Hausdorff dimensions in Engel expansions

by

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1. Introduction. Given $x$ in $(0,1]$, let $x = [d_1(x), d_2(x), \ldots]$ denote the Engel expansion of $x$, that is,

$$x = \frac{1}{d_1(x)} + \frac{1}{d_1(x)d_2(x)} + \cdots + \frac{1}{d_1(x)d_2(x)\cdots d_n(x)} + \cdots,$$

where $\{d_j(x), j \geq 1\}$ is a sequence of positive integers satisfying $d_1(x) \geq 2$ and $d_{j+1}(x) \geq d_j(x)$ for $j \geq 1$ (see [3]). In [3], János Galambos proved that for almost all $x \in (0,1]$,

$$\lim_{n \to \infty} d_n^{1/n}(x) = e.$$ 

Also he posed the following questions (see [3], P132):

(i) Find the Hausdorff dimension of the set where (2) fails.
(ii) For any $k \geq 1$, let

$$A_k = \{x \in (0,1] : \log d_n(x) \geq kn \text{ for any } n \geq 1\}.$$ 

Find the Hausdorff dimension of the set $A_k$.

For (i), the second author [4] has proved that the Hausdorff dimension of the set where (2) fails is 1.

In this paper, we get a stronger result than those in (i) and (ii). We show

Theorem. For any $\alpha \geq 1$, let

$$A(\alpha) = \{x \in (0,1] : \lim_{n \to \infty} d_n^{1/n}(x) = \alpha\}.$$ 

Then

$$\dim_H A(\alpha) = 1.$$ 

As corollaries of the Theorem, both the Hausdorff dimensions in (i) and (ii) are 1.
We use $| \cdot |$ to denote the diameter of a subset of $(0, 1]$, $\dim_H$ to denote the Hausdorff dimension, $[ \ ]$ the integer part of a real number and $\text{cl}$ the closure of a subset of $(0, 1]$ respectively.

2. Proof of the Theorem. The aim of this section is to prove the main result of this paper.

In what follows we often make use of the code space. Let $\{M_n, n \geq 1\}$ be a sequence of positive numbers such that $M_1 > 1$, $M_k < M_{k+1}$ for any $k \geq 1$. For any $n \geq 1$, let

$$D_n = \{(\sigma_1, \ldots, \sigma_n) \in \mathbb{N}^n : kM_k < \sigma_k \leq (k+1)M_k \text{ for all } 1 \leq k \leq n\}.$$ 

Define

$$D = \bigcup_{n=0}^{\infty} D_n \quad (D_0 = \emptyset).$$

For any $\sigma = (\sigma_1, \ldots, \sigma_n) \in D_n$, we use $J_{\sigma}$ to denote the following closed subinterval of $(0, 1]$:

$$J_{\sigma} = \bigcup_{k = [(n+1)M_{n+1}] + 1}^{[(n+2)M_{n+1}]} \text{cl}\{x \in (0, 1] : d_1(x) = \sigma_1, \ldots, d_n(x) = \sigma_n, d_{n+1}(x) = k\},$$

and call it an $n$-order interval.

Define

$$E = \bigcap_{n=0}^{\infty} \bigcup_{\sigma \in D_n} J_{\sigma}.$$ 

It is obvious that

$$E = \{x \in (0, 1] : nM_n < d_n(x) \leq (n+1)M_n \text{ for all } n \geq 1\}. $$

Proof of the Theorem. We divide the proof into two parts:

PART I: $\alpha > 1$. For any $n \geq 1$, let $M_n = \alpha^n$. Now we estimate the length of $J_{\sigma}$ for any $\sigma \in D_n$. Since for any $(n+1)\alpha^{n+1} < k \leq (n+2)\alpha^{n+1}$,

$$|\{x \in (0, 1] : d_1(x) = \sigma_1, \ldots, d_n(x) = \sigma_n, d_{n+1}(x) = k\}| = \frac{1}{\sigma_1 \cdots \sigma_n} \left( \frac{1}{k-1} - \frac{1}{k} \right),$$

we have

$$|J_{\sigma}| = \sum_{k = [(n+1)M_{n+1}] + 1}^{[(n+2)M_{n+1}]} \frac{1}{\sigma_1 \cdots \sigma_n} \left( \frac{1}{k-1} - \frac{1}{k} \right).$$

Therefore

$$(n+2)^{-n-2} \alpha^{-(n+1)(n+2)/2} \alpha^{-\alpha^{n+1}} \leq |J_{\sigma}| \leq \alpha^{-(n+1)(n+2)/2}. $$
Let $\mu$ be a mass distribution supported on $E$ such that for any $n \geq 0$ and $\sigma \in D_n$,

$$\mu(J_\sigma) = \frac{1}{\#D_n} \quad (\#D_0 = 1).$$

By the definition of $D_n$, it is easy to check that

$$c^{-n}\alpha^{n(n+1)/2} \leq \#D_n \leq c^n\alpha^{n(n+1)/2},$$

where $c$ is a positive constant which does not depend on $n$.

For any $x \in E$, we prove that

$$\liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \geq 1,$$

where $B(x, r)$ denotes the open ball with center at $x$ and radius $r$.

For $r < \alpha^{-3}$, choose $n \geq 3$ such that

$$\alpha^{-n(n+1)/2} < r \leq \alpha^{-(n-1)n/2}.$$

By (5), $B(x, r)$ can intersect at most $4n^n\alpha^{n-1} (n-2)$-order intervals, thus by (6) and (7),

$$\liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \geq \liminf_{n \to \infty} \frac{\log (c^{n-2}\alpha^{-(n-2)(n-1)/2}4n^n\alpha^{n-1})}{\log \alpha^{-n(n+1)/2}} = 1.$$

By [2], Proposition 2.3, (see also [1], Proposition 4.9) we have $\dim_H E = 1$. Since $E \subset A(\alpha)$, we have $\dim_H A(\alpha) = 1$.

**PART II: $\alpha = 1$.** The proof of this part is very similar to Part I; we just give an outline.

For any $n \geq 1$, let

$$M_n = \left(1 + \frac{1}{\sqrt{n}}\right)^n.$$ 

Then as in Part I, we have

$$(n + 2)^{-(n+2)} \left(\prod_{k=1}^{n+1} \left(1 + \frac{1}{\sqrt{k}}\right)^k\right)^{-1} \left(1 + \frac{1}{\sqrt{n+1}}\right)^{-(n+1)}$$

$\leq |J_\sigma| \leq \left(\prod_{k=1}^{n+1} \left(1 + \frac{1}{\sqrt{k}}\right)^k\right)^{-1},$

$$c^{-n} \prod_{k=1}^{n} \left(1 + \frac{1}{\sqrt{k}}\right)^k \leq \#D_n \leq c^n \prod_{k=1}^{n} \left(1 + \frac{1}{\sqrt{k}}\right)^k.$$

For any $x \in E$, $r < (\prod_{k=1}^{3} (1+1/\sqrt{k})^{-1}$, choose $n \geq 3$ such that

$$\left(\prod_{k=1}^{n} \left(1 + \frac{1}{\sqrt{k}}\right)^k\right)^{-1} < r \leq \left(\prod_{k=1}^{n-1} \left(1 + \frac{1}{\sqrt{k}}\right)^k\right)^{-1}.$$
By (10), \( B(x, r) \) can intersect at most \( 4n^n(1+1/\sqrt{n-1})^{n-1}(n-2) \)-order intervals, thus by (6) and (11), we have

\[
\liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \geq \liminf_{n \to \infty} \frac{\log \left( \prod_{k=1}^{n-2} \left( 1 + \frac{1}{\sqrt{k}} \right)^{-1} \right) 4n^n \left( 1 + \frac{1}{\sqrt{n-1}} \right)^{n-1}}{\log \left( \prod_{k=1}^{n} \left( 1 + \frac{1}{\sqrt{k}} \right)^{-1} \right)}.\
\]

Since \( \{(1+1/\sqrt{n})^\sqrt{n}, \ n \geq 1\} \) is an increasing sequence such that for any \( n \geq 1 \),

\[
2 \leq \left( 1 + \frac{1}{\sqrt{n}} \right)^{\sqrt{n}} \leq e,
\]
and

\[
1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} \geq \int_1^n x^{-1/2} \, dx = 2n^{1/2} - 2,
\]
we have

\[
\liminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} \geq 1,
\]
completing the proof of the Theorem.

**Corollary 1.** For any \( k \geq 1 \), \( \dim_H A_k = 1 \).

**Proof.** For any \( k \geq 1 \), choose \( M > e^k \). Let \( M_n = M^n \) for any \( n \geq 1 \). Then \( E \subset A_k \). By the proof of the Theorem, we have \( \dim_H E = 1 \), thus \( \dim_H A_k = 1 \).

From the proof of the Theorem, we can also get the following corollaries immediately.

**Corollary 2.** For any \( n \geq 2 \) and \( \alpha \geq 1 \), let

\[
B(\alpha) = \left\{ x \in (0, 1] : \lim_{n \to \infty} \frac{d_{n+1}(x)}{d_n(x) - 1} = \alpha \right\}.
\]
Then

\[
\dim_H B(\alpha) = 1.
\]

**Corollary 3.** The Hausdorff dimension of the set where (2) fails is 1.

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References


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