Eight cubes of primes and powers of 2

by

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1. Introduction. In 1951 and 1953, Linnik [16], [17] proved that each large even integer \( N \) is a sum of two primes and a bounded number of powers of 2,

\[
N = p_1 + p_2 + 2^{v_1} + \cdots + 2^{v_k},
\]

where \( p \) and \( v \), with or without subscripts, denote a prime number and a positive integer respectively. Later Gallagher [3] established a stronger result by a different method. An explicit value for the number \( k \) of powers of 2 was first established by Liu, Liu and Wang [21], who found that \( k = 54000 \) is acceptable. This value was subsequently improved by Li [12], Wang [30] and Li [13]. In 2002, Heath-Brown and Puchta [6] applied a rather different approach to this problem and showed that \( k = 13 \) is acceptable. In 2003, Pintz and Ruzsa [25] announced that \( k = 8 \) is acceptable.

In 1999, Liu, Liu and Zhan [22] proved that every large even integer \( N \) can be written as a sum of four squares of primes and a bounded number of powers of 2,

\[
N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + \cdots + 2^{v_k}.
\]

Later Liu and Liu [18] showed that \( k = 8330 \) is acceptable. This value was subsequently improved by Liu and Lü [23] and Li [14].

In 1938, Hua [7] proved that each large odd integer is the sum of nine cubes of primes. It seems reasonable to conjecture that every sufficiently large integer satisfying some necessary congruence conditions is the sum of eight cubes of primes, i.e.

\[
N = p_1^3 + p_2^3 + \cdots + p_8^3,
\]

but unfortunately, such a conjecture is out of reach at present.

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Motivated by this conjecture and the above works of Linnik and Gallagher for two primes and powers of 2, and the result of Liu, Liu and Zhan for four squares of primes and powers of 2, we extend the above results (1.1) and (1.2) to sums of eight cubes of primes and powers of 2, i.e.

\[ N = p_1^3 + \cdots + p_8^3 + 2^{\nu_1} + \cdots + 2^{\nu_k}. \]  

In 2000, Liu and Liu \cite{20} proved that such a \( k \) exists.

In this paper we bound the value of \( k \) in (1.4) by proving the following theorem.

**Theorem 1.1.** Every large even integer is a sum of eight cubes of primes and 358 powers of 2.

There are other approximations to the conjecture (1.3), and our theorem can be compared with them. In \cite{31}, Wooley got an upper bound for the exceptional set for (1.3): he showed that with at most \( O(N^{11/36+\varepsilon}) \) exceptions, all positive even integers not exceeding \( N \) can be written as in (1.3). Later Kumchev \cite{10} improved this estimate to \( O(N^{23/84+\varepsilon}) \). Roth \cite{28} proved that every large integer \( N \) can be written as

\[ N = m^3 + p_2^3 + \cdots + p_8^3 \]  

with a positive integer \( m \). Brüdern \cite{1} combined the circle method with sieves to show that (1.5) is solvable when \( m \) is a product \( P_4 \) of at most four primes. Kawada \cite{9} improved the above \( P_4 \) to \( P_3 \).

**Notation.** As usual, \( \varphi(n) \) and \( \Lambda(n) \) denote the Euler totient function and the von Mangoldt function, respectively. We write \( N \) for a large integer, and \( L = \log N \). Further, \( r \sim R \) means \( R < r \leq 2R \), and \( A \asymp B \) means \( c_1 A \leq B \leq c_2 A \). The letters \( \varepsilon \) and \( A \) denote positive constants, which are arbitrarily small and arbitrarily large, respectively.

2. **Outline of the method.** Here we outline the proof of Theorem 1.1.

In order to apply the circle method, we set

\[ P = N^{1/9-2\varepsilon}, \quad Q = N^{8/9+\varepsilon}. \]  

By Dirichlet’s lemma (\cite{29} Lemma 2.1), each \( \alpha \in [1/Q, 1+1/Q] \) may be written in the form

\[ \alpha = a/q + \lambda, \quad |\lambda| \leq 1/qQ, \]  

for some integers \( a,q \) with \( 1 \leq a \leq q \leq Q \) and \((a,q) = 1\). Denote by \( \mathcal{M}(a,q) \) the set of \( \alpha \) satisfying (2.2), and define the major arcs \( \mathcal{M} \) and the minor
arcs $C(M)$ as follows:

\[(2.3) \quad M := \bigcup_{1 \leq q \leq P} \bigcup_{1 \leq a \leq q \atop (a,q) = 1} M(a,q), \quad C(M) = \left[ \frac{1}{Q}, 1 + \frac{1}{Q} \right] \setminus M.\]

It follows from $2P \leq Q$ that the major arcs $M(a,q)$ are mutually disjoint. As in [27], let $\delta = 10^{-4}$, and

\[(2.4) \quad U = \left( \frac{N}{16(1+\delta)} \right)^{1/3}, \quad V = U^{5/6}.
\]

As usual in the circle method, let

\[(2.5) \quad S(\alpha) = \sum_{p \sim U} (\log p)e(p^3\alpha), \quad T(\alpha) = \sum_{p \sim V} (\log p)e(p^3\alpha),
\]

\[(2.6) \quad G(\alpha) = \sum_{2^v \leq N} e(2^v\alpha) = \sum_{v \leq \log_2 N} e(2^v\alpha),
\]

and

\[(2.7) \quad r_k(N) = \sum_{\substack{N = p_1^{v_1} \cdots p_8^{v_8} + 2^{v_1} + \cdots + 2^{v_k} \hfill p_1, \ldots, p_4 \sim U, p_5, \ldots, p_8 \sim V}} (\log p_1) \ldots (\log p_8).
\]

Then $r_k(N)$ can be written as

\[(2.8) \quad r_k(N) = \int_0^1 S^4(\alpha)T^4(\alpha)G^k(\alpha)e(-N\alpha) \, d\alpha
\]

\[= \left\{ \int_{M} + \int_{C(M)} \right\} S^4(\alpha)T^4(\alpha)G^k(\alpha)e(-N\alpha) \, d\alpha.
\]

To handle the integral on the major arcs, we prove the following lemma.

**Lemma 2.1.** Let $M$ be as in \[(2.3)\], with $P$ and $Q$ determined by \[(2.1)\]. Then for $N/2 \leq n \leq N$, we have

\[(2.9) \quad \int_{M} S^4(\alpha)T^4(\alpha)e(-n\alpha) \, d\alpha = \frac{1}{3^8} \mathcal{S}(n)J(n) + O(UV^4L^{-1}).
\]

Here $\mathcal{S}(n)$ is a singular series, which is defined by

\[(2.10) \quad \mathcal{S}(n) := \sum_{q=1}^{\infty} \frac{1}{\phi^8(q)} \sum_{a=1 \atop (a,q) = 1}^{q} \left( \sum_{h=1 \atop \text{Irrelevant}}^{q} e\left( \frac{ah^3}{q} \right) \right)^8 e\left( -\frac{an}{q} \right),
\]

and satisfies $\mathcal{S}(n) \gg 1$ for $n \equiv 0 \pmod{2}$. $J(n)$ is defined as

\[(2.11) \quad J(n) := \sum_{m_1 + \cdots + m_8 = n \atop U^3 < m_1, \ldots, m_4 \leq 8U^3, V^3 < m_5, \ldots, m_8 \leq 8V^3} (m_1 \cdots m_8)^{-2/3},
\]
and satisfies

\[
UV^4 \ll J(n) \ll UV^4.
\]

In this paper, the constants in the \( \gg \) and \( \ll \) symbols are of importance. If we write \( \mathcal{G}(n) > C_1 \) and \( J(n) > C_2 UV^4 \), in the following parts, we determine explicit values of \( C_1, C_2 \).

A crucial step in bounding the contributions of minor arcs is an upper bound for the number of solutions of the equation

\[
n = p_1^3 + \cdots + p_4^3 - p_5^3 - \cdots - p_8^3, \quad 0 \leq |n| \leq N.
\]

We quote the following lemma.

**Lemma 2.2.** Let \( n \equiv 0 \pmod{2} \) be an integer, and \( \rho(n) \) the number of representations of \( n \) in the form (2.13) subject to

\[
p_1, p_2, p_5, p_6 \sim U, \quad p_3, p_4, p_7, p_8 \sim V.
\]

Then for all \( 0 \leq |n| \leq N \),

\[
\rho(n) \leq bUV^4L^{-8},
\]

with \( b = 268096 \).

The inequality (2.15) is (2.6) in Ren [26], obtained by sieve methods, and the value of \( b \) is determined in Ren [27].

On the minor arcs, we also need estimates for the measure of the set

\[
E_\lambda = \{ \alpha \in (0,1] : |G(\alpha)| \geq \lambda \log_2 N \}.
\]

The following lemma is due to Heath-Brown and Puchta [6].

**Lemma 2.3.** Let

\[
G_h(\alpha) = \sum_{0 \leq n \leq h-1} e(\alpha2^n), \quad F(\xi, h) = \frac{1}{2h} \sum_{r=0}^{2h-1} \exp \left[ \xi \Re \left( G_h \left( \frac{r}{2h} \right) \right) \right].
\]

Then

\[
\text{meas}(E_\lambda) \leq N^{-E(\lambda)},
\]

where

\[
E(\lambda) = \frac{\xi \lambda}{\log 2} - \frac{\log F(\xi, h)}{h \log 2} - \frac{\varepsilon}{\log 2}
\]

for any \( h \in \mathbb{N}, \xi > 0 \) and \( \varepsilon > 0 \).

On the minor arcs, the results of Kumchev [10] on exponential sums over primes will also be applied. The following lemma is Theorem 3 of [10] for \( k = 3 \).
Lemma 2.4 (Kumchev). Let \( \alpha = a/q + \lambda \) subject to \( 1 \leq a \leq q \), \( (a, q) = 1 \), and \( |\lambda| \leq 1/qQ \), with \( Q = U^{12/7} \), and let \( S(\alpha) \) be defined in (2.5). Then

\[
S(\alpha) \ll U^{1-\varrho+\varepsilon} + \frac{q^2 U L c}{\sqrt{q(1 + |\lambda| U^3)}},
\]

with \( \varrho = 1/14 \).

We deduce Theorem 1.1 from some lemmas in Section 3. In Section 4, we give the proof of Lemma 2.1. In Sections 5 and 6, we give the value of \( C_1 \) and the proofs of three lemmas, respectively.

3. The proof of Theorem 1.1. We need the following five lemmas.

Lemma 3.1. Let

\[
\Xi(N, k) = \{(1 - \delta)N \leq n \leq N : n = N - 2^{\nu_1} - \cdots - 2^{\nu_k}\},
\]

with \( k \geq 2 \). Then for \( N \equiv 0 \pmod{2} \),

\[
\sum_{\substack{n \in \Xi(N, k) \\ n \equiv 0 \pmod{2}}} 1 \geq (1 - \varepsilon)(\log_2 N)^k.
\]

Proof. The proof is straightforward, so we omit the details. ■

Lemma 3.2. For \( n \equiv 0 \pmod{2} \), we have \( \mathcal{G}(n) > C_1 \) with

\[
C_1 = 0.00557795824;
\]

while for \( n \not\equiv 0 \pmod{2} \), we have \( \mathcal{G}(n) = 0 \).

Proof. We will prove this in Section 5. ■

Lemma 3.3. For \( (1 - \delta)N \leq n \leq N \), we have \( J(n) > C_2 U V^4 \), with

\[
C_2 = 78.15467793.
\]

Proof. We will determine the value of \( C_2 \) in Section 4. ■

Lemma 3.4. Let \( C(M) \) be as in (2.3), with \( P \) and \( Q \) determined by (2.1), and \( S(\alpha) \) be as in (2.5). Then

\[
\max_{\alpha \in C(M)} |S(\alpha)| \ll N^{1/3 - 1/42 + \varepsilon}.
\]

Proof. By Dirichlet’s lemma on rational approximations, each real number \( \alpha \in C(M) \) can be written as \( \alpha = a/q + \lambda \) with \( (a, q) = 1 \) and

\[
1 \leq q \leq Q_0 = N^{4/7}, \quad |\lambda| \leq 1/qQ_0.
\]
If \( q \leq P = N^{1/9-2\varepsilon} \), since \( \alpha \in C(M) \), we have \( |\lambda| > 1/qQ \); otherwise \( q > P \). In either case,
\[
\sqrt{q(1 + |\lambda|U^3)} > \min(P^{1/2}, (U^3/Q)^{1/2}) = N^{1/18-\varepsilon}.
\]
By Lemma 2.4, the conclusion follows.

In order to apply Lemma 2.3, we need to find an optimal \( \lambda \) such that \( E(\lambda) > 19/21 \). Thus we have to compute
\[
F(\xi, h) = \frac{1}{2^h} \sum_{r=0}^{2^h-1} \exp\left[ \xi \text{Re}\left(G_h\left(\frac{r}{2^h}\right)\right)\right],
\]
and optimize for \( \xi \) and \( h \). We can take \( \xi = 1.59, h = 23 \) in Lemma 2.3 to get

**Lemma 3.5.** Let \( E(\lambda) \) be as in Lemma 2.3. Then
\[
E(0.965411) > \frac{19}{21} + 10^{-10}.
\]

**Proof of Theorem 1.1.** Let \( N \equiv 0 \pmod{2} \), let \( \mathcal{E}_\lambda \) be as in (2.16) and \( M \) as in (2.3), with \( P \) and \( Q \) determined by (2.1). Then, by (2.8),
\[
r_k(N) = \int_0^1 S^4(\alpha)T^4(\alpha)G^k(\alpha)e(-N\alpha) \, d\alpha
\]
\[
= \int_{M} + \int_{C(M) \cap \mathcal{E}_\lambda} + \int_{C(M) \cap C(\mathcal{E}_\lambda)}.
\]
Introducing the notation \( \Xi(N, k) \) and then applying Lemma 2.1, we see that the first integral on the right-hand side of (3.6) is
\[
\sum_{n \in \Xi(N, k)} \int\limits_{M} S^4(\alpha)T^4(\alpha)e(-n\alpha) \, d\alpha
\]
\[
= \frac{1}{3^8} \sum_{n \in \Xi(N, k)} \mathcal{G}(n)J(n) + O(UV^4L^{k-1})
\]
\[
\geq \frac{1}{3^8} C_1 C_2 UV^4 \sum_{n \in \Xi(N, k)} 1 + O(UV^4L^{k-1})
\]
\[
\geq \frac{1}{3^8} C_1 C_2 (1 - \varepsilon)UV^4 \log_2 N^k,
\]
where in the last two inequalities we have used Lemmas 3.1–3.3.

With Lemma 3.4, the second integral satisfies
\[
\int_{C(M) \cap \mathcal{E}_\lambda} \ll N^{-E(\lambda)}(N^{1/3-1/42+\varepsilon})^4V^4 \log_2 N^k \ll UV^4L^{k-1}.
\]
By using the definition of $E_\lambda$ and Lemma 2.2, the last integral in (3.6) can be estimated as follows:

\begin{equation}
C(M) \cap C(E_\lambda) \leq (\lambda \log N)^k \left( \frac{1}{3} \right)^4 \left( \frac{5}{18} \right)^4 bUV^4,
\end{equation}

where in the last inequality we have used Lemma 2.2 and the definition of $\rho(n)$.

Inserting (3.7)–(3.9) into (3.6), we get

\begin{equation}
r_k(N) \geq \left( \left( \frac{1}{3} \right)^8 C_1 C_2 - \left( \frac{1}{3} \right)^4 \left( \frac{5}{18} \right)^4 b \lambda^k \right) (1 - \varepsilon) UV^4 (\log N)^k + O(UV^4 L^{k-1}).
\end{equation}

When $k \geq 358$ and $\varepsilon = 10^{-10}$, we obtain

\[ r_k(N) > 1.3 \cdot 10^{-7} UV^4 (\log N)^k. \]

Recalling the definition of $U$ and $V$, we conclude that every sufficiently large even integer $N$ can be expressed in the form (1.4). This completes the proof of Theorem 1.1.

4. The major arcs: proof of Lemma 2.1. For $\chi$ a character modulo $q$, define

\begin{equation}
C(\chi, a) := \sum_{h=1}^{q} \overline{\chi}(h) e \left( \frac{ah^3}{q} \right), \quad C(q, a) := C(\chi_0, a).
\end{equation}

If $\chi_1, \ldots, \chi_8$ are characters modulo $q$, then we write

\begin{equation}
B(n, q; \chi_1, \ldots, \chi_8) := \sum_{a=1}^{q} e \left( -\frac{an}{q} \right) C(\chi_1, a) \ldots C(\chi_8, a),
\end{equation}

\begin{equation}
B(n, q) := B(n, q; \chi_0, \ldots, \chi_0).
\end{equation}

The following lemma is important in proving Lemma 2.1.

**Lemma 4.1.** Let $\chi_i$ with $i = 1, \ldots, 8$ be primitive characters modulo $r_i$, $r_0 = [r_1, \ldots, r_8]$, and $\chi_0$ be the principal character modulo $q$. Then

\[ \sum_{q \leq z, r_0 \mid q} \frac{1}{\varphi^8(q)} |B(n, q; \chi_1 \chi_0, \ldots, \chi_8 \chi_0)| \ll r_0^{-3+\varepsilon} \log^c z. \]

**Proof.** It is similar to that of Lemma 7 in [11], so we omit the details.
To state other preliminaries, we need to introduce some extra notations. For \( i = 1, 2 \) and \( W \) equal to \( U \) or \( V \) respectively, we define
\[
V_i(\lambda) := \sum_{m \sim W} e(m^3 \lambda),
\]
\[
W_i(\chi, \lambda) := \sum_{p \sim W} (\log p) \chi(p) e(p^3 \lambda) - \delta_\chi \sum_{m \sim W} e(m^3 \lambda),
\]
where \( \delta_\chi = 1 \) or 0 according as \( \chi \) is principal or not. Define
\[
J_i(g) := \sum_{r \leq P} [g, r]^{-3+\varepsilon} \sum_{\chi \mod r} \max_{|\lambda| \leq 1/r^c} |W_i(\chi, \lambda)|,
\]
\[
K(g) := \sum_{r \leq P} [g, r]^{-3+\varepsilon} \sum_{\chi \mod r} \left( \int_{-1/r^c}^{1/r^c} |W_1(\chi, \lambda)|^2 d\lambda \right)^{1/2}.
\]

Estimates for \( J_i \) (\( i = 1, 2 \)) and \( K \) are needed in later arguments. In particular, the following three lemmas will be important to deal with enlarged major arcs.

**Lemma 4.2.** Let \( U, V \) be as in (2.4), and let \( P, Q \) satisfy (2.1). Then
\[
J_i(g) \ll g^{-3+\varepsilon} W L^c.
\]

**Lemma 4.3.** Let \( U, P, Q \) be as in Lemma 4.2. If \( g = 1 \), then (4.8) can be improved to
\[
J_1(1) \ll U L^{-A},
\]
where \( A > 0 \) is arbitrary.

**Lemma 4.4.** Let \( U, P, Q \) be as in Lemma 4.2. Then
\[
K(g) \ll g^{-3+\varepsilon} U^{-1/2} L^c.
\]

We will prove Lemmas 4.2–4.4 in Section 6.

**Proof of Lemma 2.1.** Introducing Dirichlet characters, we can rewrite the exponential sums \( S(\alpha) \) and \( T(\alpha) \) as
\[
S\left(\frac{a}{q} + \lambda\right) = C(q, a) \varphi(q)^{-1} V_1(\lambda) + \frac{1}{\varphi(q)} \sum_{\chi \mod q} C(\chi, a) W_1(\chi, \lambda),
\]
\[
T\left(\frac{a}{q} + \lambda\right) = C(q, a) \varphi(q)^{-1} V_2(\lambda) + \frac{1}{\varphi(q)} \sum_{\chi \mod q} C(\chi, a) W_2(\chi, \lambda).
\]

Thus
\[
\int M S^4(\alpha) T^4(\alpha) e(-n\alpha) d\alpha = \sum_{0 \leq i \leq 4} \sum_{0 \leq j \leq 4} C_i C_j I_{ij},
\]
Eight cubes of primes and powers of 2

where

\[
I_{ij} = \sum_{q \leq P} \frac{1}{\varphi^8(q)} \sum_{a=1}^{q} C^{8-i-j}(q, a) e\left(-\frac{an}{q}\right) \int_{-1/qQ}^{1/qQ} V_1^{4-i}(\lambda)V_2^{4-j}(\lambda)
\times \left\{ \sum_{\chi \mod q} C(\chi, a)W_1(\chi, \lambda) \right\}^{i} \left\{ \sum_{\chi \mod q} C(\chi, a)W_2(\chi, \lambda) \right\}^{j} e(-n\lambda) \, d\lambda.
\]

We will prove that \(I_{00}\) gives the main term, and the others the error term.

We begin with \(I_{00}\), which we expect to be the main term:

\[
(4.14) \quad I_{00} = \sum_{q \leq P} \frac{1}{\varphi^8(q)} \sum_{a=1}^{q} C^{8}(q, a) e\left(-\frac{an}{q}\right)
\times \int_{-1/qQ}^{1/qQ} V_1^{4}(\lambda)V_2^{4}(\lambda)e(-n\lambda) \, d\lambda.
\]

By Lemma 7.11 of [8],

\[
(4.15) \quad V_i(\lambda) = \int_{W} e(\lambda u^3) \, du + O(1) = \frac{1}{3} \sum_{W^3 < m \leq 8W^3} m^{-2/3} e(m\lambda) + O(1).
\]

Using this and the elementary estimate

\[
(4.16) \quad \sum_{W^3 < m \leq 8W^3} m^{-2/3} e(m\lambda) \ll \min(W, W^{-2}|\lambda|^{-1}),
\]

we have

\[
(4.17) \quad I_{00} = \frac{1}{3^8} \sum_{q \leq P} \frac{B(n, q)}{\varphi^8(q)} \int_{-1/qQ}^{1/qQ} \left( \sum_{U^3 < m \leq 8U^3} m^{-2/3} e(m\lambda) \right)^4
\times \left( \sum_{V^3 < m \leq 8V^3} m^{-2/3} e(m\lambda) \right)^4 e(-n\lambda) \, d\lambda
\]

\[
+ O\left( \sum_{q \leq P} \frac{|B(n, q)|}{\varphi^8(q)} \int_{-1/qQ}^{1/qQ} \left| \sum_{U^3 < m \leq 8U^3} m^{-2/3} e(m\lambda) \right|^4
\times \left| \sum_{V^3 < m \leq 8V^3} m^{-2/3} e(m\lambda) \right|^3 \, d\lambda \right).
\]

By (4.16) and Lemma 4.1 with \(r_0 = 1\), the \(O\)-term in (4.17) can be estima-
\[ \ll \sum_{q \leq P} \frac{|B(n, q)|}{\varphi^8(q)} \left( \int_0^{U^{-3}} U^4 V^3 d\lambda + \int_{U^{-3}}^{V^{-3}} U^{-8} \lambda^{-4} V^3 d\lambda \right. \\
\left. \quad + \int_{V^{-3}}^{\infty} U^{-8} \lambda^{-4} V^{-6} \lambda^{-3} d\lambda \right) \ll L^c(UV^3 + UV^3 + U^{-8} V^{12}) \ll UV^3 L^c \ll UV^4 L^{-1}. \]

Now we extend the integral in the main term of (4.17) to \([-1/2, 1/2] \); by a similar argument we see that the resulting error can be estimated as
\[ \ll \frac{L}{2^{1/2}} \int_{1/PQ}^{1/2} (U^{-2} \lambda^{-1})^4 V^4 d\lambda \ll L^c U^{-8} V^4 (PQ)^3 \ll UV^4 L^{-1}, \]
which is acceptable by the choice of \( P \) and \( Q \). Thus the main term of (4.17) becomes
\[ (4.18) \quad \frac{1}{3^8} \sum_{q \leq P} \frac{B(n, q)}{\varphi^8(q)} \sum_{m_1 + \cdots + m_8 = n, U^3 < m_1, \ldots, m_4 \leq 8U^3, V^3 < m_5, \ldots, m_8 \leq 8V^3} (m_1 \ldots m_8)^{-2/3} + O(UV^4 L^{-1}) \]
\[ = \frac{1}{3^8} \sum_{q \leq P} \frac{B(n, q)}{\varphi^8(q)} J(n) + O(UV^4 L^{-1}), \]
where \( J(n) \) is defined by (2.11).

The first sum above is \( \mathcal{D}(n) + O(L^{-1}) \). The domain of the second sum, \( J(n) \), can be written as
\[ \mathcal{D} = \{(m_1, \ldots, m_8) : U^3 < m_1, \ldots, m_4 \leq 8U^3, V^3 < m_5, \ldots, m_8 \leq 8V^3\}, \]
with \( m_1 = n - m_2 - \cdots - m_8 \).

To bound this sum from below, if we define
\[ \mathcal{D}^* = \left\{(m_2, \ldots, m_8) : \frac{8}{3} U^3 < m_2, \ldots, m_4 \leq 5U^3, V^3 < m_5, \ldots, m_8 \leq 8V^3\right\}, \]
we can deduce from \((1 - \delta)N < n \leq N\) and (2.4) that
\[ U^3 < m_1 = n - m_2 - \cdots - m_8 \leq 8U^3. \]
Thus \( \mathcal{D}^* \) is a subset of \( \mathcal{D} \), and consequently
\[ J(n) \geq \sum_{U^3 < m_1 \leq 8U^3, \frac{8}{3} U^3 < m_2, m_3 \leq 5U^3, V^3 < m_5, \ldots, m_8 \leq 8V^3} \]
\[ \quad \geq (5U^3)^{-2/3} \left\{ 5^{1/3} - \left(\frac{8}{3}\right)^{1/3}\right\}^2 \left(3^7 U^3 V^4 \right) \geq 78.15467793UV^4. \]
So, we get \( C_2 = 78.15467793 \).
It remains to estimate $I_{ij}$ ($0 \leq i, j \leq 4$, not both zero). We shall first treat $I_{44}$, the most complicated one. We have

$$I_{44} = \sum_{q \leq P} \frac{1}{\varphi^8(q)} \sum_{a=1}^{q} e\left(-\frac{an}{q}\right) \left\{ \sum_{\chi \mod q} C(\chi, a)W_{1}(\chi, \lambda) \right\}^4$$

$$\times \left\{ \sum_{\chi \mod q} C(\chi, a)W_{2}(\chi, \lambda) \right\}^4 e(-n\lambda) \, d\lambda$$

$$= \sum_{q \leq P} \frac{1}{\varphi^8(q)} \sum_{\chi_1 \mod q} \cdots \sum_{\chi_8 \mod q} \sum_{a=1}^{q} \frac{B(n, q; \chi_1, \chi^0, \ldots, \chi_8, \chi^0)}{\varphi^8(q)}$$

$$\left( \sum_{r_0 \equiv q \mod r_k} \sum_{\chi_k \mod r_k} \sum_{a=1}^{r_k} C(\chi_k, a) \right) \times \left( \sum_{r_0 \equiv q \mod r_k} \sum_{\chi_k \mod r_k} \sum_{a=1}^{r_k} C(\chi_k, a) \right) \times \left( \sum_{r_0 \equiv q \mod r_k} \sum_{\chi_k \mod r_k} \sum_{a=1}^{r_k} C(\chi_k, a) \right)$$

where $\chi_0$ is the principal character modulo $q$, $r_0 = [r_1, \ldots, r_8]$, and the sum $\sum^*$ is taken over all primitive characters. Suppose that $\chi_k$ is the primitive character modulo $r_k$ with $r_k \mid q$, inducing $\chi_k$. Thus we may write $\chi_k = \chi_k^* \chi_0^k$. It is easy to see that $W(\chi_k, \lambda) = W(\chi_k^*, \lambda)$. By Lemma 4.1 and Cauchy’s inequality, we have

$$|I_{44}| \ll L^c \sum_{r_1 \leq P} \sum_{\chi_1 \mod r_1} \max_{|\lambda| \leq 1/r_1Q} |W_{1}(\chi_1, \lambda)| \sum_{r_2 \leq P} \sum_{\chi_2 \mod r_2} \max_{|\lambda| \leq 1/r_2Q} |W_{1}(\chi_2, \lambda)|$$

$$\times \sum_{r_3 \leq P} \sum_{\chi_3 \mod r_3} \sum^* \left( \frac{1}{r_3Q} \int_{-1/r_3Q}^{1/r_3Q} |W_{1}(\chi_3, \lambda)|^2 \, d\lambda \right)^{1/2}$$

$$\times \sum_{r_4 \leq P} \sum_{\chi_4 \mod r_4} \sum^* \left( \frac{1}{r_4Q} \int_{-1/r_4Q}^{1/r_4Q} |W_{1}(\chi_4, \lambda)|^2 \, d\lambda \right)^{1/2}$$

$$\times \sum_{r_5 \leq P} \sum_{\chi_5 \mod r_5} \max_{|\lambda| \leq 1/r_5Q} |W_{2}(\chi_5, \lambda)| \sum_{r_6 \leq P} \sum_{\chi_6 \mod r_6} \max_{|\lambda| \leq 1/r_6Q} |W_{2}(\chi_6, \lambda)|$$

$$\times \sum_{r_7 \leq P} \sum_{\chi_7 \mod r_7} \sum^* \max_{|\lambda| \leq 1/r_7Q} |W_{2}(\chi_7, \lambda)| \sum_{r_8 \leq P} \sum_{\chi_8 \mod r_8} \max_{|\lambda| \leq 1/r_8Q} |W_{2}(\chi_8, \lambda)|.$$
Now we introduce an iterative procedure to bound the above sums over $r_8, \ldots, r_1$ consecutively. Since $r_0 = [r_1, \ldots, r_8] = [r_1, \ldots, r_7, r_8]$, we use Lemma 4.2 four times, Lemma 4.4 twice, Lemma 4.2 once, and Lemma 4.3 once to get

\begin{equation}
|I_{44}| \ll L^c \sum_{r_1 \leq P} \sum_{\chi_1 \mod r_1}^* \max_{|\lambda| \leq 1/r_1 Q} |W_1(\chi_1, \lambda)| \sum_{r_2 \leq P} \sum_{\chi_2 \mod r_2}^* \max_{|\lambda| \leq 1/r_2 Q} |W_1(\chi_2, \lambda)|
\end{equation}

\begin{align*}
\times \sum_{r_3 \leq P} \sum_{\chi_3 \mod r_3}^* \left( \int_{-1/r_3 Q}^{1/r_3 Q} |W_1(\chi_3, \lambda)|^2 d\lambda \right)^{1/2} \\
\times \sum_{r_4 \leq P} \sum_{\chi_4 \mod r_4}^* \left( \int_{-1/r_4 Q}^{1/r_4 Q} |W_1(\chi_4, \lambda)|^2 d\lambda \right)^{1/2} \\
\times [r_1, r_2, r_3, r_4]^{-3+\varepsilon V^4 L^4 c}
\end{align*}

\begin{equation}
\ll L^c \sum_{r_1 \leq P} \sum_{\chi_1 \mod r_1}^* \max_{|\lambda| \leq 1/r_1 Q} |W_1(\chi_1, \lambda)| \sum_{r_2 \leq P} \sum_{\chi_2 \mod r_2}^* \max_{|\lambda| \leq 1/r_2 Q} |W_2(\chi_2, \lambda)|
\end{equation}

\begin{align*}
\times [r_1, r_2]^{-3+\varepsilon U^{-1} V^4 L^6 c} \\
\ll L^c \sum_{r_1 \leq P} \sum_{\chi_1 \mod r_1}^* \max_{|\lambda| \leq 1/r_1 Q} |W_1(\chi_1, \lambda)| r_1^{-3+\varepsilon V^4 L^7 c}
\end{align*}

\begin{equation}
\ll UV^4 L^{-A+8c} \ll UV^4 L^{-1}
\end{equation}

for large $A > 0$.

To get upper bounds for other terms, we need to estimate $V_1(\lambda)$ and $V_2(\lambda)$. One easily gets

\begin{equation}
\max_{|\lambda| \leq 1/Q} |V_i(\lambda)| \ll W.
\end{equation}

By (4.15) and (4.16),

\begin{align*}
\int_{-1/Q}^{1/Q} |V_i(\lambda)|^2 d\lambda &\ll \int_{-1/Q}^{1/Q} ((\min(W^{-1}, W^{-2}|\lambda|^{-1}))^2 + O(1)) d\lambda \\
&\ll \int_{W^{-3}}^{W} W^2 d\lambda + \int_{W^{-3}}^{\infty} (W^{-2}|\lambda|^{-1})^2 d\lambda + \int_{-1/Q}^{1/Q} d\lambda \ll W^{-1},
\end{align*}

by the choices of $P$ and $Q$ in (2.1), and $W = U$ or $V$ as $i = 1, 2$. 

For all \( I_{ij}, 0 \leq i, j \leq 4 \), except \( I_{00} \) and \( I_{44} \),

\[
I_{4j} = \sum_{q \leq P} \frac{1}{\varphi(q)} \sum_{a=1}^{q} C^{4-j}(q, a) e \left( -\frac{an}{q} \right) \int_{-1/q}^{1/q} V_2^{4-j}(\lambda) \sum_{\chi \mod q} C(\chi, a) W_1(\chi, \lambda) \int_{-1/q}^{1/q} V_2^{4-j}(\lambda)
\]

and

\[
(4.22)
\]

\[
|I_{4j}| \ll L^c \sum_{r_1 \leq P} \sum_{\chi_1 \mod r_1}^{*} \max_{|\lambda| \leq 1/r_1} |W_1(\chi_1, \lambda)| \sum_{r_2 \leq P} \sum_{\chi_2 \mod r_2}^{*} \max_{|\lambda| \leq 1/r_2} |W_1(\chi_2, \lambda)|
\]

Now we use (4.21) 4 \(- j\) times, Lemma 4.2 \(j\) times, Lemma 4.4 twice, Lemma 4.2 once again, and Lemma 4.3 once to get

\[
(4.23)
\]

\[
|I_{4j}| \ll UV^4 L^{-A+(4+j)c} \ll UV^4 L^{-1}
\]

for large \( A > 0 \). We treat \(|I_{3j}|, |I_{2j}|, |I_{1j}| \) and \(|I_{0j}|\) by similar arguments:

\[
(4.24)
\]

\[
|I_{3j}| \ll L^c \sum_{r_1 \leq P} \sum_{\chi_1 \mod r_1}^{*} \max_{|\lambda| \leq 1/r_1} |W_1(\chi_1, \lambda)| \sum_{r_2 \leq P} \sum_{\chi_2 \mod r_2}^{*} \max_{|\lambda| \leq 1/r_2} |W_1(\chi_2, \lambda)|
\]

\[
(4.25)
\]

\[
|I_{3j}| \ll UV^4 L^{-A+(4+j)c} \ll UV^4 L^{-1}
\]
We need some more notation. Let
\begin{equation}
|I_{2j}| \ll L^c \sum_{r_1 \leq P \chi_1 \mod r_1} \sum_{|\lambda| \leq 1/r_1Q}^{*} \max_{|\lambda| \leq 1/r_1Q} |W_1(\chi_1, \lambda)| \leq \lambda/1 \leq r_2 Q 
\times \sum_{r_2 \leq P \chi_2 \mod r_2} \sum_{|\lambda| \leq 1/r_2Q}^{*} \max_{|\lambda| \leq 1/r_2Q} |W_1(\chi_2, \lambda)| \left( \int_{-1/Q}^{1/Q} |V_1(\lambda)|^2 d\lambda \right) 
\times \sum_{r_5 \leq P \chi_5 \mod r_5} \sum_{|\lambda| \leq 1/r_5Q}^{*} \max_{|\lambda| \leq 1/r_5Q} |W_2(\chi_5, \lambda)| \ldots 
\sum_{r_4+j \leq P \chi_4+j \mod r_4+j} \sum_{|\lambda| \leq 1/r_4+jQ}^{*} \max_{|\lambda| \leq 1/r_4+jQ} r_0^{-3+\varepsilon} |W_2(\chi_4+j, \lambda)| \left( \max_{|\lambda| \leq 1/Q} |V_2(\lambda)| \right)^{4-j} 
\ll UV^4L^{-1},
\end{equation}

\begin{equation}
|I_{1j}| \ll L^c \sum_{r_1 \leq P \chi_1 \mod r_1} \sum_{|\lambda| \leq 1/Q}^{*} \max_{|\lambda| \leq 1/Q} |W_1(\chi_1, \lambda)| \left( \max_{|\lambda| \leq 1/rQ} |V_1(\lambda)| \right)^{1/Q} 
\times \left( \int_{-1/Q}^{1/Q} |V_1(\lambda)|^2 d\lambda \right) \sum_{r_5 \leq P \chi_5 \mod r_5} \sum_{|\lambda| \leq 1/r_5Q}^{*} \max_{|\lambda| \leq 1/r_5Q} |W_2(\chi_5, \lambda)| \ldots 
\sum_{r_4+j \leq P \chi_4+j \mod r_4+j} \sum_{|\lambda| \leq 1/r_4+jQ}^{*} \max_{|\lambda| \leq 1/r_4+jQ} r_0^{-3+\varepsilon} |W_2(\chi_4+j, \lambda)| \left( \max_{|\lambda| \leq 1/Q} |V_2(\lambda)| \right)^{4-j} 
\ll UV^4L^{-1},
\end{equation}

\begin{equation}
|I_{0j}| \ll L^c \left( \max_{|\lambda| \leq 1/Q} |V_1(\lambda)| \right)^2 
\times \left( \int_{-1/Q}^{1/Q} |V_1(\lambda)|^2 d\lambda \right) \sum_{r_5 \leq P \chi_5 \mod r_5} \sum_{|\lambda| \leq 1/r_5Q}^{*} \max_{|\lambda| \leq 1/r_5Q} |W_2(\chi_5, \lambda)| \ldots 
\sum_{r_4+j \leq P \chi_4+j \mod r_4+j} \sum_{|\lambda| \leq 1/r_4+jQ}^{*} \max_{|\lambda| \leq 1/r_4+jQ} r_0^{-3+\varepsilon} |W_2(\chi_4+j, \lambda)| \left( \max_{|\lambda| \leq 1/Q} |V_2(\lambda)| \right)^{4-j} 
\ll UV^4L^{-1}
\end{equation}

for large $A > 0$.

Lemma 2.1 now follows from (4.13), (4.18), (4.19) and (4.23)–(4.27). ■

5. Estimates related to the singular series: the value of $C_1$. We need some more notation. Let $C(\chi, a), C(q, a), B(n, q; \chi_1, \ldots, \chi_8)$ and $B(n, q)$ be defined as in (4.1)–(4.3). If $\chi_1, \ldots, \chi_8$ are characters modulo $q$, then we write

\begin{equation}
A(n, q) := \frac{B(n, q)}{\varphi^4(q)}, \quad \mathcal{G}(n) := \sum_{q=1}^{\infty} A(n, q),
\end{equation}
Eight cubes of primes and powers of 2

so,

\[ S(n) := \sum_{q=1}^{\infty} \frac{1}{\varphi^8(q)} \sum_{a=1}^{q} \left( \sum_{h=1}^{q} e\left( \frac{ah^3}{q} \right) \right)^8 e\left( -\frac{an}{q} \right). \]

**Proof of Lemma 3.2** It has been shown in [7] that

\[ S(n) = \prod_{p} \left( 1 + \sum_{j=1}^{\gamma} A(n, p^j) \right), \]

where

\[ p^{\theta} \parallel k, \quad \gamma = \begin{cases} \theta + 2 & \text{if } p = 2, \ 2 \parallel k, \\ \theta + 1 & \text{otherwise}. \end{cases} \]

When \( k = 3 \), we have

(5.2) \[ S(n) = \{1 + A(n, 2)\} \{1 + A(n, 3) + A(n, 9)\} \prod_{p \geq 5} \{1 + A(n, p)\}. \]

Let \( A(n, q) \) be defined as in (5.1). We will compute \( A(n, q) \) for different \( q \).

For \( p = 2 \), one has

(5.3) \[ 1 + A(n, 2) = \begin{cases} 2, & n \equiv 0 \pmod{2}, \\ 0, & n \not\equiv 0 \pmod{2}, \end{cases} \]

by direct calculation.

For \( p = 3 \),

\[ C(3, a) = \sum_{h=1}^{2} e\left( \frac{ah^3}{3} \right) = e\left( \frac{a}{3} \right) + e\left( -\frac{a}{3} \right) = 2 \cos \frac{2\pi a}{3}, \]

so,

\[ A(n, 3) = \frac{1}{\varphi^8(3)} \sum_{a=1}^{2} \left( 2 \cos \frac{2\pi a}{3} \right)^8 e\left( -\frac{an}{3} \right) \]

\[ = \frac{1}{2^8} \left( e\left( -\frac{n}{3} \right) + e\left( -\frac{2n}{3} \right) \right) = \frac{1}{2^7} \cos \frac{2\pi n}{3}. \]

Thus,

(5.4) \[ A(n, 3) = \begin{cases} 1/2^7, & n \equiv 0 \pmod{3}, \\ -1/2^8, & n \not\equiv 0 \pmod{3}. \end{cases} \]

\[ C(9, a) = \sum_{h=1}^{9} e\left( \frac{ah^3}{9} \right) = 3 \left( e\left( \frac{a}{9} \right) + e\left( -\frac{a}{9} \right) \right) = 6 \cos \frac{2\pi a}{9}, \]
so,
\[
A(n, 9) = 1 = \frac{1}{\varphi^8(9)} \sum_{a=1}^{9} (\cos \frac{2\pi a}{9})^8 e\left(\frac{-an}{9}\right)
\]
\[
= \sum_{a=1}^{9} (\cos \frac{2\pi a}{9})^8 \cos \frac{2\pi an}{9}.
\]

For different \(n\), \(A(n, 9)\) will take five different values, and they satisfy
\[
A(n, 9) > -0.9609375.
\]

From (5.4) and (5.5) we get
\[
1 + A(n, 3) + A(n, 9) > 1 - 1/2^8 - 0.9609375 = 0.03515625.
\]

For \(p \geq 5\), if \(p \equiv 2 \pmod{3}\) and \((p, a) = 1\), we have \(C(p, a) = -1\), by Lemma 4.3 in Vaughan [29]. So,
\[
B(n, p) = \sum_{a=1}^{p-1} C^8(p, a)e\left(\frac{-an}{p}\right) = \sum_{a=1}^{p-1} e\left(\frac{-an}{p}\right)
\]
\[
= \begin{cases} 
  p - 1, & p \mid n, \\
  -1, & p \nmid n.
\end{cases}
\]

Thus,
\[
1 + A(n, p) = 1 + \frac{B(n, p)}{\varphi^8(p)} = \begin{cases} 
  1 + \frac{1}{(p - 1)^7}, & p \mid n, \\
  1 - \frac{1}{(p - 1)^8}, & p \nmid n.
\end{cases}
\]

Let \(p \equiv 1 \pmod{3}\) with \(p \geq 5\). First, when \(p = 7\),
\[
C(7, a) = \sum_{h=1}^{6} e\left(\frac{ah^3}{7}\right) = 3 \left( e\left(\frac{a}{7}\right) + e\left(-\frac{a}{7}\right) \right) = 6 \cos \frac{2\pi a}{7},
\]
so,
\[
A(n, 7) = \frac{1}{\varphi^8(7)} \sum_{a=1}^{6} \left( 6 \cos \frac{2\pi a}{7} \right)^8 e\left(\frac{-an}{7}\right)
\]
\[
= \sum_{a=1}^{6} \left( \cos \frac{2\pi a}{7} \right)^8 \cos \frac{2\pi an}{7}.
\]

For different \(n\), \(A(n, 7)\) will take four different values, and they satisfy
\[
A(n, 7) > -0.75390625.
\]
Eight cubes of primes and powers of 2

Thus

\[(5.8) \quad 1 + A(n, 7) > 1 - 0.75390625 = 0.24609375.\]

For \( p \geq 13 \) and \( p \equiv 1 \pmod{3} \), noting the elementary estimate (by Lemma 4.3 of [29])

\[|C(p, a)| \leq 2\sqrt{p} + 1,\]

we get

\[|B(n, p)| = \left| \sum_{a=1}^{p-1} C^8(p, a) e\left(-\frac{an}{p}\right) \right| \leq (2\sqrt{p} + 1)^8(p - 1).\]

Thus

\[(5.9) \quad 1 + A(n, p) > 1 - \frac{(2\sqrt{p} + 1)^8}{(p - 1)^7}.\]

Hence

\[(5.10) \quad \prod_{p \geq 5} \{1 + A(n, p)\} \geq \{1 + A(n, 7)\} \prod_{p \geq 13, p \equiv 1 \pmod{3}} \left(1 - \frac{(2\sqrt{p} + 1)^8}{(p - 1)^7}\right)\]

\[\times \prod_{p \geq 5, p \equiv 2 \pmod{3}, p \nmid n} \left(1 + \frac{1}{(p-1)^7}\right) \prod_{p \geq 5, p \equiv 2 \pmod{3}, p \nmid n} \left(1 - \frac{1}{(p-1)^8}\right)\]

\[\geq \{1 + A(n, 7)\} \prod_{p \equiv 1 \pmod{3}} \left(1 - \frac{(2\sqrt{p} + 1)^8}{(p - 1)^7}\right)\]

\[\times \prod_{p \geq 5, p \equiv 2 \pmod{3}} \left(1 - \frac{1}{(p-1)^2}\right).\]

To estimate the products above, we apply the elementary inequality

\[(5.11) \quad \frac{(2\sqrt{p} + 1)^8}{(p - 1)^7} < \frac{1}{(p-1)^2} \quad \text{for} \ p \geq 324.\]

Thus we have

\[(5.12) \quad \prod_{p \geq 5} \{1 + A(n, p)\} \geq \{1 + A(n, 7)\} \prod_{13 \leq p \leq 323, p \equiv 1 \pmod{3}} \left(1 - \frac{(2\sqrt{p} + 1)^8}{(p - 1)^7}\right)\]

\[\times \prod_{p \equiv 1 \pmod{3}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p \equiv 2 \pmod{3}} \left(1 - \frac{1}{(p-1)^2}\right)\]
\[
\{1 + A(n, 7)\} \prod_{13 \leq p \leq 323 \atop p \equiv 1 \pmod{3}} \left(1 - \frac{(2\sqrt{p} + 1)^8}{(p - 1)^7}\right)
\]
\[\times \prod_{p=3,7} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} \prod_{13 \leq p \leq 323 \atop p \equiv 1 \pmod{3}} \left(1 - \frac{1}{(p-1)^2}\right)^{-1}\]
\[\times \prod_{p \geq 3} \left(1 - \frac{1}{(p-1)^2}\right)\]
\[\geq 0.24609375 \cdot \frac{4}{3} \cdot \frac{36}{35} \cdot 0.35608989538 \cdot 0.6601\]
\[\geq 0.079331042229,
\]
where we have used \(\prod_{p \geq 3} (1 - (p-1)^{-2}) = 0.6601\ldots\) (see [5]).

This in combination with (5.2), (5.3), (5.6), (5.12) ensures that
\[\mathcal{G}(n) > 0.00557795824,
\]
when \(n \equiv 0 \pmod{2}\). The proof is complete. \(\blacksquare\)

6. Upper bounds of \(J_i(g)\) and \(K(g)\): proof of Lemmas 4.2–4.4. Lemmas 4.2 and 4.3 are similar to those in Section 5 in Liu and Liu [19], and the choices of \(P, Q\) defined in (2.1) are acceptable in these lemmas. A similar proof can also be found in [24], so we omit the details. Here we only give the proof of Lemma 4.4.

In the proof, we need a mean value theorem of Choi and Kumchev [2]:

**Lemma 6.1.** Let \(l\) be a positive integer, \(R, T, X \geq 1\) and \(\kappa = 1/\log X\). Then there is an absolute positive constant \(c\) such that
\[
\sum_{r \sim R} \sum_{\chi \mod r} \frac{A(n)\chi(n)}{n^{\kappa+i\tau}} | d\tau \ll (l^{-1} R^2 T X^{11/20} + X)(\log R T X)^c,
\]
where the implied constant is absolute.

In order to use Lemma 6.1 effectively, we need a lemma of [15]:

**Lemma 6.2.** Let \(\chi\) be a Dirichlet character modulo \(r\). Let \(2 \leq X < Y \leq 2X\), \(T_0 = (\log(Y/X))^{-1}\), \(T = X^4\) and \(\kappa = 1/\log X\). Define
\[F(s, \chi) = \sum_{X \leq n \leq 2X} A(n)\chi(n)n^{-s}.
\]
Then
\begin{equation}
\sum_{X \leq n \leq 2X} \Lambda(n) \chi(n) \ll \log(Y/X) \int_{|\tau| \leq T_0} |F(\kappa + i\tau, \chi)| \, d\tau
+ \int_{T_0 < |\tau| \leq T} \frac{|F(\kappa + i\tau, \chi)|}{|\tau|} \, d\tau + 1.
\end{equation}

The implied constants are absolute.

**Proof of Lemma 4.4**  Introduce
\[ \hat{W}_1(\chi, \lambda) := \sum_{m \sim U} \Lambda(m) \chi(m) e(\lambda m^3) - \delta_\chi \sum_{m \sim U} e(\lambda m^3). \]

When we replace \( W_1(\chi, \lambda) \) by \( \hat{W}_1(\chi, \lambda) \), the error is
\[ \hat{W}_1(\chi, \lambda) - W_1(\chi, \lambda) \ll \frac{U}{2}. \]

Thus the resulting error of \( K(g) \) is
\begin{equation}
\ll [g, r]^{-3+\varepsilon} \frac{r^{1/2} U^{1/2}}{Q^{1/2}} \ll g^{-3+\varepsilon} \frac{U^{1/2}}{Q^{1/2}} \sum_{l \leq P} \frac{l^{3-\varepsilon}}{l|g} \sum_{r \leq P} r^{-5/2+\varepsilon}
\ll g^{-3+\varepsilon} U^{-1/2} L^c.
\end{equation}

Here, in the last step, we need the definition of \( P \) and \( Q \) in (2.1).

Thus to establish Lemma 4.4, it suffices to show that
\begin{equation}
\sum_{r \sim R} [g, r]^{-3+\varepsilon} \sum_{\chi \mod r}^* \left( \int_{-1/rQ}^{1/rQ} |\hat{W}_1(\chi, \lambda)|^2 \, d\lambda \right)^{1/2} \ll g^{-3+\varepsilon} U^{-1/2} L^c
\end{equation}
for any \( R \leq P \) and some \( c > 0 \).

By Gallagher’s lemma ([4, Lemma 1]), we have
\begin{equation}
\int_{-1/rQ}^{1/rQ} |\hat{W}_1(\chi, \lambda)|^2 \, d\lambda \ll \left( \frac{1}{RQ} \right)^2 \int_{-\infty}^\infty \left| \sum_{m \sim U} (\Lambda(m) \chi(m) - \delta_\chi) \right|^2 \, dv
\ll \left( \frac{1}{RQ} \right)^2 (2U)^3 \sum_{U^{3-rQ} < m \leq Y} \left| \sum_{X < m \leq Y} (\Lambda(m) \chi(m) - \delta_\chi) \right|^2 \, dv,
\end{equation}
where
\[ X := \max\{v^{1/3}, U\}, \quad Y := \min\{(v + rQ)^{1/3}, 2U\}. \]
If $R = 1$, we have
\[
\sum_{X < n \leq Y} (A(m) \chi(m) - \delta \chi) = \sum_{X < m \leq Y} (A(m) - 1)
\]
\[
\ll (Y - X)L \ll U^{-2}QL.
\]
This contributes to (6.3) the quantity
\[
g^{-3+\varepsilon} \left( \frac{1}{Q^2} \cdot U^3 \cdot U^{-4}Q^2 \right)^{1/2} L \ll g^{-3+\varepsilon}U^{-1/2}L,
\]
which is acceptable.

For $R \geq 2$ and $r \sim R$, we have $\delta \chi = 0$. Thus, we can apply (6.1) to obtain
\[
1/r Q \ll \max_{|\tau| \leq T_0} |F(\kappa + i\tau, \chi)| d\tau \ll \frac{U^3}{(RQ)^2} \left( \sum_{T_0 < |\tau| \leq T} \frac{|F(\kappa + i\tau, \chi)|}{|\tau|} d\tau \right)^2 + \frac{U^3}{(RQ)^2},
\]
since $T_0^{-1} = \log(Y/X) \approx RQ/U^3$.

Therefore, the contribution of the first term of (6.7) to the left-hand side of (6.3) is
\[
\ll g^{-3+\varepsilon}U^{-3/2} \sum_{l \leq 2R} \left( \frac{R}{l} \right)^{-3+\varepsilon} \left( l^{-1} R^2 T'_0 U^{11/20} + U \right) L^c
\]
\[
\ll g^{-3+\varepsilon}U^{-3/2} \left( R^{-1+\varepsilon} \sum_{l \leq 2R} l^{2-\varepsilon} T'_0 U^{11/20} + U \right) L^c
\]
\[
\ll g^{-3+\varepsilon}U^{-1/2} L^c,
\]
which is acceptable by the definition of $Q$.

Set
\[
M(l, R, T', U) := \sum_{r \sim R, \chi \mod r} \sum_{T'}^{2T'} |F(\kappa + i\tau, \chi)| d\tau.
\]
The contribution of the second term of (6.7) to the left-hand side of (6.3) is
\[
\ll g^{-3+\varepsilon}U^{3/2}(RQ)^{-1} \sum_{l \leq 2R} \left( \frac{R}{l} \right)^{-3+\varepsilon} \max_{T'_0 \leq T'' \leq T} T''^{-1} M(l, R, T', U)
\]
\[
\ll g^{-3+\varepsilon}U^{3/2}(RQ)^{-1} \sum_{l \leq 2R} \left( \frac{R}{l} \right)^{-3+\varepsilon} \left( l^{-1} R^2 U^{11/20} + T_0^{-1} U \right) L^c
\]
\[
\ll g^{-3+\varepsilon}U^{-1/2} L^c,
\]
which is acceptable by the definition of $Q$. 
Finally, the contribution of the last term of (6.7) to the left-hand side of (6.3) is
\[ g^{-3+\varepsilon} U^{3/2} (RQ)^{-1} \sum_{l \leq 2R} \left( \frac{R}{l} \right)^{-3+\varepsilon} \ll g^{-3+\varepsilon} U^{-1/2} L^c. \]

Now Lemma \ref{lem:6.4} follows from (6.2), (6.3), (6.6) and (6.8)–(6.10).

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