all } j \in [i, n].$$

Thus

$$|A_i| > i \text{ for all } i \in [1, n] \iff |A'_i| > i \text{ for all } i \in [1, n].$$

Recall that $A'_n = A_n$. When $n = 2$ and $A'_2 = A_2$ is of the form (1.15), clearly

$$|A_1| \in \{3, 4\} \iff |A_1| - 1 \geq |A_2| - 2 \iff d_1 = 2 \iff k'_1 = 3.$$ 

In the case $n > 1$ and $|A_n| > n$, we have

$$|A_{n-1}| = n \iff d_{n-1} = 1 \iff k'_{n-1} = n,$$

thus $A_{n-1} = A'_{n-1}$ providing $|A_{n-1}| = n$ or $k'_{n-1} = n$.

In view of the above and Lemma 3.1, Theorem 1.2 can be reduced to the following
Theorem 4.1. Let $A_1, \ldots, A_n$ be subsets of $\mathbb{Z}$ with $|A_1| < \ldots < |A_n| < \infty$ and $|A_i| > i$ for all $i = 1, \ldots, n$. If

$$|S_{\leq}(A_i)_{i=1}^n| = 1 + \sum_{i=1}^n(|A_i| - i),$$

then $A_n \in \text{AP}$ unless we have (i) or (iii), or (ii) with $|A_1| = 3$.

Remark 4.1. Let $k$ be a positive integer. By the previous reasoning, if Theorem 4.1 holds for those subsets $A_1, \ldots, A_n$ of $\mathbb{Z}$ with $|A_1| + \ldots + |A_n| \leq k$, then so does Theorem 1.2.

5. Proof of Theorem 4.1. We proceed by induction on $k = \sum_{i=1}^n |A_i|$. Apparently $k \geq |A_1| > 1$.

If $k = 2$, then $n = 1$ and $|A_1| = 2$. In the case $n = 1$, both (4.1) and (i) hold.

Below we let $k > 2$ and $n \geq 2$, and assume that the result holds if $|A_1| + \ldots + |A_n| < k$. Now let $|A_1| + \ldots + |A_n| = k$. For all $i \in [1, n]$ we set

$$k_i = |A_i| \quad \text{and} \quad d_i = \min_{i \leq j \leq n} (k_j - j) = k_i - i.$$

Obviously $1 \leq d_1 \leq \ldots \leq d_n$. Put

$$a = \min \bigcup_{i=1}^n A_i, \quad I = \{1 \leq i \leq n : a \in A_i\}, \quad r = \min I, \quad t = \max I.$$

For $i \in I$ let

$$A'_i = \begin{cases} A_i \setminus \{a\} & \text{if } i \neq r, \\ \{a\} & \text{if } i = r; \end{cases}$$

and for $i \in \bar{I} = [1, n] \setminus I$ set

$$A'_i = \begin{cases} A_i \setminus \{a_i\} & \text{if } r < i < t \text{ and } i \notin M, \\ A_i & \text{otherwise}, \end{cases}$$

where $a_i$ is an arbitrary element of $A_i$. Write $k'_i = |A'_i|$ for $i \in [1, n] \setminus \{r\}$. Then $1 < k'_1 < \ldots < k'_{r-1} < k_r \leq k'_{r+1} < \ldots < k'_n$ and $\sum_{i \neq r} k'_i < \sum_{i=1}^n k_i = k$. For $i \in [1, n] \setminus \{r\}$ we set

$$d'_i = \begin{cases} k'_i - i & \text{if } i < r, \\ k'_i - (i - 1) & \text{if } i > r. \end{cases}$$

Let $S = S_{\leq}(A_i)_{i=1}^n$, and assume that (4.1) holds. By the Theorem of [CS] and its proof, $\bigcup_{i=1}^m A_i = A_m$ for all $m \in M$, and

$$|S_{\leq}(A'_i)_{i \neq r}| = \sum_{i \neq r} k'_i - \frac{n(n-1)}{2} + 1 = 1 + \sum_{i \neq r} d'_i.$$
Also \( t = n \) and \((r, t) \cap T \cap M = \emptyset \) (see (12) and (14) of \([CS]\)), therefore \( k'_i = k_i - 1 \) for \( i \in (r, n) \) and \( d'_i = d_i \) for all \( i \in [1, n] \setminus \{r\} \).

Clearly \( b = \max \bigcup_{i=1}^n A_i \neq a \) (otherwise \( |A_n| = |\{a\}| < n \)), \(-b = \min \bigcup_{i=1}^n (-A_i)\) and

\[
|S(\{-A_i\}_{i=1}^n)| = |S| = 1 + \sum_{i=1}^n \min_{i \leq j \leq n} (|A_j| - |j|).
\]

Like the fact that \( a \in A_t = A_n \) we should also have \(-b \in -A_n\). Thus \( b \in A_n \setminus \{a\} \).

Let \( s \) denote the least index such that \( b \in A_s \). By p. 166 of \([CS]\), there exists an \( l \in [r, n] \) such that \( k_l - l = k_r - r \) (i.e. \( d_r = \ldots = d_l \)), and \( l = s = r < n \) is impossible.

From now on we assume that none of (i)--(iii) (in Theorem 1.2) holds. Then \( k_n > n + 1 \). If \( k_{n-1} = n \), then \( n - 1 \in M \) and \( \bigcup_{i=1}^{n-1} A_i = A_{n-1} \subseteq A_n \), thus (iii) holds by Lemma 3.1. Now that (iii) fails, we must have \( k_{n-1} > n \).

We claim that \( A^* = A_n \setminus \{a\} \in \text{AP} \). For this conclusion, it suffices to work under the condition \( A^* \notin \text{AP} \).

**Case 1.** \( r < n - 1 \). Apparently \( n > 2 \), \( k'_n = k_n - 1 > n = (n - 1) + 1 \) and \( k'_{n-1} = k_{n-1} - 1 > n - 1 = (n - 2) + 1 \). As \( A'_n = A^*_n \notin \text{AP} \), by the induction hypothesis, \( n - 1 = 2, r = 1, k'_2 = 3 \) and \( A'_3 = A_3 \setminus \{a\} \) is of the form (1.15).

Note that \( k_2 = k'_2 + 1 = 4 \) and \( k_3 = k'_3 + 1 = 5 \). If \( k_1 > 2 \), then \( k_1 = 3 \) and \( M = \{3\} \), hence \( S = 3^A \) and \( A_3 \in \text{AP} \) by Example 1.2. Thus \( k_2 = 4 \) and \( k_3 = 5 \). Observe that \( |S| = 1 + (2 - 1) + (4 - 2) + (5 - 3) = 6 \).

If \( 1 \leq i < j \leq 4 \), then \( x_i \) or \( x_j \) lies in \( A_2 \) (since \( A_2 \subseteq A_3 \) and \( k_3 - k_2 = 1 \)), therefore \( a + x_i + x_j \in S \). Thus \( S \) contains the following 5 integers:

\[
a + x_1 + x_2, \ a + x_1 + x_3, \ a + x_1 + x_4 = a + x_2 + x_3, \ a + x_2 + x_4, \ a + x_3 + x_4.
\]

Suppose that \( A_1 = \{a, x_i\} \) where \( 1 \leq i \leq 4 \). If \( i \in \{3, 4\} \), then both \( x_4 + x_3 + x_1 \) and \( x_4 + x_3 + x_2 \) belong to \( S \), this contradicts the fact that \( |S| = 6 < 5 + 2 \). So \( i \in \{1, 2\} \), and \( S \) consists of the above 5 integers and the number \( x_i + x_3 + x_4 \). Apparently \( S \) also contains \( x_1 + x_2 + x_3 \) and \( x_1 + x_2 + x_4 \). Since \( a + x_2 + x_3 < x_1 + x_2 + x_3 < x_1 + x_2 + x_4 < x_i + x_3 + x_4 \), we must have \( x_1 + x_2 + x_3 = a + x_2 + x_4 \) and \( x_1 + x_2 + x_4 = a + x_3 + x_4 \). Thus \( x_4 - x_3 = x_1 - a = x_3 - x_2 \) and hence \( A_n = A_3 \in \text{AP} \).

**Case 2.** \( A_{n-1} \subset A^*_n \). As \( n - 1 \in M, a \notin A_{n-1} \in \bigcup_{i=1}^{n-1} A_i \) and so \( r = n \). Clearly \( k_1 < \ldots < k_{n-1} < k^*_n = |A^*_n| = k_n - 1 \). Let \( S^* \) denote the set \( S(A_1, \ldots, A_{n-1}, A^*_n) \). Then \( a + \min S(\{A_i\}_{i=1}^{n-1}) = \min S < \min S^* \). So \( |S^*| \leq |S| - 1 = \sum_{i=1}^n (k_i - i) \) and hence \( |S^*| = |S| - 1 = 1 + \sum_{i=1}^{n-1} (k_i - i) + (k^*_n - n) \).

Recall that \( k^*_n = k_n - 1 > k_{n-1} \geq n + 1 \). By the induction hypothesis, \( n = 2, k_1 = 3, A^*_2 \) has the form (1.15) and hence \( k_2 = 5 \). For any two distinct
elements $x$ and $y$ of $A^*_2$ we have $x+y \in S^*$ since one of them belongs to $A_1$. All the $1 + (3 - 1) + (4 - 2) = 5$ elements of $S^*$ are as follows:

\[ x_1 + x_2, \ x_1 + x_3, \ x_1 + x_4 = x_2 + x_3, \ x_2 + x_4, \ x_3 + x_4. \]

As $|a + A_1| = 3$, $\max(a + A_1) < x_1 + x_4$ and $|S| = 1 + (3 - 1) + (5 - 2) = 6$, we must have

\[ S = (a + A_1) \cup \{x_1 + x_4 : i = 1, 2, 3\}. \]

Evidently $x_4 \in A_1$ and $x_1 + x_3 = a + x_4$ since $x_1 + x_3 \in a + A_1$, also $x_3 \in A_1$ and $x_1 + x_2 = a + x_3$ since $x_1 + x_2 \in a + A_1$. So $x_4 - x_3 = x_1 - a = x_3 - x_2$ and hence $A_n = A_2 \in \text{AP}$. 

CASE 3. $r = n - 1$, or $r = n$ and $A_{n-1} = A^*_n$. Let $\bar{r} = n$ if $r = n - 1$, and $\bar{r} = n - 1$ if $r = n$. Clearly $A^*_r = A^*_n$ and $k^*_r = |A^*_n| = k_n - 1 > n = (n - 1) + 1$.

Let us handle the case $n = 2$. Note that $k_1 = k_{n-1} > n = 2$. If $A_1 = A_2$, then $\min(-A_1) = \min(-A_2)$ and $\max(-A_1) < \max(-A_2) = -a$, hence $-A_2 \in \text{AP}$ (i.e. $A_2 \in \text{AP}$) by Lemma 3.2 since $\text{(ii)}$ fails. When $r = 1$, we have $\min A_1 = \min A_2$, if $s = 1$ (i.e. $\max A_1 \neq \max A_2$) then $A_2 \in \text{AP}$ by Lemma 3.2. In the case $r = s = 1$, we have $l > 1$ because $l = r = s < n$ is impossible, hence $k_1 = k_2 - 1$ since $k_r - r = k_l - l$, thus $S = 2^\wedge A_2$ and $A_2 \in \text{AP}$ by Example 1.2. (Recall that $\text{(ii)}$ fails.)

Let $n - 1 = 2, k_1 = k'_1 = 3$ and $A^*_1$ have the form $(1.15)$. Observe that $n = 3 < k_{n-1} = k_2 \leq k_3 - 1 = |A^*_3| = |A^*_r| = 4$. So $M = \{3\}$ and hence $A_3 \in \text{AP}$ by Example 1.2.

Now we assume that $n > 2$, and $n \neq 3$ or $k'_1 \neq 3$ or $A^*_r$ is not of the form $(1.15)$. As $A^*_r = A^*_n \not\in \text{AP}$, by the induction hypothesis, $k_{n-2} = k'_{n-2} = n - 1$, also $A_{n-2} = A^*_{n-2}$ and $A^*_n \setminus A_{n-2} = A^*_r \setminus A^*_{n-2}$ form arithmetic progressions with the same difference $d$. Since $k_{n-2} = n - 1 < n < k_{n-1}$, we have $n - 2 \in M$ and hence $\bigcup_{i=1}^{n-2} A_i = A_{n-2} \subseteq A^*_n$. Let $A^*_{n-1} = A_{n-1} \setminus \{a\}$, $k^*_{n-1} = |A^*_{n-1}|$ and $S^* = S(A_1, \ldots, A_{n-2}, A^*_{n-1}, A^*_n)$. Then

\[ 1 < k_1 < \ldots < k_{n-2} = n - 1 < k^*_{n-1} \leq k^*_n < k_n, \]

\[ d^*_n = k^*_n - n = k_n - 1 - n = d_n - 1 > 0, \]

\[ d^*_{n-1} = \min\{k^*_{n-1} - (n - 1), k^*_n - n\} = k_n - n = d_{n-1} - 1 > 0, \]

\[ d^*_i = \min\{k_i - i, \ldots, k_{n-2} - (n - 2), d^*_{n-1}\} = 1 = d_i \text{ for } i \in [1, n - 2]. \]

Write $A_{n-2} = \{x_1, \ldots, x_{n-1}\}$ and $A^*_n \setminus A_{n-2} = \{y_1, \ldots, y_{k_{n-1} - (n-1)}\}$ where $x_1 < \ldots < x_{n-1}$ and $y_1 < \ldots < y_{k_{n-1} - n}$. In view of Example 1.2, $S(\{A_i\}_{i=1}^{n-2}) = (n - 2)^\wedge A_{n-2} = \{x - x_i : 1 \leq i \leq n - 1\}$ where $x = \sum_{i=1}^{n-1} x_i$. As $A^*_{n-1} \subseteq A^*_n$ all elements of $S^*$ have the form $x - x_i + y_i + z$ where $1 \leq i \leq n - 1$, $1 \leq j \leq k_n - n$ and $z \in \{x_i, y_1, \ldots, y_{k_n} \setminus \{y_j\}\}$, they are all greater than $x - x_{n-1} + y_1 + a$. If $x - x_{n-1} + y_2 + a = x - x_i + y_j + z$ where $i, j, z$ are as above, then $j = 1$ and $z = x_i$ since $a + y_2 < \min\{x_i + y_2, y_1 + y_2\}$, hence $-x_{n-1} + y_2 + a = -x_i + y_1 + x_i = y_1$ and $x_{n-1} - a = y_2 - y_1 = d = x_{n-1} - x_{n-2}$;
this is impossible. So \( x - x_{n-1} + y_1 + a, x - x_{n-1} + y_2 + a \not\in S^* \). However, both \( x - x_{n-1} + y_1 + a \) and \( x - x_{n-1} + y_2 + a \) lie in \( S \), for, \( a \in A_{n-1} \) if \( r = n - 1 \), and \( y_1, y_2 \in A_{n-1} \) if \( A_{n-1} = A_n^* \). Therefore

\[
|S^*| \leq |S| - 2 = 1 + \sum_{i=1}^{n} d_i - 2 = 1 + \sum_{i=1}^{n} d_i^*.
\]

If \( A_{n-1} = A_n^* \), then \( k_{n-1} = k_{n-1} > n \). Since \( A_n^* \not\in \text{AP} \), by Remark 4.1 and the induction hypothesis we have either

(i*) \( k_n - 1 = k_n^* = n + 1 \) and hence \( k_{n-1} = n + 1 \), or

(iii*) \( |A_{n-1}^*| = n \) (whence \( r = n - 1 \)), and \( A_{n-1}^* \) and \( A_n \setminus A_{n-1} = A_n^* \setminus A_{n-1}^* \) form arithmetic progressions with the same difference.

Assume (i*). Let \( B_1 = \ldots = B_{n-2} = A_{n-2} \) and \( B_{n-1} = B_n = A_n \). As \( M = \{n-2, n\} \), by the idea in Example 1.2 or the proof of Corollary 1.6, \( S = S(B_1) \) and \( |S(B_1)| = 1 + \sum_{i=1}^{n} \min_{i \leq j \leq n} (|B_j| - j). \) The dual sequence of \( (B_i) \) is the sequence \( A_n \setminus A_{n-2}, A_n \) with \( |A_n \setminus A_{n-2}| = n + 2 - (n-1) = 3, |A_n| = n + 2 > 4 \) and \( |A_n \setminus A_{n-2}| + |A_n| < (n+1) + k_n \leq k = k_1 + \ldots + k_n \). In view of Lemma 3.3 and the induction hypothesis, we have \( A_n \in \text{AP} \).

Now we consider the case (iii*). Clearly \( k_{n-1} = n + 1 \) and \( k_n - k_{n-1} \geq 2 \), so \( n - 1 \in M \) and \( A_{n-2} \subset A_{n-1} \subset A_n \). Write \( A_{n-1} = \{a, x_1, \ldots, x_{n-1}, y_j\} \) where \( 1 \leq j \leq k_n - n \). Then \( A_n \setminus A_{n-1} = \{y_1, \ldots, y_{k_n-n}\} \setminus \{y_j\} \). Since \( d(A_{n-1}^*) = d(A_n \setminus A_{n-1}) \geq d \), we must have \( y_j \in \{x_1 - d, x_{n-1} + d\} \). Now that \( d(A_n \setminus A_{n-1}) = d(A_{n-1}^*) = d \), \( j \) must be 1 or \( k_n - n \). If \( y_1 \in A_{n-1} \) (i.e. \( j = 1 \)), then \( y_1 + d = y_2 \neq x_1 \) and hence \( y_1 = x_{n-1} + d \), thus \( A_n^* = \{x_1, \ldots, x_{n-1}, y_1, \ldots, y_{k_n-n}\} \in \text{AP} \). If \( y_{k_n-n} \in A_{n-1} \) (i.e. \( j = k_n - n \)), then \( y_{k_n-n} - d = y_{k_n-n-1} \neq x_{n-1} \) and hence \( y_{k_n-n} = x_{1} - d \), thus \( A_n^* = \{y_1, \ldots, y_{k_n-n}, x_1, \ldots, x_{n-1}\} \in \text{AP} \).

By the above, we do have \( A_n \setminus \{a\} \in \text{AP} \) in either case. As \( -b = \min \bigcup_{i=1}^{n} (-A_i) \), by analogy \( -A_n \setminus \{-b\} \in \text{AP} \). Because \( k_n > n + 1 \geq 3 \), and \( A_n \setminus \{\min A_n\} \) and \( A_n \setminus \{\max A_n\} \) are both in \( \text{AP} \), the set \( A_n \) must form an arithmetic progression.

The induction step is now complete and the proof of Theorem 4.1 is finished.

References


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