ACTA ARITHMETICA
145.2 (2010)

On the infinite product exponents of meromorphic modular forms for certain arithmetic groups

by

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1. Introduction. By using Ramanujan's theta operator, Bruinier, Kohnen and Ono [6] obtained formulae for the infinite product exponents of meromorphic modular forms for $SL_2(\mathbb{Z})$ which are determined by the divisors of the modular forms. Ahlgren [1] generalized this result to Hecke subgroups $\Gamma_0(p)$ of $SL_2(\mathbb{Z})$ with $p \in \{2, 3, 5, 7, 13\}$. Let $\Gamma_0^+(N)$ be the group generated by the Hecke subgroup $\Gamma_0(N)$ and the Fricke involution $W_N := \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. In this paper, we generalize the above-mentioned results to the groups $\Gamma_0^+(N)$ of genus zero, that is, to the groups $\Gamma_0^+(N)$, where N lies in

$$\Phi = \{1, \ldots, 21, 23, \ldots, 27, 29, \ldots, 32, 35, 36, 39, 41, 47, 49, 50, 59, 71\}.$$

As an application we reprove the formula for the number of representations of a given integer as a sum of four squares. The infinite product exponents of modular forms are also related to Borcherds product. Borcherds [2] showed that the infinite product exponents of certain modular functions for $SL_2(\mathbb{Z})$ come from Fourier coefficients of weakly holomorphic modular forms. On the other hand, in [15, Theorem 1] Zagier proved that the generating series for the traces of singular moduli of modular functions on $SL_2(\mathbb{Z})$ is a weakly holomorphic modular form of weight $3/2$ on $\Gamma_0(4)$. Here we give a connection between the traces of singular moduli of modular functions on $\Gamma_0^+(N)$ and the exponents of another modular function on $\Gamma_0^+(N)$. Moreover we express the Hurwitz–Kronecker class number via traces of singular moduli.

To state our results we require some notations. Throughout, we agree that $q = e^{2\pi i z}$ for $z \in \mathfrak{H}$ (= complex upper half plane) and N is a natural integer contained in Φ, and denote the sum of divisors of n by $\sigma(n)$. We

2010 *Mathematics Subject Classification*: Primary 11F03, 11F11; Secondary 11F37.
Key words and phrases: meromorphic modular forms, infinite product exponents.
Kim was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2007-313-C00008).

DOI: 10.4064/aa145-2-5

let $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$. Finally, for all positive integers n let $j^+_{N,n}$ be the modular function for $\Gamma_0^+(N)$ which is holomorphic on $\mathfrak{H}^* - \Gamma_0^+(N)\infty$ and has the Fourier expansion at ∞ of the form

$$j^+_{N,n}(z) = \frac{1}{q^n} + \sum_{m=1}^{\infty} a_{j^+_{N,n}}(m)q^m.$$

Let h_s (resp. $\hbar_s$) denote the width of a cusp s in $\Gamma_0(N)$ (resp. $\Gamma_0^+(N)$). Let γ be an element of $SL_2(\mathbb{Z})$ such that $\gamma\infty = s$. There exists a positive integer w_s such that

$$(1) \qquad \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}\gamma = \gamma' U \quad \text{for some } \gamma' \in SL_2(\mathbb{Z}) \text{ and } U = \begin{pmatrix} x & y \\ 0 & w_s \end{pmatrix}.$$

We observe that the quantities h_s, $\hbar_s$ and w_s are independent of the choice of $\gamma \in SL_2(\mathbb{Z})$ and only depend on the $\Gamma_0(N)$-class of s. More explicitly, if s is a cusp represented by a rational number a/c where c is a divisor of N and a is coprime to c, then it is not difficult to verify that

$$\begin{aligned} &w_s = N/c, \qquad \hbar_s = N/(c^2, N), \\ &h_s = \hbar_s \qquad \text{unless } N = 4 \text{ and } c = 2, \text{ in which case } h_s = 1/2. \end{aligned}$$

Thus the quantity $h_s(1 + N/w_s^2)$ which appears in Theorem 1.1 below depends only on the $\Gamma_0^+(N)$-class of s. Let $1/e_\tau$ be the cardinality of the quotient $\Gamma_0^+(N)_\tau/\{\pm1\}$ for each $\tau \in \mathfrak{H}$, where $\Gamma_0^+(N)_\tau$ is the stabilizer of τ in $\Gamma_0^+(N)$. Let $\pi : \mathfrak{H}^* \to \Gamma_0^+(N)\backslash\mathfrak{H}^*$ be the canonical quotient map. With these notations we can state the first main result.

THEOREM 1.1. *Suppose that $f = \sum_{n=h}^{\infty} a_f(n)q^n$ is a meromorphic modular form of weight k for $\Gamma_0^+(N)$, normalized so that $a_f(h) = 1$. Let*

$$f = q^h \prod_{n=1}^{\infty} (1 - q^n)^{c_f(n)}$$

for some complex numbers $c_f(n)$. Then for each integer $n \geq 1$ we have

$$\sum_{d|n} c_f(d)d = k\sigma(n) + kN\sigma\left(\frac{n}{N}\right) + \sum_{\pi(\tau)\in\Gamma_0^+(N)\backslash\mathfrak{H}} e_\tau j^+_{N,n}(\tau) \operatorname{ord}_\tau f$$
$$+ \sum_{s\in S_{\Gamma_0^+(N)}-\{\infty\}} j^+_{N,n}(s)\left(\operatorname{ord}_s f - kh_s\left(\frac{1}{24} + \frac{N}{24w_s^2}\right)\right),$$

where $\operatorname{ord}_\tau f$ is the standard order of vanishing of f, $\operatorname{ord}_s f$ is the order of vanishing of f at s, and $S_{\Gamma_0^+(N)}$ is the set of $\Gamma_0^+(N)$-inequivalent cusps.

REMARK 1.2. (i) Suppose that the genus of $\Gamma_0(N)$ is zero. The same formula holds when $\Gamma_0^+(N)$ (respectively, $j^+_{N,n}$) is replaced by $\Gamma_0(N)$ (respectively, $j_{N,n}$) in Theorem 1.1, where $j_{N,n}$ is the modular function for $\Gamma_0(N)$ which is holomorphic on $\mathfrak{H}^* - \Gamma_0(N)\infty$ and has the Fourier expansion at ∞ of the form

$$j_{N,n}(z) = \frac{1}{q^n} + \sum_{m=1}^{\infty} a_{j_{N,n}}(m) q^m.$$

(ii) We mention that Choi [7] obtained another formula for the infinite product exponents of modular forms on $\Gamma_0(N)$ when N is square free.

Let Θ be the Jacobi theta function:

$$\Theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2} := \sum_{n=0}^{\infty} r_1(n) q^n.$$

The number $r_k(n)$ of representations of an integer n as a sum of k squares is the coefficient of q^n in the modular form Θ^k. As an application of Theorem 1.1 we derive a formula for the number $r_4(n)$.

Theorem 1.3. *For each integer $n \geq 1$ we have*

$$r_4(n) = 8\sigma(n) - 32\sigma(n/4),$$

where $\sigma(n/4) = 0$ if 4 does not divide n.

We note that there are alternative proofs of this theorem. See [9, Theorem 386], [5, Proposition 11], [10, Theorem 5.33] and [13, Exercise III.5.2].

Let d denote a positive integer congruent to 0 or 3 modulo 4. We denote by $\mathcal{Q}_d$ the set of positive definite binary quadratic forms

$$Q = [a, b, c] = aX^2 + bXY + cY^2 \quad (a, b, c \in \mathbb{Z})$$

of discriminant $-d$, with the usual action of the modular group $\Gamma(1) = SL_2(\mathbb{Z})$. To each $Q \in \mathcal{Q}_d$, we associate its unique root $\alpha_Q \in \mathfrak{H}$. Let $j(z)$ be the elliptic modular invariant defined on $\mathfrak{H}$ with a Fourier expansion

$$j(z) = q^{-1} + 744 + 196884q + \cdots.$$

The Hurwitz–Kronecker class number $H(d)$ and the trace $\mathbf{t}_J(d)$ for $J(z) = j(z) - 744$ are defined as

$$H(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma(1)} e_{\alpha_Q}, \qquad \mathbf{t}_J(d) = \sum_{Q \in \mathcal{Q}_d/\Gamma(1)} e_{\alpha_Q} J(\alpha_Q).$$

Let $M^!_{1/2}$ be the vector space of weakly holomorphic modular forms (that is, meromorphic with poles only at the cusps) of weight $1/2$ for $\Gamma_0(4)$ whose Fourier coefficients satisfy the Kohnen's "plus space" condition (i.e. the nth coefficients vanish unless $n \equiv 0$ or 1 modulo 4). For each nonnegative integer $d \equiv 0, 3 \pmod 4$ there is a unique modular form $f_d \in M^!_{1/2}$ having a Fourier development of the form

$$f_d(z) = q^{-d} + \sum_{D>0} A(D,d) q^D. \tag{2}$$

We define a function $\mathcal{H}_d(X)$ by

$$\mathcal{H}_d(X) = \prod_{Q \in \mathcal{Q}_d/\Gamma} (X - J(\alpha_Q))^{e_{\alpha_Q}}.$$

Then Borcherds' theorem [2] states that

$$\mathcal{H}_d(J(z)) = q^{-H(d)} \prod_{\nu=1}^{\infty} (1-q^\nu)^{A(\nu^2,d)}. \tag{3}$$

Here we observe that the exponents of the modular function $\mathcal{H}_d(J(z))$ come from the coefficients of the weakly holomorphic modular forms f_d.

Let d denote a positive integer such that $-d$ is congruent to a square modulo $4N$. Let $\mathcal{Q}_{d,N} = \{[a,b,c] \in \mathcal{Q}_d \mid a \equiv 0 \pmod{N}\}$ on which $\Gamma_0^+(N)$ acts. Let f be a weakly holomorphic modular function for $\Gamma_0^+(N)$. We define the class number $H_N^+(d)$ and the trace $\mathbf{t}_f^+(d)$ by

$$H_N^+(d) = \sum_{Q \in \mathcal{Q}_{d,N}/\Gamma_0^+(N)} e_{\alpha_Q}, \qquad \mathbf{t}_f^+(d) = \sum_{Q \in \mathcal{Q}_{d,N}/\Gamma_0^+(N)} e_{\alpha_Q} f(\alpha_Q).$$

The higher level analogue of (3) was obtained in [11] by means of Borcherds lifting [3] when the level is prime. More generally, using Bruinier and Funke's work [4] on traces of singular moduli we derive the following result.

THEOREM 1.4. *Let N be a positive integer for which $\Gamma_0^+(N)$ is of genus zero. Let j_N^+ be the corresponding Hauptmodul. For each cusp s in $S_N := S_{\Gamma_0(N)} - \{\infty, 0\}$, define $k_s = (c, N/c)$ if s is represented by a rational number a/c such that c is a divisor of N and a is coprime to c. Then for each positive integer d such that $-d$ is congruent to a square modulo $4N$ we have the following infinite product expansion:*

$$\prod_{s \in S_N} (j_N^+(z) - j_N^+(s))^{-\frac{1}{2}k_s H(d/k_s^2)} \prod_{Q \in \mathcal{Q}_{d,N}/\Gamma_0^+(N)} (j_N^+(z) - j_N^+(\alpha_Q))^{e_{\alpha_Q}} = q^{-H_N^+(d) + \sum_{s \in S_N} \frac{1}{2} k_s H(d/k_s^2)} \prod_{\nu=1}^{\infty} (1-q^\nu)^{-B(\nu^2,d)} \tag{4}$$

where the class number $H(d/k_s^2)$ is defined to be zero if $k_s^2 \nmid d$, and $B(\nu^2, d)$ is the coefficient of q^d in a certain weakly holomorphic modular form of weight $3/2$ for $\Gamma_0(4N)$.

REMARK 1.5. For each $N \in \Phi$, we set

$$\mathcal{H}_d^{j_N^+}(X) = \prod_{Q \in \mathcal{Q}_{d,N}/\Gamma_0^+(N)} (X - j_N^+(\alpha_Q))^{e_{\alpha_Q}}$$

and take

$$f = \mathcal{H}_d^{j_N^+}(j_N^+(z)) = \prod_{Q \in \mathcal{Q}_{d,N}/\Gamma_0^+(N)} (j_N^+(z) - j_N^+(\alpha_Q))^{e_{\alpha_Q}}$$

so that f (or some power of f) is a meromorphic modular form of weight 0 for $\Gamma_0^+(N)$. If we apply Theorem 1.1 to f (or some power of f), then the exponent $c_f(n)$ satisfies the following identity:

$$\mathbf{t}^+_{j^+_{N,n}}(d) = \sum_{\nu \mid n} c_f(\nu)\nu.$$

Let N be a prime power and assume that the discriminant d satisfies the Heegner condition (i.e. if we write $d = d_K \cdot c^2$, then $(c, N) = 1$). In this case we note that $(k_s, N) > 1$ for $s \in S_N$. Thus $H(d/k_s^2) = 0$ and the product identity reduces to a simpler form

$$f = \mathcal{H}_d^{j_N^+}(j_N^+(z)) = q^{-H_N^+(d)} \prod_{\nu=1}^{\infty} (1 - q^\nu)^{-B(\nu^2, d)}, \tag{5}$$

so that the exponent of $f = \mathcal{H}_d^{j_N^+}(j_N^+(z))$ arises from weakly holomorphic modular forms.

In Theorem 1.4, if $N = 4$ and the discriminant is a multiple of 4, we can get more information on the exponent, as shown in Lemma 3.4, which enables us to express the Hurwitz–Kronecker class numbers via traces of singular moduli, as in the following theorem.

THEOREM 1.6.

$$H(d) = \begin{cases} -\dfrac{1}{24\sigma(n)} \mathbf{t}^+_{j^+_{4,n}}(4d) & \textit{if } n \textit{ is odd}, \\ -\dfrac{1}{24(\sigma(n) - 4\sigma(n/2) + 4\sigma(n/4))} (\mathbf{t}^+_{j^+_{4,n}}(4d) - 2\mathbf{t}_{n/2}(d)) & \textit{if } n \textit{ is even}. \end{cases}$$

Here $\mathbf{t}_{n/2}(d)$ *stands for* $\mathbf{t}^+_{j^+_{1,n/2}}(d)$.

We remark that Theorem 1.6 is an extension of [8, Theorem 1.2], but the style of the proof is different.

2. Proofs of Theorems 1.1 and 1.3. We denote by E_2 the weight two Eisenstein series

$$E_2(z) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n) q^n.$$

We define the usual slash operator $f|_k\gamma$ by

$$(f|_k\gamma)(z) = (\det\gamma)^{k/2}(cz+d)^{-k} f(\gamma z),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{Q})$ and $\gamma z = \frac{az+b}{cz+d}$. The action of Ramanujan's theta-operator is defined by

$$\theta\Big(\sum_{n=n_0}^{\infty} a(n)q^n\Big) := \sum_{n=n_0}^{\infty} na(n)q^n.$$

We construct a modular form that will be important in this paper.

LEMMA 2.1. *Let N be a positive integer and suppose that f is a meromorphic modular form of weight k for $\Gamma_0^+(N)$. Then*

$$F := \frac{\theta f}{f} - \frac{k(E_2(z) + NE_2(Nz))}{24}$$

is a meromorphic modular form of weight 2 for $\Gamma_0^+(N)$.

Proof. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0^+(N)$. Since $f(\gamma z) = (\det\gamma)^{-k/2}(cz+d)^k f(z)$, we have

$$\frac{\theta f}{f}(\gamma z) = (\det\gamma)^{-1}\bigg[(cz+d)^2\frac{\theta f}{f}(z) + \frac{kc}{2\pi i}(cz+d)\bigg]. \tag{6}$$

We note that the transformation formula for E_2 is given by

$$E_2(\gamma z) = (cz+d)^2 E_2(z) - \frac{6ci}{\pi}(cz+d) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{7}$$

and the function $E_2^*(z) := E_2(z) - 3/(\pi \mathrm{Im}(z))$ is $SL_2(\mathbb{Z})$-invariant under the action of the slash operator. Hence $E_2^*(z) + (E_2^*|_2 W_N)(z) = E_2(z) + NE_2(Nz) - 6/(\pi\mathrm{Im}(z))$ is $\Gamma_0^+(N)$-invariant and therefore for $\gamma \in \Gamma_0^+(N)$,

$$\begin{aligned} &E_2(\gamma z) + NE_2(N\gamma z) \\ &\quad = (\det\gamma)^{-1}\bigg[(cz+d)^2(E_2(z) + NE_2(Nz)) - \frac{12ci}{\pi}(cz+d)\bigg]. \end{aligned} \tag{8}$$

Consequently, the two functional equations (6) and (8) show that

$$F(\gamma z) = (\det\gamma)^{-1}(cz+d)^2 F(z) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^+(N).$$

Now we investigate the behavior of $F(z)$ at the cusp s. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ with $\gamma\infty = s$. We recall from (1) that

$$\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}\gamma = \gamma' U \quad \text{for some } \gamma' = \begin{pmatrix} * & * \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ and } U = \begin{pmatrix} x & y \\ 0 & w_s \end{pmatrix}.$$

Then we have

$$F|_2\gamma = \frac{\theta(f|_k\gamma)}{f|_k\gamma} - \frac{kE_2(z)}{24} - \frac{kNE_2(Uz)}{24w_s^2}, \tag{9}$$

which means that $F(z)$ is holomorphic at the cusp s. Indeed, we see that

$$\frac{\theta f}{f}|_2\gamma = \frac{\theta(f|_k\gamma)}{f|_k\gamma} + \frac{kc}{2\pi i(cz+d)}$$

and

$$\begin{aligned} E_2(N\gamma z) = E_2(\gamma' Uz) &= (c'Uz+d')^2E_2(Uz) - \frac{6c'i}{\pi}(c'Uz+d') \\ &= \frac{(cz+d)^2E_2(Uz)}{w_s^2} - \frac{6c'i(cz+d)}{\pi w_s} \end{aligned}$$

since $c'Uz + d' = (cz+d)/w_s$. These two identities, (7) and the fact that $c'/w_s = c/N$ imply (9). Consequently, the lemma follows. ∎

A meromorphic weight two modular form $G(z)$ for $\Gamma_0^+(N)$ has a Fourier expansion at each cusp $s \in S_{\Gamma_0^+(N)}$ of the form

$$(G|_2\gamma)(z) = \sum_{n \geq N_0} a_n q_{h_s}^n$$

where γ is an element of $SL_2(\mathbb{Z})$ such that $\gamma\infty = s$. As usual, adjoining the cusps to $\Gamma_0^+(N)\backslash\mathfrak{H}$ we obtain a compact Riemann surface $X_0^+(N)$. Then for the corresponding differential $\omega_G = G(z)dz$ on $X_0^+(N)$, using the canonical quotient map $\pi : \mathfrak{H}^* \to X_0^+(N)$ and the Residue Theorem we obtain

$$\sum_{p \in X_0^+(N)} \operatorname{Res}_p \omega_G = 0.$$

We note that for each $s \in S_{\Gamma_0^+(N)}$ and $\tau \in \mathfrak{H}$,

$$\operatorname{Res}_{\pi(s)} \omega_G = \frac{h_s}{2\pi i}a_0, \qquad \operatorname{Res}_{\pi(\tau)} \omega_G = e_\tau \operatorname{Res}_\tau G(z).$$

We are ready to prove our first main theorem which gives a formula for the exponents in the infinite product expansion of any modular form for $\Gamma_0^+(N)$.

For each $n \in \mathbb{Z}_{>0}$ let $G_n(z) = j_{N,n}^+(z)F(z)$ and $\omega_n = G_n(z)dz$. Then ω_n is a differential on $X_0^+(N)$. We calculate the residue of ω_n at each point $\pi(\tau)$ for $\tau \in \mathfrak{H}^*$. Using [6, Proposition 2.1] and the definitions of $F(z)$ and $E_2(z)$

we obtain

$$\begin{aligned}
G_n(z) &= j_{N,n}^+(z)F(z)\\
&= \left(\frac{1}{q^n}+O(q)\right)\\
&\quad\times\left(\operatorname{ord}_\infty f-\frac{k+kN}{24}+\sum_{m=1}^{\infty}\left(-\sum_{d|m}c_f(d)d+k\sigma(m)+kN\sigma\left(\frac{m}{N}\right)\right)q^m\right)\\
&= \text{negative powers of } q+\left(-\sum_{d|n}c_f(d)d+k\sigma(n)+kN\sigma\left(\frac{n}{N}\right)\right)\\
&\quad+\text{higher terms in } q,
\end{aligned}$$

which means

$$\operatorname{Res}_{\pi(\infty)}\omega_n=\frac{1}{2\pi i}\left(-\sum_{d|n}c_f(d)d+k\sigma(n)+kN\sigma\left(\frac{n}{N}\right)\right).$$

For a cusp $s\in S_{\Gamma_0^+(N)}-\{\infty\}$ and $\gamma\in SL_2(\mathbb{Z})$ with $\gamma\infty=s$, we let

$$j_{N,n}^+(\gamma z)=\sum_{m=0}^{\infty}b_{n,s}(m)q_{h_s}^m.$$

The constant term $j_{N,n}^+(s):=b_{n,s}(0)$ is independent of the choice of γ. Then we obtain

$$\begin{aligned}
(G_n|_2\gamma)(z) &= ((j_{N,n}^+F)|_2\gamma)(z)=j_{N,n}^+(\gamma z)(F|_2\gamma)(z)\\
&= \left(\sum_{m=0}^{\infty}b_{n,s}(m)q_{h_s}^m\right)\left(\frac{\theta(f|_k\gamma)}{f|_k\gamma}-\frac{k}{24}E_2(z)-\frac{kN}{24w_s^2}E_2(U(z))\right)\quad\text{by (9)}\\
&= j_{N,n}^+(s)\left(\frac{\operatorname{ord}_s f}{h_s}-\frac{k}{24}-\frac{kN}{24w_s^2}\right)+\text{higher terms in } q_{h_s},
\end{aligned}$$

which means

$$\operatorname{Res}_{\pi(s)}\omega_n=\frac{h_s}{2\pi i}j_{N,n}^+(s)\left(\frac{\operatorname{ord}_s f}{h_s}-\frac{k}{24}-\frac{kN}{24w_s^2}\right).$$

Let $\tau\in\mathfrak{H}$. Since $j_{N,n}^+(z)$ and $kE_2(z)+NE_2(Nz)$ is holomorphic at τ we obtain

$$\operatorname{Res}_{\pi(\tau)}\omega_n=e_\tau\operatorname{Res}_\tau j_{N,n}^+(z)\frac{\theta f}{f}=e_\tau j_{N,n}^+(\tau)\frac{\operatorname{ord}_\tau f}{2\pi i}.$$

Consequently, the Residue Theorem implies Theorem 1.1.

Next, we start with the proof of Theorem 1.3. We need three identities related to $r_4(n)$. Combining the identities we will find our formula. It is well known that Θ^4 is a modular form of weight 2 for $\Gamma_0(4)$. Moreover by [12,

p. 133] it has the following Fourier expansions at the cusps 0 and 1/2, with $\alpha = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\beta = \left(\begin{smallmatrix} -1 & 0 \\ -2 & -1 \end{smallmatrix}\right)$:

$$\text{at } s = 0: \quad (\Theta^4|_2\alpha)(z) = -1/4 + \text{higher terms in } q_4, \tag{10}$$

$$\text{at } s = 1/2: \quad (\Theta^4|_2\beta)(z) = 2^4 q + 2^6 q^3 + \text{higher terms in } q. \tag{11}$$

As usual, adjoining the cusps to $\Gamma_0(N)\backslash\mathfrak{H}$ we obtain a compact Riemann surface $X_0(N)$. Applying the Residue Theorem to the differential form $j_{4,n}\Theta^4(z)dz$ on $X_0(N)$ and using (10) and (11) we obtain the first identity

$$r_4(n) = j_{4,n}(0),$$

where $j_{4,n}$ is the modular function defined in Remark 1.2.

Now, $F(z) := (\theta\Theta^4)/\Theta^4 - E_2(z)/6$ is a modular form of weight 2 for $\Gamma_0(4)$. Applying the Residue Theorem to the differential form $j_{4,n}F(z)dz$ on $X_0(N)$ and using (10) and (11) we have the second identity

$$\sum_{d|n} c_{\Theta^4}(d)d = 4\sigma(n) + \frac{5j_{4,n}(1/2)}{6} - \frac{2j_{4,n}(0)}{3},$$

where $\Theta^4(z) = \prod_{n=1}^{\infty}(1-q^n)^{c_{\Theta^4}(n)}$. Lastly by Remark 1.2 we have the third identity

$$\sum_{d|n} c_{\Theta^4}(d)d = 2\sigma(n) + 8\sigma\left(\frac{n}{4}\right) + \frac{5j_{4,n}(1/2)}{6} - \frac{5j_{4,n}(0)}{12}.$$

Combining the three identities we obtain Theorem 1.3.

3. Proofs of Theorems 1.4 and 1.6. According to Bruinier and Funke's result [4], the generating series for the traces $\mathbf{t}^+_{j^+_{N,n}}(d)$ is the holomorphic part of a harmonic weak Maass form of weight 3/2. Furthermore in [8] it was shown that by adding a suitable linear combination of weight 3/2 Eisenstein series, one can always obtain a generating series that is a weakly holomorphic modular form. More precisely, by making use of [8, Theorem 3.2] if we proceed as in [8, Section 4], then we can obtain the following lemma.

LEMMA 3.1. *For each cusp $s \in S_N = S_{\Gamma_0(N)} - \{\infty, 0\}$, let k_s be the positive rational number defined in Theorem* 1.4. *Then*

$$\begin{aligned}\mathfrak{G}_n := {} & -\sum_{\nu|n} \nu q^{-\nu^2} + \left(\sigma(n) + N\sigma(n/N) - \frac{1}{24}\sum_{s\in S_N} j^+_{N,n}(s)(\hbar_s - k_s)\right) \\ & + \sum_{m>0}\left(\boldsymbol{t}^+_{j^+_{N,n}}(m) - \frac{1}{2}\sum_{s\in S_N} j^+_{N,n}(s)k_s H(m/k_s^2)\right)q^m\end{aligned}$$

is a weakly holomorphic modular form of weight 3/2.

For each positive integer ν, we inductively define g_{ν^2} such that

$$\mathfrak{G}_n = -\sum_{\nu|n} \nu g_{\nu^2}. \tag{12}$$

We denote by $B(\nu^2, d)$ the coefficient of q^d in g_{ν^2}. It then follows from (12) and the definition of $\mathfrak{G}_n$ that

$$\mathbf{t}^+_{j^+_{N,n}}(d) = \frac{1}{2}\sum_{s\in S_N} j^+_{N,n}(s) k_s H(d/k_s^2) - \sum_{\nu|n} \nu B(\nu^2, d). \tag{13}$$

Let $z, \tau \in \mathfrak{H}$. It is well known [14] that

$$j^+_N(z) - j^+_N(\tau) = q^{-1}\exp\Big(-\sum_{n=1}^{\infty} j^+_{N,n}(\tau)\frac{q^n}{n}\Big) \qquad (\mathrm{Im}(z) \gg 0). \tag{14}$$

We observe that

$$\prod_{Q\in\mathcal{Q}_{d,N}/\Gamma_0^+(N)} (j^+_N(z) - j^+_N(\alpha_Q))^{e_{\alpha_Q}} \tag{15}$$

$$= q^{-H^+_N(d)}\exp\Big(-\sum_{n=1}^{\infty} \mathbf{t}^+_{j^+_{N,n}}(d)\frac{q^n}{n}\Big)$$

by (14) and definition of $H^+_N(d)$ and $\mathbf{t}^+_{j^+_{N,n}}(d)$

$$= q^{-H^+_N(d)}\exp\Big(-\sum_{n=1}^{\infty}\sum_{s\in S_N}\frac{1}{2}j^+_{N,n}(s)k_s H(d/k_s^2)\frac{q^n}{n}\Big)\exp\Big(\sum_{n=1}^{\infty}\sum_{\nu|n}\nu B(\nu^2, d)\frac{q^n}{n}\Big)$$

by (13).

Lemma 3.2.

$$\exp\Big(-\sum_{n=1}^{\infty}\sum_{s\in S_N}\frac{1}{2}j^+_{N,n}(s)k_s H(d/k_s^2)\frac{q^n}{n}\Big) = q^{\sum_{s\in S_N}\frac{1}{2}k_s H(d/k_s^2)}\prod_{s\in S_N}(j^+_N(z) - j^+_N(s))^{\frac{1}{2}k_s H(d/k_s^2)}.$$

Proof. Let $s \in S_N$ be a cusp represented by a/c, where c is a divisor of N and a is coprime to c. Choose integers b and d such that $ad - bc = 1$. Then $a/c = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\infty$ with $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. In (14) if we replace τ by $\frac{a\tau+b}{c\tau+d}$ and send $\tau \to i\infty$, then we come up with

$$j^+_N(z) - j^+_N(s) = q^{-1}\exp\Big(-\sum_{n=1}^{\infty} j^+_{N,n}(s)\frac{q^n}{n}\Big), \tag{16}$$

from which the lemma easily follows. ∎

LEMMA 3.3.
$$\exp\Big(\sum_{n=1}^{\infty}\sum_{\nu\mid n}\nu B(\nu^2,d)\frac{q^n}{n}\Big)=\prod_{\nu=1}^{\infty}(1-q^\nu)^{-B(\nu^2,d)}.$$

Proof. Indeed,
$$\begin{aligned}\exp\Big(\sum_{n=1}^{\infty}\sum_{\nu\mid n}\nu B(\nu^2,d)\frac{q^n}{n}\Big)&\\&=\exp\Big(\sum_{n=1}^{\infty}\sum_{\nu=1}^{\infty}\nu B(\nu^2,d)\frac{q^{\nu n}}{\nu n}\Big)\\&=\exp\Big(\sum_{\nu=1}^{\infty}\Big(-B(\nu^2,d)\sum_{n=1}^{\infty}\frac{-(q^\nu)^n}{n}\Big)\Big)\\&=\exp\Big(\sum_{\nu=1}^{\infty}\log(1-q^\nu)^{-B(\nu^2,d)}\Big)=\prod_{\nu=1}^{\infty}(1-q^\nu)^{-B(\nu^2,d)}.\ \blacksquare\end{aligned}$$

Now Theorem 1.4 immediately follows from (15) and Lemmas 3.2 and 3.3. Next, to prove Theorem 1.6 we need the following lemma.

LEMMA 3.4. *Let d be a positive integer congruent to* $0,1$ *modulo* 4. *Then*
$$\begin{aligned}\mathcal{H}_d(J(2z))&=q^{-2H(d)}\prod_{\nu=1}^{\infty}(1-q^{2\nu})^{A(\nu^2,d)}\\&=(j_4^+(z)-j_4^+(1/2))^{-H(d)}\mathcal{H}_{4d}^{j_4^+}(j_4^+(z)).\end{aligned}\tag{17}$$

Proof. The first equality follows from (3). For the second equality we note that $J(2z)$ is a modular function for $\Gamma_0^+(4)$. Now we investigate the poles and zeros of the modular function $\mathcal{H}_d(J(2z))$ on the modular curve $\Gamma_0^+(4)\backslash\mathfrak{H}^*$. There are two inequivalent cusps ∞ and $1/2$ under $\Gamma_0^+(4)$. Then $\mathcal{H}_d(J(2z))$ has a pole of order $H(d)$ (resp. $2H(d)$) at the cusp $1/2$ (resp. ∞). And it has a usual simple zero at α_Q for each $Q=[4a,2b,c]\in\mathcal{Q}_{4d,4}/\Gamma_0^+(4)$. Since our modular curve $\Gamma_0^+(4)\backslash\mathfrak{H}^*$ has genus zero, there is a unique rational function in j_4^+ with prescribed zeros and poles. Now the assertion is immediate. $\blacksquare$

Taking the logarithmic derivative of (17) we come up with
$$\begin{aligned}\sum_{Q\in\mathcal{Q}_d/\Gamma(1)}e_{\alpha_Q}\frac{\theta(J(2z))}{J(2z)-J(\alpha_Q)}&\\&=-H(d)\frac{\theta(j_4^+(z))}{j_4^+(z)-j_4^+(1/2)}+\sum_{Q\in\mathcal{Q}_{4d,4}/\Gamma_0^+(4)}e_{\alpha_Q}\frac{\theta(j_4^+(z))}{j_4^+(z)-j_4^+(\alpha_Q)}.\end{aligned}$$

This combined with the logarithmic derivatives of (14) and (16) yields

$$-2\sum_{m\geq 1}\mathbf{t}_m(d)q^{2m} = H(d)\sum_{n\geq 1} j_{4,n}^{+}(1/2)q^n - \sum_{n\geq 1}\mathbf{t}^{+}_{j_{4,n}^{+}}(4d)q^n. \tag{18}$$

LEMMA 3.5. *$j_{4,n}^{+}(1/2) = -24(\sigma(n) - 4\sigma(n/2) + 4\sigma(n/4))$ where $\sigma(x)$ is defined to be zero if $x \notin \mathbb{Z}$.*

Proof. We recall that

$$-\frac{\theta(j_4^{+}(z))}{j_4^{+}(z) - j_4^{+}(1/2)} = 1 + \sum_{n=1}^{\infty} j_{4,n}^{+}(1/2)q^n. \tag{19}$$

We observe that the left hand side of (19) belongs to the space $M_2(\Gamma_0^{+}(4))$. Note that $M_2(\Gamma_0(4))$ is two-dimensional and spanned by $E_2^{(2)}(z)$ and $E_2^{(2)}(2z)$ where $E_2^{(2)}(z) = E_2(z) - 2E_2(2z)$. We see that

$$E_2^{(2)}(z) + E_2^{(2)}(z)|_2 W_4 = E_2^{(2)}(z) - 2E_2^{(2)}(2z) = E_2(z) - 4E_2(2z) + 4E_2(4z)$$

is invariant under the action of W_4 and spans the space $M_2(\Gamma_0^{+}(4))$. Here the first equality of the above identity follows from (7). The left hand side of (19) is then expressed as

$$-\frac{\theta(j_4^{+}(z))}{j_4^{+}(z) - j_4^{+}(1/2)} = E_2(z) - 4E_2(2z) + 4E_2(4z). \tag{20}$$

Now (19) combined with (20) yields the identity

$$1 + \sum_{n=1}^{\infty} j_{4,n}^{+}(1/2)q^n = E_2(z) - 4E_2(2z) + 4E_2(4z),$$

from which the assertion is immediate. ■

Now (18) together with Lemma 3.5 yields Theorem 1.6.

4. Example. Let $\eta(z)$ be the Dedekind eta-function defined by

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n)$$

and let

$$\begin{aligned} j_4^{+}(z) &= \left(\frac{\eta(z)}{\eta(4z)}\right)^8 + 8 + 4^4\left(\frac{\eta(4z)}{\eta(z)}\right)^8 \\ &= 1/q + 276q + 2048q^2 + 11202q^3 + 49152q^4 \\ &\quad + 184024q^5 + 614400q^6 + 1881471q^7 + \cdots \end{aligned}$$

be the Hauptmodul for $\Gamma_0^{+}(4)$. In Table 1 we shall compute the values of the class numbers $H(d)$ and the traces $\mathbf{t}(d)$, $\mathbf{t}^{+}_{j_{4,n}^{+}}(d)$ for $1 \leq n \leq 3$.

Table 1. Values of class numbers and traces

d	$H(d)$	$\mathbf{t}(d)$	d	$\mathcal{Q}_{d,4}/\Gamma_0^+(4)$	$\mathbf{t}^+_{j_4^+}(d)$	$\mathbf{t}^+_{j_{4,2}^+}(d)$	$\mathbf{t}^+_{j_{4,3}^+}(d)$
			7	$\{[4,5,2]\}$	-23	-23	733
3	1/3	-248	12	$\{[4,2,1]\}$	-8	-488	-32
			15	$\{[4,-1,1],[8,7,2]\}$	-1	-1	-13114
4	1/2	492	16	$\{[4,0,1],[4,-4,2]\}$	-12	996	-48

By Lemma 3.1, we have weakly holomorphic modular forms

$$\begin{aligned}\mathfrak{G}_n = &-\sum_{\nu\mid n}\nu q^{-\nu^2} + \sigma(n) + 4\sigma(n/4) - \frac{1}{24}j^+_{4,n}(1/2)(1-2)\\ &+\sum_{m>0}(\mathbf{t}^+_{j^+_{4,n}}(m) - j^+_{4,n}(1/2)H(m/4))q^m.\end{aligned}$$

And by Lemma 3.5 we compute that $j^+_{4,1}(1/2) = -24$, $j^+_{4,2}(1/2) = 24$, $j^+_{4,3}(1/2) = -96$, etc. Utilizing the cusp values of $j^+_{4,n}$ and Table 1 we have

$$\begin{aligned}\mathfrak{G}_1 &= -q^{-1} + 0 - 23q^7 - q^{15} + \cdots,\\ \mathfrak{G}_2 &= -2q^{-4} - q^{-1} + 4 - 23q^7 - 496q^{12} - q^{15} + 984q^{16} + \cdots,\\ \mathfrak{G}_3 &= -3q^{-9} - q^{-1} + 0 + 733q^7 - 13114q^{15} + \cdots, \text{ etc.,}\end{aligned}$$

from which it follows that

$$\begin{aligned}g_1 &= q^{-1} + 0 + 23q^7 + q^{15} + \cdots,\\ g_4 &= q^{-4} - 2 + 248q^{12} - 492q^{16} + \cdots,\\ g_9 &= q^{-9} + 0 - 252q^7 + 4371q^{15} + \cdots, \text{ etc.}\end{aligned}$$

Thus Theorem 1.4 yields the following product identities.

If $d = 7$,

$$j_4^+(z) + 23 = q^{-1}(1-q)^{-23}(1-q^2)^0(1-q^3)^{252}\cdots.$$

If $d = 12$,

$$(j_4^+(z) + 24)^{-1/3}(j_4^+(z) + 8) = q^{-1+1/3}(1-q)^0(1-q^2)^{-248}(1-q^3)^0\cdots.$$

From Table 1, we can check that

$$-\frac{1}{24}\mathbf{t}^+_{j_4^+}(4d) = -\frac{1}{96}\mathbf{t}^+_{j^+_{4,3}}(4d) = H(d)$$

and

$$\frac{1}{24}(\mathbf{t}^+_{j^+_{4,2}}(4d) - 2\mathbf{t}(d)) = H(d),$$

as desired in Theorem 1.6.

Moreover in (17) if we put $d = 3$, we obtain the following product identity by making use of Table 1, (2) and [15]:

$$(j_4^+(z)+24)^{-H(3)} \prod_{Q\in\mathcal{Q}_{12,4}/\Gamma_0^+(4)} (j_4^+(z)-j_4^+(\alpha_Q))^{e_{\alpha_Q}} = q^{-2H(3)}\prod_{\nu=1}^{\infty}(1-q^{2\nu})^{A(\nu^2,3)} = \mathcal{H}_3(J(2z)),$$

which reads

$$(j_4^+(z)+24)^{-1/3}(j_4^+(z)+8) = q^{-2/3}(1-q^2)^{-248}(1-q^4)^{26752}(1-q^6)^{-4096248} = (J(2z))^{1/3}.$$

Acknowledgments. We would like to thank KIAS (Korea Institute for Advanced Study) for its hospitality. We also thank the referee for valuable comments.

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Received on 1.10.2009
and in revised form on 11.3.2010 (6164)