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Reciprocity formulae for general multiple Dedekind–Rademacher sums and enumeration of lattice points

by

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1. Introduction and statement of main results. The multiple Dedekind–Rademacher sums that we introduce in this work are a generalization of the Dedekind–Rademacher sums studied by Hall, Wilson and Zagier in [4] (from dimension $n = 3$ to arbitrary dimension n).

It should be noted that these sums which are expressed in terms of values of Bernoulli functions are different from those introduced by Zagier [10] and from Dedekind cotangent sums studied by Beck [1], which are expressed in terms of the cotangent function.

We recall that the Bernoulli polynomials are defined by the generating function

$$\frac{ze^{zx}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!}, \qquad |z| < 2\pi, \tag{1.1}$$

and that the Bernoulli functions are defined by

$$\overline{B}_n(x) := \begin{cases} 0 & \text{if } x \in \mathbb{Z},\ n = 1, \\ B_n(\{x\}) & \text{otherwise.} \end{cases} \tag{1.2}$$

For each pair (a, b) of coprime integers, the classical Dedekind sum $S(a, b)$ is defined by

$$S(a,b) = \sum_{t=0}^{a-1} \overline{B}_1\left(\frac{bt}{a}\right)\overline{B}_1\left(\frac{t}{a}\right). \tag{1.3}$$

Dedekind [2] introduced this sum in connection with the transformation formula for the Dedekind η-function. But the story of the Dedekind sum

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started with a paper of L. Kronecker [5] written in 1885, seven years before the widely cited paper of R. Dedekind [2] on Riemann's work. In that paper, Kronecker obtains a Dedekind sum by calculating the logarithm of the quadratic residue symbol using the Gauss lemma. Rademacher introduced the homogeneous sum $S(a,b,c)$ given by

$$S(a,b,c) := \sum_{t=0}^{a-1} \overline{B}_1\left(\frac{bt}{a}\right)\overline{B}_1\left(\frac{ct}{a}\right). \tag{1.4}$$

That is the Rademacher generalization of the classical Dedekind sum.

The most fundamental and important theorems for any generalized Dedekind sums are the reciprocity laws. The famous reciprocity laws for the classical Dedekind and Dedekind–Rademacher sums are as follows (we write $\mathbb{N}^*$ for the set of all integers $n > 0$):

THEOREM 1.1 (Dedekind–Rademacher).

(i) (Dedekind) *If $a, b \in \mathbb{N}^*$ are relatively prime then*

$$S(a,b) + S(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right). \tag{1.5}$$

(ii) (Rademacher) *If $a, b, c \in \mathbb{N}^*$ are pairwise coprime then*

$$S(a,b,c) + S(b,c,a) + S(c,a,b) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{c}{ab} + \frac{b}{ac}\right). \tag{1.6}$$

The standard reference for ordinary Dedekind sums is Grosswald and Rademacher [3].

Let us now introduce our main object of study, the multiple Dedekind–Rademacher sums.

For a fixed integer $n \geq 2$, we let $r_1, \ldots, r_n$ be positive integers and $a_1, \ldots, a_n$ be pairwise coprime positive integers, and we set, for any $1 \leq k \leq n$,

$$\vec{A}_k = (a_1, \ldots, \check{a}_k, \ldots, a_n), \qquad \vec{R}_k = (r_1, \ldots, \check{r}_k, \ldots, r_n) \in \mathbb{N}^{n-1}.$$

We define *multiple Dedekind–Rademacher sums* by the formula

$$S_n(\vec{A}_k, \vec{R}_k) := \sum_{t=0}^{a_k-1} \prod_{1\leq j\neq k\leq n} \overline{B}_{r_j}\left(\frac{a_j t}{a_k}\right). \tag{1.7}$$

It is easy to see for $n = 3$, $k = 3$, $\vec{A}_3 = (\check{a}, b, c)$ and $\vec{R}_3 = (1,1)$ that

$$S(a,b,c) = S_3(\vec{A}_3, \vec{R}_3).$$

Our aim here is to prove that these sums satisfy certain reciprocity relations under cyclic permutations of $(a_1, \ldots, \check{a}_k, \ldots, a_n)$. We state the reciprocity law in terms of the generating function defined by

$$(1.8)\qquad \mathfrak{S}_n(\vec{A}_k,\vec{\Phi}_k) := \sum_{\vec{R}_k\in\mathbb{N}^{n-1}} \frac{S_n(\vec{A}_k,\vec{R}_k)}{r_1!\ldots\check{r}_k!\ldots r_n!} \prod_{1\le j\ne k\le n} \left(\frac{2\pi i\varphi_j}{a_j}\right)^{r_j-1}$$

where $\vec{\Phi}_k = (\phi_1,\ldots,\check{\phi}_k,\ldots,\phi_n)$ is an $n-1$-uple of reals. We prove the following generalized result:

THEOREM 1.2. *Let n be an integer ≥ 2, let $a_1,\ldots,a_n$ be pairwise coprime positive integers and let $\varphi_1,\ldots,\varphi_n$ be real variables in the interval $]-1,1[$ with sum zero. Then*

$$(1.9)\qquad \sum_{k=1}^{n} \mathfrak{S}_n(\vec{A}_k,\vec{\Phi}_k) = -\frac{1}{(2i)^{n-1}} \frac{\sin\big(\pi\sum_{j=1}^{n}\frac{\varphi_j}{a_j}\big)}{\prod_{j=1}^{n}\sin\big(\pi\frac{\varphi_j}{a_j}\big)} + \frac{1}{(2i)^{n-1}} \sum_{k=1}^{n}\prod_{1\le j\ne k\le n}\cot\left(\pi\frac{\varphi_j}{a_j}\right).$$

By our result we recover the well known theorem of Hall, Wilson and Zagier in [4], which is stated for $n=3$. Precisely, we have the following interesting relations deduced directly from the last theorem using the expansion of $\sin(\sum_{j=1}^{n} U_j)$ for $n=2,3,4,5$.

COROLLARY 1.3. *Under the hypotheses of Theorem 1.2, we have the simple identities*

n	2	3	4	5
$\sum_{k=1}^{n}\mathfrak{S}_n(\vec{A}_k,\vec{\Phi}_k)$	0	$-\frac{1}{4}$	$\frac{i}{8}\sum_{j=1}^{4}\cot(\pi\frac{\varphi_j}{a_j})$	$-\frac{1}{2^4}\sum_{1\le i<j\le 5}\cot(\pi\frac{\varphi_i}{a_i})\cot(\pi\frac{\varphi_j}{a_j})$

Proof of Theorem 1.1. We show how to obtain Theorem 1.1 from our main result, Theorem 1.2. We put $a_1=a$, $a_2=b$, $a_3=c$ and we define

$$S_{r_1,r_2}(a,b,c) := S_3(\vec{A}_3,\vec{R}_3),\qquad S_{r_2,r_3}(b,c,a) := S_3(\vec{A}_1,\vec{R}_1),$$
$$S_{r_3,r_1}(c,a,b) := S_3(\vec{A}_2,\vec{R}_2).$$

Then we can write

$$\mathfrak{S}_3(\vec{A}_3,\vec{\Phi}_3) = \sum_{r_1,r_2\ge 0} \frac{S_{r_1,r_2}(a,b,c)}{r_1!r_2!}\left(\frac{2\pi i\varphi_1}{a}\right)^{r_1-1}\left(\frac{2\pi i\varphi_2}{b}\right)^{r_2-1},$$

$$\mathfrak{S}_3(\vec{A}_1,\vec{\Phi}_1) = \sum_{r_2,r_3\ge 0} \frac{S_{r_2,r_3}(b,c,a)}{r_2!r_3!}\left(\frac{2\pi i\varphi_2}{b}\right)^{r_2-1}\left(\frac{2\pi i\varphi_3}{c}\right)^{r_3-1},$$

$$\mathfrak{S}_3(\vec{A}_2,\vec{\Phi}_2) = \sum_{r_3,r_1\ge 0} \frac{S_{r_3,r_1}(c,a,b)}{r_3!r_1!}\left(\frac{2\pi i\varphi_3}{c}\right)^{r_3-1}\left(\frac{2\pi i\varphi_1}{a}\right)^{r_1-1}.$$

Theorem 1.2 shows that

$$\mathfrak{S}_3(\vec{A}_1,\vec{\Phi}_1)+\mathfrak{S}_3(\vec{A}_2,\vec{\Phi}_2)+\mathfrak{S}_3(\vec{A}_3,\vec{\Phi}_3)=-\frac{1}{4}. \tag{1.10}$$

Write

$$\begin{aligned}\mathfrak{S}_3(\vec{A}_k,\vec{\Phi}_k) &= (\text{part of total degree } -2 \text{ in terms of } \varphi_1,\varphi_2,\varphi_3)\\ &\quad+(\text{part of total degree } -1 \text{ in terms of } \varphi_1,\varphi_2,\varphi_3)\\ &\quad+(\text{part of total degree } 0 \text{ in terms of } \varphi_1,\varphi_2,\varphi_3)\\ &\quad+(\text{analytic part of total degree } \geq 1 \text{ in terms of } \varphi_1,\varphi_2,\varphi_3).\end{aligned}$$

For each $k=1,2,3$ we compute the homogeneous part of degree zero in φ_1,φ_2 and φ_3 on the left-hand side of (1.10), and we find that the *part of total degree* $l=0$ *in terms of* $\varphi_1,\varphi_2,\varphi_3$ of

$$\mathfrak{S}_3(\vec{A}_1,\vec{\Phi}_1)+\mathfrak{S}_3(\vec{A}_2,\vec{\Phi}_2)+\mathfrak{S}_3(\vec{A}_3,\vec{\Phi}_3)$$

is equal to $-\frac{1}{4}$. Precisely,

$$\begin{aligned}\mathfrak{S}_3(\vec{A}_3,\vec{\Phi}_3) &= S_{1,1}(a,b,c)+\frac{S_{0,2}(a,b,c)}{0!2!}\,\frac{a}{b}\cdot\frac{\varphi_2}{\varphi_1}+\frac{S_{2,0}(a,b,c)}{2!0!}\,\frac{b}{a}\cdot\frac{\varphi_1}{\varphi_2},\\ \mathfrak{S}_3(\vec{A}_2,\vec{\Phi}_2) &= S_{1,1}(c,a,b)+\frac{S_{0,2}(c,a,b)}{0!2!}\,\frac{c}{a}\cdot\frac{\varphi_1}{\varphi_3}+\frac{S_{2,0}(c,a,b)}{2!0!}\,\frac{a}{c}\cdot\frac{\varphi_3}{\varphi_1},\\ \mathfrak{S}_3(\vec{A}_1,\vec{\Phi}_1) &= S_{1,1}(b,c,a)+\frac{S_{0,2}(b,c,a)}{0!2!}\,\frac{b}{c}\cdot\frac{\varphi_3}{\varphi_2}+\frac{S_{2,0}(b,c,a)}{2!0!}\,\frac{c}{b}\cdot\frac{\varphi_2}{\varphi_3}.\end{aligned}$$

Now by using the additive distribution formula (2.1) and the definition (1.4), we obtain

$$\begin{aligned}&S_{1,1}(a,b,c)+\frac{S_{0,2}(a,b,c)}{0!2!}\,\frac{a}{b}\cdot\frac{\varphi_2}{\varphi_1}+\frac{S_{2,0}(a,b,c)}{2!0!}\,\frac{b}{a}\cdot\frac{\varphi_1}{\varphi_2}\\ &\qquad= S(a,b,c)+\frac{1}{2}\left(\frac{a}{bc}\cdot\frac{\varphi_2}{\varphi_1}+\frac{b}{ac}\cdot\frac{\varphi_1}{\varphi_2}\right)\overline{B}_2(0),\\ &S_{1,1}(c,a,b)+\frac{S_{0,2}(c,a,b)}{0!2!}\,\frac{c}{a}\cdot\frac{\varphi_1}{\varphi_3}+\frac{S_{2,0}(c,a,b)}{2!0!}\,\frac{a}{c}\cdot\frac{\varphi_3}{\varphi_1}\\ &\qquad= S(c,a,b)+\frac{1}{2}\left(\frac{c}{ab}\cdot\frac{\varphi_1}{\varphi_3}+\frac{a}{bc}\cdot\frac{\varphi_3}{\varphi_1}\right)\overline{B}_2(0),\\ &S_{1,1}(b,c,a)+\frac{S_{0,2}(b,c,a)}{0!2!}\,\frac{b}{c}\cdot\frac{\varphi_3}{\varphi_2}+\frac{S_{2,0}(b,c,a)}{2!0!}\,\frac{c}{b}\cdot\frac{\varphi_2}{\varphi_3}\\ &\qquad= S(b,c,a)+\frac{1}{2}\left(\frac{b}{ac}\cdot\frac{\varphi_3}{\varphi_2}+\frac{c}{ab}\cdot\frac{\varphi_2}{\varphi_3}\right)\overline{B}_2(0).\end{aligned}$$

Finally, the homogeneous part of degree zero of

$$\mathfrak{S}_3(\vec{A}_1,\vec{\Phi}_1)+\mathfrak{S}_3(\vec{A}_2,\vec{\Phi}_2)+\mathfrak{S}_3(\vec{A}_3,\vec{\Phi}_3)$$

is

$$S(a,b,c)+S(b,c,a)+S(c,a,b)-\frac{1}{12}\left(\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right)$$

since $\varphi_1+\varphi_2+\varphi_3=0$ and $\overline{B}_2(0)=\frac{1}{6}$. This leads to

$$S(a,b,c)+S(b,c,a)+S(c,a,b)-\frac{1}{12}\left(\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right)=-\frac{1}{4}$$

by comparing both sides of (1.10). ∎

Remark 1.4. In general, for any integer $N\geq 0$, by comparing the coefficients of degree $N-(n-1)$ of both sides of (1.9) we get relations for non-trivial Dedekind–Rademacher sums $S_n(\vec{A}_k,\vec{R}_k)$, $1\leq k\leq n$, with $\sum_{1\leq j\neq k\leq n} r_j=N$, and $r_j\geq 1$. This lets us express all the sums $S_n(\vec{A}_k,\vec{R}_k)$ as linear combinations of those.

2. Bernoulli functions and proofs of the main results.

We use throughout the notation $e(z)=e^{2\pi iz}$ $(z\in\mathbb{C})$.

2.1. Bernoulli functions: An overview.

We will need the additive distribution formula satisfied by Bernoulli functions, which was proved by Raabe in [6]:

$$\forall a\in\mathbb{N}^*,\ \forall x\in\mathbb{R},\quad \sum_{t=0}^{a-1}\overline{B}_m\left(x+\frac{t}{a}\right)=a^{1-m}\overline{B}_m(ax). \tag{2.1}$$

We recall the difference and addition formulas satisfied by Bernoulli polynomials:

$$B_m(X+1)-B_m(X)=mX^{m-1}=\sum_{k=0}^{m-1}\binom{m}{k}B_k(X)\quad (m\in\mathbb{N}), \tag{2.2}$$

$$B_m(X+Y)=\sum_{k=0}^{m}\binom{m}{k}X^{m-k}B_k(Y)\quad (m\in\mathbb{N}). \tag{2.3}$$

Also we need the Fourier expansion formula for $\overline{B}_m(x)$ [4]:

$$\overline{B}_m(x)=-\frac{m!}{(2\pi i)^m}\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{e^{2\pi ikx}}{k^m}\quad (m\in\mathbb{N}^*,\ x\in\mathbb{R}). \tag{2.4}$$

Now we consider for all $\phi\in\mathbb{R}\setminus\mathbb{Z}$ the function

$$F(z,\phi)=\begin{cases}\dfrac{1}{2i}\cot(\pi\phi) & \text{if } z\in\mathbb{Z},\\[2mm] \dfrac{e(\{z\}\phi)}{e(\phi)-1} & \text{if } z\notin\mathbb{Z}.\end{cases} \tag{2.5}$$

By the definition (1.1) of $B_m(x)$, for all $\phi \in \mathbb{R} \setminus \mathbb{Z}$ we have

$$F(z,\phi) = \sum_{m=0}^{+\infty} \frac{\overline{B}_m(z)}{m!}(2\pi i\phi)^{m-1} \qquad (z \in \mathbb{R}). \tag{2.6}$$

2.2. Proof of Theorem 1.2. We begin by proving the following lemma:

LEMMA 2.1. *Under the hypothesis of Theorem* 1.2, *we have*

$$\sum_{k=1}^{n} \mathfrak{S}_n(\vec{A}_k, \vec{\Phi}_k) = \sum_{\substack{(t_1,\dots,t_n)\in\mathbb{N}^n \\ 0\le t_i\le a_i-1}} \sum_{k=1}^{n} \prod_{\substack{j=1 \\ j\neq k}}^{n} F\left(\frac{t_k}{a_k} - \frac{t_j}{a_j}, \phi_j\right).$$

Proof. Using the distribution property (2.1) of $\overline{B}_m$, we can rewrite $\mathfrak{S}_n(\vec{A}_k, \vec{\Phi}_k)$ as

$$\begin{aligned} \mathfrak{S}_n(\vec{A}_k, \vec{\Phi}_k) &= \sum_{\vec{R}_k\in\mathbb{N}^{n-1}} S_n(\vec{A}_k, \vec{R}_k) \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{r_j!}\left(\frac{2\pi i\phi_j}{a_j}\right)^{r_j-1} \\ &= \sum_{\substack{0\le t_i\le a_i-1 \\ 1\le i\le n}} \sum_{\vec{R}_k\in\mathbb{N}^{n-1}} \prod_{\substack{j=1 \\ j\neq k}}^{n} \frac{1}{r_j!}\overline{B}_{r_j}\left(\frac{t_k}{a_k} - \frac{t_j}{a_j}\right)(2\pi i\phi_j)^{r_j-1} \\ &= \sum_{\substack{0\le t_i\le a_i-1 \\ 1\le i\le n}} \prod_{\substack{j=1 \\ j\neq k}}^{n} \sum_{r_j=0}^{\infty} \frac{1}{r_j!}\overline{B}_{r_j}\left(\frac{t_k}{a_k} - \frac{t_j}{a_j}\right)(2\pi i\phi_j)^{r_j-1}. \end{aligned}$$

Using the Laurent expansion (2.6) of F, we thus obtain

$$\mathfrak{S}_n(\vec{A}_k, \vec{\Phi}_k) = \sum_{\substack{0\le t_i\le a_i-1 \\ 1\le i\le n}} \prod_{\substack{j=1 \\ j\neq k}}^{n} F\left(\frac{t_k}{a_k} - \frac{t_j}{a_j}, \phi_j\right),$$

from which the lemma follows. ∎

LEMMA 2.2. *Let* $(u_1,\dots,u_n) \in \mathbb{R}^n$ *be such that* $u_k - u_j \notin \mathbb{Z}$ *for all distinct* $k,j \in \{1,\dots,n\}$. *Then*

$$\sum_{k=1}^{n} \prod_{\substack{j=1 \\ j\neq k}}^{n} F(u_k - u_j, \phi_j) = 0.$$

Proof. The Fourier expansions (2.4) and (2.6) give the identity

$$F(z,\phi) = \frac{1}{2\pi i}\sum_{k\in\mathbb{Z}} \frac{e(kz)}{\phi - k} = \frac{1}{2\pi i}\sum_{\lambda=\phi \bmod \mathbb{Z}} \frac{e((\phi-\lambda)z)}{\lambda},$$

where the last sum, obtained by writing $\lambda = \phi - k$, is to be interpreted as a Cauchy principal value (sum over $|\lambda| < L$ and let $L \to \infty$). Then

$$\prod_{\substack{j=1\\ j\neq k}}^{n} F(u_k-u_j,\phi_j)=\frac{1}{(2\pi i)^{n-1}}\sum_{(\lambda_1,\ldots,\check{\lambda}_k,\ldots,\lambda_n)}^{*}\frac{e((u_k-u_j)(\phi_j-\lambda_j))}{\lambda_j},$$

where the sum $\sum^*$ is over all $(\lambda_1,\ldots,\check{\lambda}_k,\ldots,\lambda_n)\equiv(\phi_1,\ldots,\check{\phi}_k,\ldots,\phi_n)$ (mod $\mathbb{Z}^{n-1}$). Now since $\sum_{j=1}^n \phi_j=0$, by setting $\lambda_n=-\sum_{j=1}^{n-1}\lambda_j$ one has

$$\sum_{\substack{j=1\\ j\neq k}}^{n}(u_k-u_j)(\phi_j-\lambda_j)=\sum_{j=1}^{n}(u_k-u_j)(\phi_j-\lambda_j)=-\sum_{j=1}^{n}u_j(\phi_j-\lambda_j),$$

which is independent of k.

We also observe that

$$\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{\lambda_j}=\sum_{k=1}^{n}\lambda_k\prod_{j=1}^{n}\frac{1}{\lambda_j}=0.$$

We conclude that

$$\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}F(u_k-u_j,\phi_j)$$
$$=\frac{1}{(2\pi i)^{n-1}}\sum_{k=1}^{n}\sum_{(\lambda_1,\ldots,\check{\lambda}_k,\ldots,\lambda_n)}^{*}\frac{e(-\sum_{j=1}^{n}u_j(\phi_j-\lambda_j))}{\prod_{\substack{j=1\\ j\neq k}}^{n}\lambda_j}=0. \qquad \blacksquare$$

In order to prove the formula (1.9), we compute the sum

$$\sum_{\substack{(t_1,\ldots,t_n)\in\mathbb{N}^n\\ 0\le t_i\le a_i-1}}\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}F\left(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\right). \tag{2.7}$$

Thanks to Lemma 2.2, we know that the contribution in the sum (2.7) coming from $t_k/a_k-t_j/a_j\notin\mathbb{Z}$ for all distinct $k,j\in\{1,\ldots,n\}$ and all $t_k,t_j\in\mathbb{Z}$, is zero. Let $(a_1,\ldots,a_n)$ and $(t_1,\ldots,t_n)$ be fixed. We set

$$\sum=\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\neq k}}^{n}F\left(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\right).$$

Our aim here is to calculate the quantity $\sum$.

We assume first that $(t_1,\ldots,t_n)$ is of the form

$$(t_1,\ldots,t_n)=(0,\ldots,0,t_{j_0+1},\ldots,t_n)\in\prod_{j=1}^{j_0}\{0\}\times\prod_{j=j_0+1}^{n}\,]0,a_{j-1}]\cap\mathbb{N}.$$

Throughout, we assume that every sum (resp. product) over the empty set is zero (resp. 1).

We will observe easily that the general case follows without difficulty. For simplicity, in our compution we can assume that

$$\frac{t_j}{a_j} \leq \frac{t_{j+1}}{a_{j+1}} \quad (j_0 < j < n).$$

We decompose the sum $\sum$ into two sums $\sum_1 + \sum_2$ over $k \leq j_0$ and $k > j_0$, respectively. Let us put $h_j = t_j/a_j$ $(j = 1, \ldots, n)$. On the one hand, we have

$$\begin{aligned}
\sum\nolimits_1 &:= \sum_{k=1}^{j_0} \prod_{\substack{j=1 \\ j\neq k}}^{n} F(h_k - h_j, \phi_j) = \sum_{k=1}^{j_0} \prod_{\substack{j=1 \\ j\neq k}}^{n} F(-h_j, \phi_j) \\
&= \sum_{k=1}^{j_0} \prod_{\substack{j=1 \\ j\neq k}}^{j_0} F(0, \phi_j) \prod_{j>j_0} F(-h_j, \phi_j) \\
&= \sum_{k=1}^{j_0} \frac{1}{F(0,\phi_k)} \prod_{j\leq j_0} F(0, \phi_j) \prod_{j>j_0} F(-h_j, \phi_j) \\
&= \prod_{j\leq j_0} F(0, \phi_j) \sum_{j\leq j_0} \frac{1}{F(0,\phi_j)} \prod_{j>j_0} \frac{e(\{-h_j\}\phi_j)}{e(\phi_j)-1} \\
&= \prod_{j\leq j_0} F(0, \phi_j) \sum_{j\leq j_0} \frac{1}{F(0,\phi_j)} \prod_{j>j_0} \frac{e((1-h_j)\phi_j)}{e(\phi_j)-1} \\
&= \prod_{j\leq j_0} \frac{1}{2} \frac{e(\phi_j)+1}{e(\phi_j)-1} \sum_{j\leq j_0} \frac{1}{F(0,\phi_j)} \frac{e(\sum_{j>j_0} \phi_j) \cdot e(-\sum_{j>j_0} h_j\phi_j)}{\prod_{j>j_0}(e(\phi_j)-1)}.
\end{aligned}$$

So after rearrangement

$$\begin{aligned}
\sum\nolimits_1 &= \frac{e(-\sum_{j=1}^{n} h_j\phi_j)}{\prod_{j=1}^{n}(e(\phi_j)-1)} e\Big(\sum_{j>j_0} \phi_j\Big) \prod_{j\leq j_0} \frac{1}{2}(e(\phi_j)+1) \sum_{j\leq j_0} \frac{1}{F(0,\phi_j)} \\
&= \frac{e(-\sum_{j=1}^{n} h_j\phi_j)}{\prod_{j=1}^{n}(e(\phi_j)-1)} \prod_{j\leq j_0} \frac{1}{2}(1+e(-\phi_j)) \sum_{j\leq j_0} \frac{1}{F(0,\phi_j)}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sum\nolimits_2 &= \sum_{k>j_0} \prod_{\substack{j=1 \\ j\neq k}}^{n} F(h_k - h_j, \phi_j) = \sum_{k>j_0} \frac{e(\sum_{j\neq k,\, 1\leq j\leq n}\{h_k - h_j\}\phi_j)}{\prod_{j\neq k}(e(\phi_j)-1)} \\
&= \frac{1}{\prod_{j=1}^{n}(e(\phi_j)-1)} \sum_{k>j_0} (e(\phi_k)-1) \\
&\quad \times e\Big(\sum_{j\leq k}(h_k - h_j)\phi_j\Big) e\Big(\sum_{j>k}(1 - h_j + h_k)\phi_j\Big)
\end{aligned}$$

$$= \frac{1}{\prod_{j=1}^{n}(e(\phi_j)-1)} \sum_{k>j_0} (e(\phi_k)-1) e\Big(-\sum_{j\le n} h_j\phi_j\Big) e\Big(\sum_{j>k}\phi_j\Big)$$
$$= \frac{e(-\sum_{j=1}^{n} h_j\phi_j)}{\prod_{j=1}^{n}(e(\phi_j)-1)} \sum_{k>j_0} (e(\phi_k)-1) e\Big(\sum_{j>k}\phi_j\Big).$$

Thus

$$\sum\nolimits_2 = \frac{e(-\sum_{j=1}^{n} h_j\phi_j)}{\prod_{j=1}^{n}(e(\phi_j)-1)} \Big(e\Big(\sum_{j>j_0}\phi_j\Big)-1\Big).$$

We have thus shown that

$$\sum = \sum\nolimits_1 + \sum\nolimits_2$$
$$= \frac{e(-\sum_{j=1}^{n} h_j\phi_j)}{\prod_{j=1}^{n}(e(\phi_j)-1)} \Big(\prod_{j\le j_0} \frac{1}{2}(e(-\phi_j)+1) \sum_{j\le j_0} \frac{1}{F(0,\phi_j)} + e\Big(\sum_{j>j_0}\phi_j\Big)-1\Big).$$

For every $\vec{t}=(t_1,\dots,t_n)$, we set

$$J(\vec{t}) = \{j : t_j = 0\}.$$

By permutation, we can consider the previous case and so $\sum$ equals

$$\frac{e\big(-\sum_{j=1}^{n} \frac{t_j}{a_j}\phi_j\big)}{\prod_{j=1}^{n}(e(\phi_j)-1)} \Big(e\Big(-\sum_{j\in J}\phi_j\Big) - 1 + \prod_{j\in J} \frac{1}{2}(e(-\phi_j)+1) \sum_{j\in J} \frac{1}{F(0,\phi_j)}\Big).$$

We now consider the sum of $\sum$ over all $\vec{t}\in\prod_{j=1}^{n}[0,a_j-1]\cap\mathbb{N}$. Let us write

$$\sum_{\vec{t}}\sum = \sum_{J\subset[\![1,n]\!]} \sum_{\substack{\vec{t}\\ J(\vec{t})=J}} \sum.$$

We see first that

$$\sum_{\substack{\vec{t}\\ J(\vec{t})=J}} e\Big(-\sum_{j=1}^{n}\frac{t_j}{a_j}\phi_j\Big) = \prod_{j\notin J}\sum_{t_j=1}^{a_j-1} e\Big(-\frac{\phi_j}{a_j}t_j\Big)$$
$$= \prod_{j\notin J} \frac{e\big(-\frac{\phi_j}{a_j}\big)-e(-\phi_j)}{1-e\big(-\frac{\phi_j}{a_j}\big)}$$
$$= \prod_{j=1}^{n} \frac{e\big(-\frac{\phi_j}{a_j}\big)-e(-\phi_j)}{1-e\big(-\frac{\phi_j}{a_j}\big)} \prod_{j\in J} \frac{1-e\big(-\frac{\phi_j}{a_j}\big)}{e\big(-\frac{\phi_j}{a_j}\big)-e(-\phi_j)}.$$

On the other hand, we have

$$\sum_J \prod_{j\in J} \frac{1-e\big(-\frac{\phi_j}{a_j}\big)}{e\big(-\frac{\phi_j}{a_j}\big)-e(-\phi_j)} e(-\phi_j) = \sum_J \prod_{j\in J} \frac{1-e\big(-\frac{\phi_j}{a_j}\big)}{e\big((1-\frac{1}{a_j})\phi_j\big)-1}$$
$$= \prod_{i=1}^n \left(1+\frac{1-e\big(-\frac{\phi_i}{a_i}\big)}{e\big((1-\frac{1}{a_i})\phi_i\big)-1}\right)$$
$$= \prod_{i=1}^n \frac{e(\phi_i)-1}{e(\phi_i)-e\big(\frac{\phi_i}{a_i}\big)}.$$

It remains to give a formula for

$$S = \sum_J \prod_{j\in J} \frac{1}{2}(e(-\phi_j)+1) \frac{1-e\big(-\frac{\phi_j}{a_j}\big)}{e\big(-\frac{\phi_j}{a_j}\big)-e(-\phi_j)} \sum_{j\in J} \frac{1}{F(0,\phi_j)}.$$

By putting

$$\alpha_j = \frac{1}{2}(1+e(-\phi_j)) \cdot \frac{1-e\big(-\frac{\phi_j}{a_j}\big)}{e\big(-\frac{\phi_j}{a_j}\big)-e(-\phi_j)},$$
$$\beta_j = \frac{1}{F(0,\phi_j)} = 2\frac{e(\phi_j)-1}{e(\phi_j)+1},$$

we may write

$$S = \sum_J \prod_{j\in J} \alpha_j \sum_{j\in J} \beta_j.$$

We set $\beta = \sum_{j\in J} \beta_j$. It is clear that

$$S = H'(1),$$

where

$$H(y) = \sum_J \Big(\prod_{j\in J} \alpha_j\Big) y^\beta = \sum_J \prod_{j\in J} \alpha_j y^{\beta_j} = \prod_{i=1}^n (1+\alpha_i y^{\beta_i}).$$

Hence

$$S = \prod_{i=1}^n (1+\alpha_i) \sum_{i=1}^n \frac{\alpha_i \beta_i}{1+\alpha_i}.$$

Since

$$1+\alpha_j = \frac{1}{2}\,\frac{A_j}{e\big(-\frac{\phi_j}{a_j}\big)-e(-\phi_j)}$$

with

$$A_j = 2e\left(-\frac{\phi_j}{a_j}\right) - 2e(-\phi_j) + (1+e(-\phi_j))\left(1 - e\left(-\frac{\phi_j}{a_j}\right)\right)$$
$$= e\left(-\frac{\phi_j}{a_j}\right) - e(-\phi_j) + 1 + e\left(-\left(1+\frac{1}{a_j}\right)\phi_j\right)$$
$$= \left(1 + e\left(-\frac{\phi_j}{a_j}\right)\right)(1 - e(-\phi_j)),$$

it follows that

$$1 + \alpha_j = \frac{1}{2}\,\frac{\left(1 + e\left(-\frac{\phi_j}{a_j}\right)\right)(1 - e(-\phi_j))}{e\left(-\frac{\phi_j}{a_j}\right) - e(-\phi_j)}.$$

On the other hand, we have

$$\alpha_j\beta_j = \frac{1}{2}(1 + e(-\phi_j))\frac{1 - e\left(-\frac{\phi_j}{a_j}\right)}{e\left(-\frac{\phi_j}{a_j}\right) - e(-\phi_j)}\,\frac{2(e(\phi_j) - 1)}{e(\phi_j) + 1}$$
$$= (1 - e(-\phi_j))\frac{1 - e\left(-\frac{\phi_j}{a_j}\right)}{e\left(-\frac{\phi_j}{a_j}\right) - e(-\phi_j)}.$$

Hence, we obtain the following expression:

$$\frac{\alpha_j\beta_j}{1+\alpha_j} = \frac{1 - e\left(-\frac{\phi_j}{a_j}\right)}{1 + e\left(-\frac{\phi_j}{a_j}\right)},$$

and consequently

$$S = \prod_{j=1}^{n}\frac{1}{2}\frac{\left(1 + e\left(-\frac{\phi_j}{a_j}\right)\right)(1 - e(-\phi_j))}{e\left(-\frac{\phi_j}{a_j}\right) - e(-\phi_j)}\sum_{j=1}^{n} 2\frac{1 - e\left(-\frac{\phi_j}{a_j}\right)}{1 + e\left(-\frac{\phi_j}{a_j}\right)}.$$

Finally, by using Lemma 2.2,

$$\sum_{\substack{0\le t_j\le a_j-1\\ 1\le j\le n}}\sum_{k=1}^{n}\prod_{\substack{j=1\\ j\ne k}}^{n} F\left(\frac{t_k}{a_k} - \frac{t_j}{a_j}, \phi_j\right) = \prod_{j=1}^{n}\frac{e\left(-\frac{\phi_j}{a_j}\right) - e(-\phi_j)}{1 - e\left(-\frac{\phi_j}{a_j}\right)}\prod_{j=1}^{n}\frac{1}{e(\phi_j) - e\left(\frac{\phi_j}{a_j}\right)}$$
$$- \prod_{j=1}^{n}\frac{1}{1 - e\left(-\frac{\phi_j}{a_j}\right)} + \prod_{j=1}^{n}\frac{1}{2}\frac{\left(1 + e\left(-\frac{\phi_j}{a_j}\right)\right)}{1 - e\left(-\frac{\phi_j}{a_j}\right)}\sum_{j=1}^{n} 2\frac{1 - e\left(-\frac{\phi_j}{a_j}\right)}{1 + e\left(-\frac{\phi_j}{a_j}\right)},$$

which can also be rewritten as

$$\sum_{\substack{0\le t_j\le a_j-1\\1\le j\le n}}\sum_{k=1}^{n}\prod_{\substack{j=1\\j\neq k}}^{n}F\Big(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\Big)$$
$$=\frac{1}{2^{n-1}}\sum_{k=1}^{n}\prod_{1\le j\neq k\le n}\frac{e\big(\frac{\phi_j}{a_j}\big)+1}{e\big(\frac{\phi_j}{a_j}\big)-1}-\frac{e\big(\sum_{j=1}^{n}\frac{\phi_j}{a_j}\big)-1}{\prod_{j=1}^{n}e\big(\frac{\phi_j}{a_j}\big)-1}.$$

Then we obtain our formula in terms of sin and cot:

$$\sum_{\substack{0\le t_j\le a_j-1\\1\le j\le n}}\sum_{k=1}^{n}\prod_{\substack{j=1\\j\neq k}}^{n}F\Big(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\Big)$$
$$=-\frac{1}{(2i)^{n-1}}\frac{\sin\big(\pi\sum_{j=1}^{n}\frac{\varphi_j}{a_j}\big)}{\prod_{j=1}^{n}\sin\big(\pi\frac{\varphi_j}{a_j}\big)}+\frac{1}{(2i)^{n-1}}\sum_{k=1}^{n}\prod_{1\le j\neq k\le n}\cot\Big(\pi\frac{\varphi_j}{a_j}\Big).$$

Now, by using Lemma 2.1, we obtain

$$\sum_{k=1}^{n}\mathfrak{S}_n(\vec{A}_k,\vec{\Phi}_k)=\sum_{k=1}^{n}\prod_{\substack{j=1\\j\neq k}}^{n}F\Big(\frac{t_k}{a_k}-\frac{t_j}{a_j},\phi_j\Big).$$

This finishes the proof of Theorem 1.2. ■

3. Connection to multiple Rademacher sums. Let $a_1,\dots,a_n$ be pairwise coprime positive integers. Inspired by Rademacher's work [7] we define *multiple Rademacher sums* by

$$B(a_1,\dots,a_n)=\sum_{i=1}^{n}\sum_{t_i=0}^{a_i-1}\frac{t_i}{a_i}\prod_{\substack{j=1\\j\neq i}}^{n}\Big[\frac{t_ia_j}{a_i}\Big]. \tag{3.1}$$

We have a connection between $B(a_1,\dots,a_n)$ and our multiple Dedekind–Rademacher sums $S_n(\vec{A}_k,\vec{R}_k)$ given by the following result:

THEOREM 3.1. *For any pairwise coprime positive integers $a_1,\dots,a_n$, we have*

$$B(a_1,\dots,a_n)$$
$$=A_n\sum_{i=1}^{n}\frac{1}{a_i}\sum_{J\subset\{1,\dots,n\}\setminus\{i\}}F(J)\sum_{k=0}^{|\overline{J}|}\binom{|\overline{J}|+1}{k}\sum_{L\subset J}2^{|L|}S_k(a_i;(a_l)_{l\in L})$$
$$-\frac{A_n}{2}\sum_{i=1}^{n}\frac{1}{a_i}\sum_{J\subset\{1,\dots,n\}\setminus\{i\}}F(J)(|\overline{J}|+1),$$

where

$$A_n = \prod_{j=1}^{n} a_j, \qquad F(J) = \frac{1}{2^{|J|}} \cdot \frac{1}{|\overline{J}| + 1} \prod_{j \in J} \left(\frac{-1}{a_j} \right)$$

and

$$\overline{J} = \{1, \ldots, n\} \setminus J, \qquad S_k(a_i; (a_l)_{l \in L}) = \sum_{t_i=0}^{a_i-1} \overline{B}_k \left(\frac{t_i}{a_i} \right) \prod_{l \in L} \overline{B}_1 \left(\frac{t_i a_l}{a_i} \right).$$

Proof. Let us put $u_i = t_i/a_i$, $i = 1, \ldots, n$. We have

$$u_i \prod_{\substack{j=1 \\ j \neq i}}^{n} [u_i a_j] = u_i \prod_{\substack{j=1 \\ j \neq i}}^{n} a_j \cdot \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(u_i - \frac{1}{a_j} \{u_i a_j\} \right) = \frac{A_n}{a_i} \cdot \prod_{j=1}^{n} \left(u_i - \frac{1}{a_j} \{u_i a_j\} \right)$$

$$= \frac{A_n}{a_i} \sum_{J \subset \{1, \ldots, n\}} u_i^{|\overline{J}|} \prod_{j \in J} \left(\frac{-1}{a_j} \right) \{u_i a_j\}.$$

Note that if $i \in J$ then

$$\prod_{j \in J} \left(\frac{-1}{a_j} \right) \{u_i a_j\} = 0,$$

so

$$u_i \prod_{\substack{j=1 \\ j \neq i}}^{n} [u_i a_j] = \frac{A_n}{a_i} \sum_{J \subset \{1, \ldots, n\} \setminus \{i\}} u_i^{|\overline{J}|} \prod_{j \in J} \left(\frac{-1}{a_j} \right) \{u_i a_j\}.$$

It follows that

$$B(a_1, \ldots, a_n) = A_n \sum_{i=1}^{n} \frac{1}{a_i} \sum_{J \subset \{1, \ldots, n\} \setminus \{i\}} \prod_{j \in J} \left(\frac{-1}{a_j} \right) \sum_{t_i=0}^{a_i-1} u_i^{|\overline{J}|} \prod_{j \in J} \{u_i a_j\}. \tag{3.2}$$

Now, we set $m = |\overline{J}| + 1$, $u = \{u_i\}$ and $v = \{u_i a_j\}$. Then we use the difference formula (2.2) and the identity

$$\{v\} = \overline{B}_1(v) + \frac{1}{2}, \qquad v \in \mathbb{R} \setminus \mathbb{Z}.$$

We deduce that, for all $t_i = 1, \ldots, a_i - 1$ and all $J \subset \{1, \ldots, n\} \setminus \{i\}$,

$$u_i^{|\overline{J}|} = \frac{1}{|\overline{J}| + 1} \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}| + 1}{k} \overline{B}_k(u_i),$$

$$\prod_{j \in J} \{u_i a_j\} = \prod_{j \in J} \left(\overline{B}_1(u_i a_j) + \frac{1}{2} \right)$$

$$= \sum_{L \subset J} \left(\frac{1}{2} \right)^{|J| - |L|} \prod_{l \in L} \overline{B}_1(u_i a_l).$$

By substituting these equalities in (3.2), we obtain

$$
\begin{aligned}
&B(a_1,\dots,a_n)\\
&\quad = A_n \sum_{i=1}^{n} \frac{1}{a_i} \sum_{J\subset\{1,\dots,n\}\setminus\{i\}} \frac{1}{|\overline{J}|+1} \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} \prod_{j\in J} \frac{-1}{a_j}\\
&\qquad \times \sum_{L\subset J} \left(\frac{1}{2}\right)^{|J|-|L|} \sum_{t_i=1}^{a_i-1} \overline{B}_k(u_i) \prod_{l\in L} \overline{B}_1(u_i a_l)\\
&\quad = A_n \sum_{i=1}^{n} \frac{1}{a_i} \sum_{J\subset\{1,\dots,n\}\setminus\{i\}} \frac{1}{|\overline{J}|+1} \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} \prod_{j\in J} \frac{-1}{a_j}\\
&\qquad \times \sum_{L\subset J} \left(\frac{1}{2}\right)^{|J|-|L|} \Big(\sum_{t_i=0}^{a_i-1} \overline{B}_k(u_i) \prod_{l\in L} \overline{B}_1(u_i a_l) - \overline{B}_k(0) \prod_{l\in L} \overline{B}_1(0)\Big)\\
&\quad = A_n \sum_{i=1}^{n} \frac{1}{a_i} \sum_{J\subset\{1,\dots,n\}\setminus\{i\}} F(J) \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k}\\
&\qquad \times \sum_{L\subset J} 2^{|L|} \Big(\sum_{t_i=0}^{a_i-1} \overline{B}_k(u_i) \prod_{l\in L} \overline{B}_1(u_i a_l) - \overline{B}_k(0) \prod_{l\in L} \overline{B}_1(0)\Big).
\end{aligned}
$$

We can rewrite this equality as follows:

$$
\begin{aligned}
&B(a_1,\dots,a_n)\\
&\quad = A_n \sum_{i=1}^{n} \frac{1}{a_i} \sum_{J\subset\{1,\dots,n\}\setminus\{i\}} F(J) \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} \sum_{L\subset J} 2^{|L|} S_k(a_i;(a_l)_{l\in L})\\
&\qquad - A_n \sum_{i=1}^{n} \frac{1}{a_i} \sum_{J\subset\{1,\dots,n\}\setminus\{i\}} F(J) \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} \overline{B}_k(0).
\end{aligned}
$$

Indeed,

$$
\begin{aligned}
\sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} \overline{B}_k(0) &= \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} B_k(0) - \binom{|\overline{J}|+1}{1} B_1(0)\\
&= (|\overline{J}|+1)0^{|\overline{J}|} - B_1(0)(|\overline{J}|+1)\\
&= \frac{|\overline{J}|+1}{2}, \qquad \text{since } |\overline{J}| \geq 1.
\end{aligned}
$$

Finally, we obtain our desired formula. ∎

4. Connection to the enumeration of lattice points in pyramids in dimension $n+1$. Let $a_1, \dots, a_n$ be pairwise coprime positive integers. Let P denote the integral point $(0, \dots, 0, A_n)$ in the Euclidean space $\mathbb{R}^{n+1}$ where $A_n = a_1 \dots a_n$. For $I \subset [\![1, n]\!]$ let $P_I = (x_1^{(I)}, \dots, x_n^{(I)}, 0)$, where $x_i^{(I)} = a_i$ or 0 according to whether $i \in I$ or not. Denote by $N(a_1, \dots, a_n)$ the number of lattice points $(x_1, \dots, x_n, x_{n+1})$ with $0 < x_{n+1} < A_n$ in the pyramid with vertices P and $P_I, I \subset [\![1, n]\!]$.

Sun [9] proves the following relation:

$$N(a_1, \dots, a_n) = A_n\Big(\prod_{i=1}^{n}(a_i - 1) - B(a_1, \dots, a_n)\Big). \tag{4.1}$$

By combining Theorem 3.1 with (4.1) it is obvious that our multiple Dedekind–Rademacher sums appear in lattice-point enumeration functions. More precisely, we obtain

THEOREM 4.1. *Let $a_1, \dots, a_n$ be pairwise coprime positive integers. Then*

$$\begin{aligned} N(a_1, \dots, a_n) & \\ &= A_n \prod_{i=1}^{n}(a_i - 1) \\ &\quad - A_n \sum_{i=1}^{n} \frac{1}{a_i} \sum_{J \subset \{1,\dots,n\}\setminus\{i\}} F(J) \sum_{k=0}^{|\overline{J}|} \binom{|\overline{J}|+1}{k} \sum_{L \subset J} 2^{|L|} S_k(a_i; (a_l)_{l \in L}) \\ &\quad + \frac{A_n}{2} \sum_{i=1}^{n} \frac{1}{a_i} \sum_{J \subset \{1,\dots,n\}\setminus\{i\}} F(J)(|\overline{J}|+1). \end{aligned}$$

COROLLARY 4.2 (Explicit formulas). *Let a_1, a_2 be coprime positive integers. Then*

$$\begin{aligned} N(a_1) &= \frac{a_1(a_1-1)}{2}, \\ N(a_1, a_2) &= a_1 a_2\left(\frac{1}{3}a_1 a_2 - \frac{a_1+a_2}{4} + \frac{1}{4}\right) - \frac{a_1^2 + a_2^2 - 1}{12}. \end{aligned}$$

Proof. (1) For $n = 1$, thanks to Theorem 3.1 we have

$$\begin{aligned} B(a_1) &= F(\emptyset) \sum_{k=0}^{1} \binom{2}{k} \sum_{t_1=0}^{a_1-1} \overline{B}_k\left(\frac{t_1}{a_1}\right) - \frac{1}{2} \cdot F(\emptyset) \cdot 2 \\ &= \frac{1}{2} \sum_{k=0}^{1} \binom{2}{k} \cdot a_1^{1-k} \overline{B}_k(0) - \frac{1}{2}. \end{aligned}$$

Hence

$$B(a_1) = \frac{a_1 - 1}{2}.$$

Using (4.1) we obtain

$$N(a_1) = \frac{a_1(a_1 - 1)}{2}.$$

(2) For $n = 2$, from our Theorem 3.1 we obtain

$$\begin{aligned}
&B(a_1, a_2)\\
&= a_2\Bigg(F(\emptyset)\sum_{k=0}^{2}\binom{3}{k}\sum_{t_1=0}^{a_1-1}\overline{B}_k\left(\frac{t_1}{a_1}\right)\\
&\quad + F(\{2\})\left(\sum_{k=0}^{1}\binom{2}{k}\sum_{t_1=0}^{a_1-1}\overline{B}_k\left(\frac{t_1}{a_1}\right) + 2\sum_{k=0}^{1}\binom{2}{k}S_k(a_1, a_2)\right)\Bigg)\\
&\quad + a_1\Bigg(F(\emptyset)\sum_{k=0}^{2}\binom{3}{k}\sum_{t_2=0}^{a_2-1}\overline{B}_k\left(\frac{t_2}{a_2}\right)\\
&\quad + F(\{1\})\left(\sum_{k=0}^{1}\binom{2}{k}\sum_{t_2=0}^{a_2-1}\overline{B}_k\left(\frac{t_2}{a_2}\right) + 2\sum_{k=0}^{1}\binom{2}{k}S_k(a_2, a_1)\right)\Bigg)\\
&\quad - \frac{a_1a_2}{2}\left(\frac{1}{a_1}\cdot F(\emptyset)\cdot 3 + \frac{1}{a_1}\cdot F(\{2\})\cdot 2 + \frac{1}{a_2}\cdot F(\emptyset)\cdot 3 + \frac{1}{a_2}\cdot F(\{1\})\cdot 2\right).
\end{aligned}$$

We remark that $F(\emptyset) = \frac{1}{3}$, $F(\{1\}) = -\frac{1}{4a_1}$, and $F(\{2\}) = -\frac{1}{4a_2}$. By using the Raabe formula (2.1), we obtain

$$\begin{aligned}
B(a_1, a_2) = a_1a_2\Bigg(&\frac{1}{3a_1}\sum_{k=0}^{2}\binom{3}{k}a_1^{1-k}\overline{B}_k(0)\\
&- \frac{1}{4a_1a_2}\left(\sum_{k=0}^{1}\binom{2}{k}a_1^{1-k}\overline{B}_k(0) + 2\cdot\overline{B}_1(0) + 4S(a_1, a_2)\right)\Bigg)\\
&+ a_1a_2\Bigg(\frac{1}{3a_2}\sum_{k=0}^{2}\binom{3}{k}a_2^{1-k}\overline{B}_k(0)\\
&- \frac{1}{4a_1a_2}\left(\sum_{k=0}^{1}\binom{2}{k}a_2^{1-k}\overline{B}_k(0) + 2\cdot\overline{B}_1(0) + 4S(a_2, a_1)\right)\Bigg)\\
&- \frac{1}{2}(a_1 + a_2) + \frac{1}{2}.
\end{aligned}$$

We rewrite this formula as follows:

$$\begin{aligned}B(a_1,a_2) = -\frac{1}{2}(a_1+a_2) + \frac{1}{2}a_1a_2\Bigg(&\frac{1}{3a_1}\Bigg(\sum_{k=0}^{3}\binom{3}{k}a_1^{1-k}B_k(0) - 3B_1(0)\Bigg)\\ &- \frac{1}{4a_1a_2}\Bigg(\sum_{k=0}^{2}\binom{2}{k}a_1^{1-k}B_k(0) - 2B_1(0) + 4S(a_1,a_2)\Bigg)\Bigg)\\ &+ a_1a_2\Bigg(\frac{1}{3a_1}\Bigg(\sum_{k=0}^{3}\binom{3}{k}a_1^{1-k}B_k(0) - 3B_1(0)\Bigg)\\ &- \frac{1}{4a_1a_2}\Bigg(\sum_{k=0}^{2}\binom{2}{k}a_1^{1-k}B_k(0) - 2B_1(0) + 4S(a_1,a_2)\Bigg)\Bigg).\end{aligned}$$

We use the addition formula (2.3) to obtain

$$\begin{aligned}&B(a_1,a_2) = -\frac{1}{2}(a_1+a_2)\\ &+ \frac{1}{2}a_1a_2\Bigg(\frac{1}{3a_1}\Bigg(\frac{B_3(a_1)}{a_1^2} + \frac{3}{2}\Bigg) - \frac{1}{4a_1a_2}\Bigg(\frac{B_2(a_1)-B_2(0)}{a_1} + 1 + 4S(a_1,a_2)\Bigg)\Bigg)\\ &+ a_1a_2\Bigg(\frac{1}{3a_2}\Bigg(\frac{B_3(a_2)}{a_1^2} + \frac{3}{2}\Bigg) - \frac{1}{4a_1a_2}\Bigg(\frac{B_2(a_2)-B_2(0)}{a_2} + 1 + 4S(a_2,a_1)\Bigg)\Bigg).\end{aligned}$$

We deduce from $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + \frac{1}{6}$, and $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$ that

$$\begin{aligned}B(a_1,a_2) = -\frac{1}{2}(a_1+a_2) + \frac{1}{2} + \frac{1}{6}\left(\frac{a_2}{a_1} + \frac{a_1}{a_2}\right) - \frac{1}{4}(a_1+a_2)\\ - (S(a_1,a_2) + S(a_2,a_1)).\end{aligned}$$

By using our main Theorem 1.2, we obtain

$$B(a_1,a_2) = \frac{2}{3}a_1a_2 - \frac{3(a_1+a_2)}{4} + \frac{3}{4} + \frac{a_1^2+a_2^2-1}{12a_1a_2}.$$

From (4.1) we obtain the desired result

$$N(a_1,a_2) = a_1a_2\left(\frac{1}{3}a_1a_2 - \frac{a_1+a_2}{4} + \frac{1}{4}\right) - \frac{a_1^2+a_2^2-1}{12}. \ \blacksquare$$

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