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On a Diophantine problem with two primes and s powers of two

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1. Introduction. In this paper we are interested in the numbers of the form

$$\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 2^{m_1} + \cdots + \mu_s 2^{m_s}, \tag{1.1}$$

where p_1, p_2 are prime numbers, $m_1, \ldots, m_s$ are positive integers, and the coefficients λ_1, λ_2 and μ_1, ..., μ_s are real numbers satisfying suitable relations.

This is clearly a variation of the so-called Goldbach–Linnik problem, i.e. to prove that every sufficiently large even integer is a sum of two primes and s powers of two, where s is a fixed integer. Concerning this problem the first result was proved by Linnik himself [14, 15] who remarked that a suitable s exists but he gave no explicit estimate of its size. Other results were proved by Gallagher [6], Liu–Liu–Wang [16, 17, 18], Wang [29] and Li [12, 13]. Now the best conditional result is due to Pintz–Ruzsa [21] and Heath-Brown–Puchta [9] ($s = 7$ suffices under the assumption of the Generalized Riemann Hypothesis), while, unconditionally, it is due to Heath-Brown–Puchta [9] ($s = 13$ suffices). Elsholtz, in unpublished work, improved it to $s = 12$. We should also remark that Pintz–Ruzsa announced a proof for $s = 8$ in their paper [22] which is as yet unpublished. Looking for the size of the exceptional set of the Goldbach problem we recall the fundamental paper by Montgomery–Vaughan [19] in which they showed that the number of even integers up to X that are not the sum of two primes is $\ll X^{1-\delta}$. Pintz recently announced that $\delta = 1/3$ is admissible in the previous estimate. Concerning the exceptional set for the Goldbach–Linnik problem, the authors of this paper in a joint work with Pintz [11] proved that for every

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$s \geq 1$, there are $\ll X^{3/5}(\log X)^{10}$ even integers in $[1, X]$ that are not the sum of two primes and s powers of two. This obviously corresponds to the case $\lambda_1 = \lambda_2 = \mu_1 = \cdots = \mu_s = 1$.

In Diophantine approximation several results were proved concerning the linear forms with primes that, in some sense, can be considered as the real analogues of the binary and ternary Goldbach problems. On this topic we recall the papers by Vaughan [26, 27, 28], Harman [8], Brüdern–Cook–Perelli [2], and Cook–Harman [4].

Concerning the problem in (1.1), we can consider it as a real analogue of the Goldbach–Linnik problem. Since the quality of our result depends on rational approximations to λ_1/λ_2, we need to introduce the set of irrational numbers with a suitable Diophantine property. More precisely, we let $\mathfrak{R}$ denote the set of irrational numbers ξ such that the denominators q_m of the convergents to ξ, arranged in increasing order of magnitude, satisfy $q_{m+1} \ll q_m^{1+\varepsilon}$. By Roth's Theorem, all algebraic numbers belong to $\mathfrak{R}$, and almost all real numbers, in the sense of the Lebesgue measure, also belong to $\mathfrak{R}$. We denote by $\mathfrak{R}'$ the set of irrational numbers that do not belong to $\mathfrak{R}$.

We have the following

Theorem. *Suppose that λ_1, λ_2 are real numbers such that λ_1/λ_2 is negative and irrational with $\lambda_1 > 1$, $\lambda_2 < -1$ and $|\lambda_1/\lambda_2| \geq 1$. Further suppose that $\mu_1, \dots, \mu_s$ are nonzero real numbers such that $\lambda_i/\mu_i \in \mathbb{Q}$ for $i \in \{1, 2\}$, and denote by a_i/q_i their reduced representations as rational numbers. Let moreover η be a sufficiently small positive constant such that $\eta < \min(\lambda_1/a_1; |\lambda_2/a_2|)$. Finally, for $\lambda_1/\lambda_2 \in \mathfrak{R}'$, let*

$$s_0 = 2 + \left\lceil \frac{\log(C(q_1, q_2)\lambda_1) - \log \eta}{-\log(0.91237810306)} \right\rceil, \tag{1.2}$$

while, for $\lambda_1/\lambda_2 \in \mathfrak{R}$, let

$$s_0 = 2 + \left\lceil \frac{\log(C(q_1, q_2)\lambda_1) - \log \eta}{-\log(0.83372131685)} \right\rceil, \tag{1.3}$$

where

$$C(q_1, q_2) = (\log 2 + C \cdot \mathfrak{S}'(q_1))^{1/2}(\log 2 + C \cdot \mathfrak{S}'(q_2))^{1/2} \tag{1.4}$$

with $C = 10.0219168340$ and

$$\mathfrak{S}'(n) = \prod_{\substack{p \mid n \\ p > 2}} \frac{p-1}{p-2}. \tag{1.5}$$

Then for every real number γ and every integer $s \geq s_0$ the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 2^{m_1} + \cdots + \mu_s 2^{m_s} + \gamma| < \eta \tag{1.6}$$

has infinitely many solutions in primes p_1, p_2 *and positive integers* $m_1, \dots, m_s$.

Using the notation $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$, $\boldsymbol{\mu} = (\mu_1, \mu_2)$, we notice that inequality (1.6) is homogeneous in the quantities $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, γ and η, and by a suitable rescaling we can take, say, λ_2 as close to -1 as we please. The hypothesis $\lambda_2/\mu_2 \in \mathbb{Q}$ (which is needed in the proof of Lemma 4) prevents us from simply taking $\lambda_2 = -1$.

The only result on this problem we know of is by Parsell [20]; our values in (1.2)–(1.3) improve Parsell's

$$s_0 = 2 + \left\lceil \frac{\log(2C_1(q_1, q_2)|\lambda_1\lambda_2|) - \log \eta}{-\log(0.954)} \right\rceil, \tag{1.7}$$

where

$$C_1(q_1, q_2) = 25(\log 2q_1)^{1/2}(\log 2q_2)^{1/2}. \tag{1.8}$$

Checking the proof in [20] one can see that (1.8) is in fact

$$C_1(q_1, q_2, \epsilon) = (1 + C_1 \cdot \mathfrak{S}'(q_1))^{1/2}(1 + C_1 \cdot \mathfrak{S}'(q_2))^{1/2} + \epsilon, \tag{1.9}$$

and $C_1 = 11.4525218267$. Comparing only denominators in (1.2)–(1.3) with the denominator in (1.7), we see that our gain is about 50% when $\lambda_1/\lambda_2 \in \mathfrak{R}'$ and about 75% when $\lambda_1/\lambda_2 \in \mathfrak{R}$. In practice, the following examples show that gains are actually slightly larger. For instance, taking $\lambda_1 = \sqrt{3} = \mu_1^{-1}$, $\lambda_2 = -\sqrt{2} = \mu_2^{-1}$ and $\eta = 1$, we get $s_0 = 19$, while for $\lambda_1 = \pi = \mu_1^{-1}$, $\lambda_2 = -\sqrt{2} = \mu_2^{-1}$ and $\eta = 1$, we get $s_0 = 41$. Parsell's estimates (1.7) and (1.9) respectively give $s_0 = 90$ and $s_0 = 102$.

Moreover we remark that the work of Rosser–Schoenfeld [23] on $n/\varphi(n)$ (see Lemma 2 below) gives for $\mathfrak{S}'(q)$ a sharper estimate than $2\log(2q)$, used in (1.8), for large values of q.

With respect to [20], our main gain comes from enlarging the size of the major arc since this lets us use sharper estimates on the minor arc. In particular, on the major arc we replace the technique used in [20] with a well-known argument involving the Selberg integral; this also simplifies the actual work to get a "good" major arc contribution.

On the minor arc we use Brüdern–Cook–Perelli's [2] and Cook–Harman's [4] technique to deal with the exponential sum on primes ($S(\alpha)$), while in order to work with the exponential sum over powers of two ($G(\alpha)$), we apply Pintz–Ruzsa's [21] algorithm to estimate the measure of the subset of the minor arc on which $|G(\alpha)|$ is "large". These two ingredients lead to a sharper estimate on the minor arc and let us improve the size of the denominators in (1.2)–(1.3). It is in this step that we have to distinguish between whether λ_1/λ_2 belongs to $\mathfrak{R}$ or to $\mathfrak{R}'$; this leads to two different estimates for the minor arc and, *a fortiori*, using Pintz–Ruzsa's algorithm (see Lemma 5), to two different constants in (3.10)–(3.11) and (1.2)–(1.3).

A second, less important, gain arises from our Lemma 4 which improves the values in (1.4) compared with the ones in (1.9) (obtained in [20, Lemma 3]). Such an improvement comes from using the Prime Number Theorem (to get $\log 2$ instead of 1) and Khalfalah–Pintz's [10] computational estimates for the number of representations of an integer as a difference of powers of two (see Lemma 1).

Finally we remark that assuming a suitable form of the twin-prime conjecture, i.e. $B = 1 + \epsilon$ in Lemma 3, we find that (1.4) holds with $C = 2.5585042083$.

As a consequence of the Theorem we have

Corollary. *Suppose that λ_1, λ_2 are real numbers such that λ_1/λ_2 is negative and irrational. Further suppose $\mu_1, \dots, \mu_s$ are nonzero real numbers such that $\lambda_i/\mu_i \in \mathbb{Q}$ for $i \in \{1, 2\}$, and denote by a_i/q_i their reduced representations as rational numbers. Let moreover η be a sufficiently small positive constant such that $\eta < \min(|\lambda_1/a_1|; |\lambda_2/a_2|)$ and $\tau \geq \eta > 0$. Finally let $s_0 = s_0(\boldsymbol{\lambda}, \boldsymbol{\mu}, \eta)$ be as defined in* (1.2)–(1.3)*. Then for every real number γ and every integer $s \geq s_0$ the inequality*

$$|\lambda_1 p_1 + \lambda_2 p_2 + \mu_1 2^{m_1} + \dots + \mu_s 2^{m_s} + \gamma| < \tau \tag{1.10}$$

has infinitely many solutions in primes p_1, p_2 and positive integers $m_1, \dots, m_s$.

This Corollary immediately follows from the Theorem, since multiplying both sides of (1.10) by a suitable constant, we can always reduce ourselves to the case $\lambda_1 > 1$, $\lambda_2 < -1$ and $|\lambda_1/\lambda_2| \geq 1$. Hence the Theorem ensures that (1.6) has infinitely many solutions and the Corollary immediately follows from the condition $\tau \geq \eta$.

We finally remark that the condition about the rationality of the two ratios λ_i/μ_i, $i = 1, 2$, which, at first sight, could appear "weird", is in fact quite natural in the sense that otherwise the numbers $\lambda x + \mu y$, $x, y \in \mathbb{Z}$, are dense in $\mathbb{R}$ by Kronecker's Theorem (see also the remark after Lemma 4).

2. Definitions. Let ϵ be a sufficiently small positive constant, X be a large parameter, $M = |\mu_1| + \dots + |\mu_s|$ and $L = \log_2(\epsilon X/(2M))$, where $\log_2 v$ is the base 2 logarithm of v. We will use the Davenport–Heilbronn variation of the Hardy–Littlewood method to count the number $\mathfrak{N}(X)$ of solutions of the inequality (1.6) with $\epsilon X \leq p_1, p_2 \leq X$ and $1 \leq m_1, \dots, m_s \leq L$. Let now $e(u) = \exp(2\pi i u)$ and

$$S(\alpha) = \sum_{\epsilon X \leq p \leq X} \log p \, e(p\alpha) \quad \text{and} \quad G(\alpha) = \sum_{1 \leq m \leq L} e(2^m \alpha).$$

For $\alpha \neq 0$, we also define

$$K(\alpha,\eta) = \left(\frac{\sin \pi\eta\alpha}{\pi\alpha}\right)^2$$

and hence both

$$\widehat{K}(t,\eta) = \int_{\mathbb{R}} K(\alpha,\eta)e(t\alpha)\,d\alpha = \max(0;\eta - |t|) \tag{2.1}$$

and

$$K(\alpha,\eta) \ll \min(\eta^2;\alpha^{-2}) \tag{2.2}$$

are well-known facts. Letting

$$I(X;\mathbb{R}) = \int_{\mathbb{R}} S(\lambda_1\alpha)S(\lambda_2\alpha)G(\mu_1\alpha)\cdots G(\mu_s\alpha)e(\gamma\alpha)K(\alpha,\eta)\,d\alpha,$$

it follows from (2.1) that

$$I(X;\mathbb{R}) \ll \eta \log^2 X \cdot \mathfrak{N}(X).$$

We will prove that

$$I(X;\mathbb{R}) \gg_{s,\boldsymbol{\lambda},\epsilon} \eta^2 X(\log X)^s, \tag{2.3}$$

thus obtaining

$$\mathfrak{N}(X) \gg_{s,\boldsymbol{\lambda},\epsilon} \eta X(\log X)^{s-2}$$

and hence the Theorem follows. To prove (2.3) we first dissect the real line into the major, minor and trivial arcs, by choosing $P = X^{1/3}$ and letting

$$\mathfrak{M} = \{\alpha \in \mathbb{R} : |\alpha| \leq P/X\}, \quad \mathfrak{m} = \{\alpha \in \mathbb{R} : P/X < |\alpha| \leq L^2\}, \tag{2.4}$$

and $\mathfrak{t} = \mathbb{R} \setminus (\mathfrak{M} \cup \mathfrak{m})$. Accordingly, we write

$$I(X;\mathbb{R}) = I(X;\mathfrak{M}) + I(X;\mathfrak{m}) + I(X;\mathfrak{t}). \tag{2.5}$$

We will prove that the inequalities

$$I(X;\mathfrak{M}) \geq c_1\eta^2 XL^s, \tag{2.6}$$

$$|I(X;\mathfrak{t})| = o(XL^s) \tag{2.7}$$

hold for all sufficiently large X, and

$$|I(X;\mathfrak{m})| \leq c_2(s)\eta XL^s, \tag{2.8}$$

where $c_2(s) > 0$ depends on s, $c_2(s) \to 0$ as $s \to +\infty$, and $c_1 = c_1(\epsilon,\boldsymbol{\lambda}) > 0$ is a constant such that

$$c_1\eta - c_2(s) \geq c_3\eta \tag{2.9}$$

for some absolute positive constant c_3 and $s \geq s_0$. Inserting (2.6)–(2.9) into (2.5), we finally conclude that (2.3) holds, thus proving the Theorem.

3. Lemmas. Let $1 \le n \le (1-\epsilon)X/2$ be an integer and p, p' two prime numbers. We define the twin prime counting function as follows:

$$Z(X;2n) = \sum_{\epsilon X \le p \le X} \sum_{\substack{p' \le X \\ p'-p=2n}} \log p \log p'. \tag{3.1}$$

Moreover we denote by $\mathfrak{S}(n)$ the singular series and set $\mathfrak{S}(n) = 2c_0\mathfrak{S}'(n)$ where $\mathfrak{S}'(n)$ is defined in (1.5) and

$$c_0 = \prod_{p>2} \Big(1 - \frac{1}{(p-1)^2}\Big). \tag{3.2}$$

Notice that $\mathfrak{S}'(n)$ is a multiplicative function. According to Gourdon–Sebah [7], we also have $0.66016181584 < c_0 < 0.66016181585$.

Let further $k \ge 1$ be an integer and $r_{k,k}(m)$ be the number of representations of an integer m as $\sum_{i=1}^{k} 2^{u_i} - \sum_{i=1}^{k} 2^{v_i}$, where $1 \le u_i, v_i \le L$ are integers, so that $r_{k,k}(m) = 0$ for sufficiently large $|m|$. Define

$$S(k,L) = \sum_{m \in \mathbb{Z}\setminus\{0\}} r_{k,k}(m)\mathfrak{S}(m).$$

The first lemma is about the behaviour of $S(k,L)$ for sufficiently large X.

Lemma 1 (Khalfalah–Pintz [10, Theorem 2]). *For any given $k \ge 1$, there exists $A(k) \in \mathbb{R}$ such that*

$$\lim_{L \to +\infty} \Big(\frac{S(k,L)}{2L^{2k}} - 1\Big) = A(k).$$

Moreover they also proved numerical estimates for $A(k)$ when $1 \le k \le 7$. We will just need

$$A(1) < 0.2792521041. \tag{3.3}$$

The second lemma is an upper bound for the multiplicative part of the singular series.

Lemma 2. *For $n \in \mathbb{N}$, $n \ge 3$, we have*

$$\mathfrak{S}'(n) < \frac{n}{c_0\varphi(n)} < \frac{e^{\gamma}\log\log n}{c_0} + \frac{2.50637}{c_0 \cdot \log\log n},$$

where $\gamma = 0.5772156649\ldots$ is the Euler constant.

Proof. Let $n \ge 3$. The first estimate follows immediately after remarking that

$$\mathfrak{S}'(n) = \prod_{\substack{p \mid n \\ p>2}} \frac{(p-1)^2}{p(p-2)} \prod_{\substack{p \mid n \\ p>2}} \frac{p}{p-1} < \prod_{p>2} \frac{(p-1)^2}{p(p-2)} \prod_{p \mid n} \frac{p}{p-1} = \frac{1}{c_0}\,\frac{n}{\varphi(n)}.$$

The second estimate is a direct application of Theorem 15 of Rosser and Schoenfeld [23]. ∎

Letting $f(1) = f(2) = 1$ and $f(n) = n/(c_0\varphi(n))$ for $n \geq 3$, we can say that the inequality $\mathfrak{S}'(n) \leq f(n)$ is sharper than Parsell's estimate $\mathfrak{S}'(n) \leq 2\log(2n)$ (see page 7 of [20]) for every $n \geq 1$. Since it is clear that computing the exact value of $f(n)$ for large values of n is not easy (it requires the knowledge of every prime factor of n), we also remark that the second estimate in Lemma 2 leads to a sharper bound than $\mathfrak{S}'(n) \leq 2\log(2n)$ for every $n \geq 14$.

The next lemma is a famous result of Bombieri and Davenport.

Lemma 3 (Theorem 2 of Bombieri–Davenport [1]). *There exists a positive constant B such that, for every positive integer n,*

$$Z(X; 2n) < B\mathfrak{S}(n)X,$$

where $Z(X; 2n)$ and $\mathfrak{S}(n)$ are defined in (1.5) *and* (3.1)–(3.2), *provided that X is sufficiently large.*

Chen [3] proved that $B = 3.9171$ can be used in Lemma 3. The assumption of a suitable form of the twin prime conjecture, i.e. $Z(X; 2n) \sim \mathfrak{S}(n)X$ as $X \to +\infty$, implies that in this case we can take $B = 1 + \epsilon$ for every positive ϵ.

Now we state some lemmas we need to estimate $I(X; \mathfrak{m})$. The first one is

Lemma 4. *Let X be a sufficiently large parameter and let $\lambda, \mu \neq 0$ be two real numbers such that $\lambda/\mu \in \mathbb{Q}$. Let $a, q \in \mathbb{Z}\setminus\{0\}$ with $q > 0$, $(a, q) = 1$ be such that $\lambda/\mu = a/q$. Let further $0 < \eta < |\lambda/a|$. Then*

$$\int_{\mathbb{R}} |S(\lambda\alpha)G(\mu\alpha)|^2 K(\alpha, \eta)\, d\alpha < \eta X L^2((1-\epsilon)\log 2 + C \cdot \mathfrak{S}'(q)) + \mathcal{O}_{M,\epsilon}(\eta X L),$$

where $C = 10.0219168340$.

Proof. First of all we remark that the constant C is in fact $2B(1+A(1))$, where $B = 3.9171$ is the constant in Lemma 3 and $A(1)$ is estimated in (3.3). This should be compared with the value $C_1 = 11.4525218267$ obtained in [20]. Assuming the twin prime conjecture in Lemma 3 and taking $B = 1 + 10^{-20}$, we get $C = 2.5585042083$. Letting now

$$I = \int_{\mathbb{R}} |S(\lambda\alpha)G(\mu\alpha)|^2 K(\alpha, \eta)\, d\alpha,$$

by (2.1) we immediately have

$$I = \sum_{\epsilon X \leq p_1, p_2 \leq X} \sum_{1 \leq m_1, m_2 \leq L} \log p_1 \log p_2 \max(0; \eta - |\lambda(p_1 - p_2) + \mu(2^{m_1} - 2^{m_2})|). \tag{3.4}$$

Let $\delta = \lambda(p_1 - p_2) + \mu(2^{m_1} - 2^{m_2})$. For a sufficiently small $\eta > 0$, we claim that

$$|\delta| < \eta \quad \text{is equivalent to} \quad \delta = 0. \tag{3.5}$$

Recall our hypothesis on a and q, and assume that $\delta \neq 0$ in (3.5). For $\eta < |\lambda/a|$ this leads to a contradiction. In fact we have

$$\frac{1}{|a|} > \frac{\eta}{|\lambda|} > \left|(p_1 - p_2) + \frac{q}{a}(2^{m_1} - 2^{m_2})\right| = \left|\frac{a(p_1 - p_2) + q(2^{m_1} - 2^{m_2})}{a}\right| \geq \frac{1}{|a|},$$

since $a(p_1 - p_2) + q(2^{m_1} - 2^{m_2}) \neq 0$ is a linear integral combination. Invoking (3.5) in (3.4), for $\eta < |\lambda/a|$ we can write

$$I = \eta \sum_{\substack{\epsilon X \leq p_1, p_2 \leq X \\ \lambda(p_1 - p_2) + \mu(2^{m_1} - 2^{m_2}) = 0}} \sum_{1 \leq m_1, m_2 \leq L} \log p_1 \log p_2. \tag{3.6}$$

The diagonal contribution in (3.6) is equal to

$$\eta \sum_{\epsilon X \leq p \leq X} \log^2 p \sum_{1 \leq m \leq L} 1 = \eta X L^2 (1 - \epsilon) \log 2 + \mathcal{O}_{M,\epsilon}(\eta X L) \tag{3.7}$$

where we used the Prime Number Theorem instead of trivially estimating the contribution of $\log p_i$ as in [20].

Now we have to estimate the contribution I' of the nondiagonal solutions of $\delta = 0$ and we will achieve this by connecting I' with the singular series of the twin prime problem. Recalling that $\lambda/\mu = a/q \neq 0$, $(a, q) = 1$, by Lemma 3 and the fact that $Z(X; (q/a)(2^{m_2} - 2^{m_1})) \neq 0$ if and only if $a \mid (2^{m_2} - 2^{m_1})$, we have, since $\mathfrak{S}(v) = \mathfrak{S}(2^u v)$ for all $u, v \in \mathbb{N}$, $u \geq 1$,

$$\begin{aligned} I' &\leq 2\eta \sum_{1 \leq m_1 < m_2 \leq L} Z\Big(X; \frac{q}{a}(2^{m_2} - 2^{m_1})\Big) \\ &< 2BX\eta \sum_{1 \leq m_1 < m_2 \leq L} \mathfrak{S}\Big(\frac{q}{a}(2^{m_2} - 2^{m_1})\Big). \end{aligned} \tag{3.8}$$

Using the multiplicativity of $\mathfrak{S}'(n)$ (defined in (1.5)), we get

$$\mathfrak{S}'\Big(\frac{q}{a}(2^{m_2} - 2^{m_1})\Big) \leq \mathfrak{S}'(q)\mathfrak{S}'\Big(\frac{2^{m_2} - 2^{m_1}}{a}\Big) \leq \mathfrak{S}'(q)\mathfrak{S}'(2^{m_2} - 2^{m_1}),$$

and so, by Lemma 1, (3.3) and (3.8), we can write, for every sufficiently large X,

$$\begin{aligned} I' &\leq 2BX\eta\mathfrak{S}'(q) \sum_{1 \leq m_1 < m_2 \leq L} \mathfrak{S}(2^{m_2} - 2^{m_1}) = BX\eta\mathfrak{S}'(q) S(1, L) \\ &< 2B(1 + A(1))\mathfrak{S}'(q) X\eta L^2. \end{aligned} \tag{3.9}$$

Hence, by (3.6)–(3.7) and (3.9), we finally get

$$I < \eta X L^2((1-\epsilon)\log 2 + 2B(1+A(1))\mathfrak{S}'(q)) + \mathcal{O}_{M,\epsilon}(\eta X L),$$

proving Lemma 4. ∎

We remark that if in Lemma 4 we consider also the case $\lambda/\mu \in \mathbb{R}\setminus\mathbb{Q}$, we can just find $\eta = \eta(X) \to 0$ as $X \to +\infty$ and this implies that $s_0 \approx |\log \eta| \to +\infty$ (see (1.2)–(1.3) for the precise definition of s_0). This essentially depends on the fact that, for $\lambda/\mu \in \mathbb{R}\setminus\mathbb{Q}$ and $m, n \in \mathbb{Z}$, there is no function $f(X)$ such that $|\lambda m + \mu n| \ge f(X)$ and $f(X) \to c > 0$ as $X \to +\infty$ since the set of values of $\lambda m + \mu n$ is dense in $\mathbb{R}$. A different, but related, way to see this phenomenon is to remark that the inequality $|\alpha n + m| < \eta$ is equivalent to either $\|n\alpha\| < \eta$ or $\|n\alpha\| > 1-\eta$, where $\|u\|$ is the distance of u from the nearest integer. When α is irrational, this has $\sim 2\eta X$ solutions with $n \le X$, since the sequence $\|n\alpha\|$ is uniformly distributed modulo 1.

To estimate the contribution of $G(\alpha)$ on the minor arc we use Pintz–Ruzsa's method as developed in [21, §§3–7].

Lemma 5 (Pintz–Ruzsa [21, §7]). *Let $0 < c < 1$. Then there exists $\nu = \nu(c) \in (0,1)$ such that*

$$|E(\nu)| := |\{\alpha \in (0,1) : |G(\alpha)| > \nu L\}| \ll_{M,\epsilon} X^{-c}.$$

To obtain explicit values for ν we have to modify the Pintz–Ruzsa algorithm since in this application the estimate has to be performed for a different choice of parameters than the ones used in [21]. We have used the PARI/GP [25] language and the gp2c compiling tool to compute fifty decimal digits (but we write here just ten) of the constant involved in the above lemma. We will give two different estimates that we will use in the case when λ_1/λ_2 belongs to $\mathfrak{R}$ or $\mathfrak{R}'$. If we run the program in our cases, Lemma 5 gives the following results:

$$|G(\alpha)| \le 0.83372131685 \cdot L \tag{3.10}$$

if $\alpha \in [0,1]\setminus E$ where $|E| \ll_{M,\epsilon} X^{-2/3-10^{-20}}$, to be used when $\lambda_1/\lambda_2 \in \mathfrak{R}$, and

$$|G(\alpha)| \le 0.91237810306 \cdot L \tag{3.11}$$

if $\alpha \in [0,1]\setminus E$ where $|E| \ll_{M,\epsilon} X^{-4/5-10^{-20}}$, to be used when $\lambda_1/\lambda_2 \in \mathfrak{R}'$.

The computing time to get (3.10)–(3.11) on a double quad-core PC of the NumLab laboratory of the Department of Pure and Applied Mathematics of the University of Padova was 24 minutes and 40 seconds in the first case (but to get 30 correct digits just 3 minutes and 24 seconds suffice), and 29 minutes in the second case (but just 3 minutes and 50 seconds to get 30 correct digits). The PARI/GP source code of our program together with the cited numerical values are available at www.math.unipd.it/˜languasc/PintzRuzsaMethod.html.

Now we state some lemmas we will use to work on the major arc. Let $\theta(x)=\sum_{p\le x}\log p$, let

$$J(X,h)=\int_{\epsilon X}^{X}(\theta(x+h)-\theta(x)-h)^2\,dx \tag{3.12}$$

be the Selberg integral, and set

$$U(\alpha)=\sum_{\epsilon X\le n\le X}e(\alpha n).$$

Applying to $S(\alpha)-U(\alpha)$ the famous Gallagher lemma ([5, Lemma 1]) on the truncated L^2-norm of exponential sums, one gets the following well-known statement which we cite from Brüdern–Cook–Perelli [2, Lemma 1].

Lemma 6. *For $1/X\le Y\le 1/2$ we have*

$$\int_{-Y}^{Y}|S(\alpha)-U(\alpha)|^2\,d\alpha\ll_\epsilon \frac{\log X}{Y}+Y^2X+Y^2J(X,1/Y),$$

where $J(X,h)$ is defined in (3.12).

To estimate the Selberg integral, we use the next result.

Lemma 7 (Saffari–Vaughan [24, §6]). *For any $A>0$ there exists $B=B(A)>0$ such that*

$$J(X,h)\ll_\epsilon \frac{h^2X}{(\log X)^A}$$

uniformly for $h\ge X^{1/6}(\log X)^B$.

4. The major arc. Letting

$$T(\alpha)=\int_{\epsilon X}^{X}e(t\alpha)\,dt\ll_\epsilon \min(X,1/|\alpha|), \tag{4.1}$$

we first write

$$\begin{aligned}I(X;\mathfrak{M})&=\int_{\mathfrak{M}}T(\lambda_1\alpha)T(\lambda_2\alpha)G(\mu_1\alpha)\cdots G(\mu_s\alpha)e(\gamma\alpha)K(\alpha,\eta)\,d\alpha\\&\quad+\int_{\mathfrak{M}}(S(\lambda_1\alpha)-T(\lambda_1\alpha))T(\lambda_2\alpha)G(\mu_1\alpha)\cdots G(\mu_s\alpha)e(\gamma\alpha)K(\alpha,\eta)\,d\alpha\\&\quad+\int_{\mathfrak{M}}S(\lambda_1\alpha)(S(\lambda_2\alpha)-T(\lambda_2\alpha))G(\mu_1\alpha)\cdots G(\mu_s\alpha)e(\gamma\alpha)K(\alpha,\eta)\,d\alpha\\&=J_1+J_2+J_3,\end{aligned} \tag{4.2}$$

say. We will prove that

$$J_1 \geq \frac{1-c_4\epsilon}{\lambda_1}\eta^2 X L^s \tag{4.3}$$

for some positive c_4, and

$$J_2 + J_3 = o(\eta^2 X L^s), \tag{4.4}$$

showing that

$$I(X;\mathfrak{M}) \geq \frac{1-c_5\epsilon}{\lambda_1}\eta^2 X L^s$$

for some positive c_5, which implies that (2.6) holds with $c_1 = (1-c_5\epsilon)/\lambda_1$.

Estimation of J_2 and J_3. We first estimate J_3. We remark that, by the partial summation formula, we have $T(\alpha) - U(\alpha) \ll 1 + X|\alpha|$. So, recalling $P = X^{1/3}$, (2.4) and $|S(\lambda_1\alpha)| \ll X$, we get

$$\begin{aligned}\int_{\mathfrak{M}} |T(\lambda_2\alpha) - U(\lambda_2\alpha)|\,|S(\lambda_1\alpha)|\,d\alpha &\ll X \int_{\mathfrak{M}} (1 + X|\lambda_2\alpha|)\,d\alpha \\ &\ll_{\boldsymbol{\lambda}} X^{2/3}.\end{aligned}$$

Hence, using the trivial estimates $|G(\mu_i\alpha)| \leq L$ and $K(\alpha,\eta) \ll \eta^2$, we can write

$$\begin{aligned}J_3 = &\int_{\mathfrak{M}} S(\lambda_1\alpha)(S(\lambda_2\alpha) - U(\lambda_2\alpha))G(\mu_1\alpha)\cdots G(\mu_s\alpha)e(\gamma\alpha)K(\alpha,\eta)\,d\alpha \\ &+ \mathcal{O}_{\boldsymbol{\lambda},M}(\eta^2 X^{2/3} L^s).\end{aligned}$$

Now, using (2.4), the Cauchy–Schwarz inequality, the Prime Number Theorem, Lemmas 6–7 with $A = 3$, $Y = P/X$, $P = X^{1/3}$, and again the trivial estimates $|G(\mu_i\alpha)| \leq L$ and $K(\alpha,\eta) \ll \eta^2$, we obtain

$$\begin{aligned}J_3 &\ll \eta^2 L^s \Big(\int_{\mathfrak{M}} |S(\lambda_2\alpha) - U(\lambda_2\alpha)|^2\,d\alpha\Big)^{1/2}\Big(\int_{\mathfrak{M}} |S(\lambda_1\alpha)|^2\,d\alpha\Big)^{1/2} \\ &\quad + \mathcal{O}_{\boldsymbol{\lambda},M}(\eta^2 X^{2/3} L^s) \\ &\ll_{\boldsymbol{\lambda},M,\epsilon} \eta^2 L^s \frac{X^{1/2}}{(\log X)^{3/2}}\Big(\int_0^1 |S(\alpha)|^2\,d\alpha\Big)^{1/2} + \eta^2 X^{2/3} L^s \\ &\ll_{\boldsymbol{\lambda},M,\epsilon} \eta^2 X L^{s-1} = o(\eta^2 X L^s).\end{aligned}$$

The integral J_2 can be estimated analogously using (4.1) instead of the Prime Number Theorem. Hence (4.4) holds.

Estimation of J_1. Recalling that $P = X^{1/3}$ and using (2.4), (4.1) and (4.2) we obtain

$$J_1 = \sum_{1\leq m_1\leq L} \cdots \sum_{1\leq m_s\leq L} J(\mu_1 2^{m_1} + \cdots + \mu_s 2^{m_s} + \gamma, \eta) + \mathcal{O}_\epsilon(\eta^2 X^{2/3} L^s), \tag{4.5}$$

where $J(u,\eta)$ is defined by

$$\begin{aligned} J(u,\eta) &:= \int_{\mathbb{R}} T(\lambda_1\alpha)T(\lambda_2\alpha)e(u\alpha)K(\alpha,\eta)\,d\alpha \\ &= \int_{\epsilon X}^{X}\int_{\epsilon X}^{X} \widehat{K}(\lambda_1 u_1+\lambda_2 u_2+u,\eta)\,du_1\,du_2 \end{aligned}$$

and the second relation follows by interchanging the order of integration. For the sake of simplicity let

$$J_0(u,\eta) := \int_0^X\int_0^X \widehat{K}(\lambda_1 u_1+\lambda_2 u_2+u,\eta)\,du_1\,du_2,$$

where $\lambda_1 > -\lambda_2 > 1$, $|u| \le \epsilon X$, $0 < \eta \le \epsilon X$, and $\epsilon > 0$ is sufficiently small in terms of λ_1 and λ_2. The trivial change of variables $y_1 = \lambda_1 u_1$ and $y_2 = -\lambda_2 u_2$ yields

$$\begin{aligned} J_0(u,\eta) &= -\frac{1}{\lambda_1\lambda_2}\int_0^{\lambda_1 X}\int_0^{-\lambda_2 X} \widehat{K}(y_1-y_2+u,\eta)\,dy_1\,dy_2 \\ &= -\frac{1}{\lambda_1\lambda_2}\int_0^{\lambda_1 X} dy_1 \int_0^{-\lambda_2 X} \max(0;\eta-|y_1-y_2+u|)\,dy_2. \end{aligned}$$

We may obviously assume that $X \ge (\lambda_1+\lambda_2)^{-1}(\eta+|u|)$, so that the lines $y_2 = y_1+u+j\eta$, for $j \in \{-1, 0, 1\}$, intersect the boundary of the rectangle $[0,\lambda_1 X]\times[0,-\lambda_2 X]$ on its upper horizontal side. Exploiting the fact that the integrand vanishes outside the set $y_1+u-\eta \le y_2 \le y_1+u+\eta$, we may replace the condition $y_1 \in [0,\lambda_1 X]$ with $y_1 \in [0,-\lambda_2 X-\eta-u]\cup[-\lambda_2 X-\eta-u,-\lambda_2 X+\eta-u] = I_1\cup I_2$, say. If $y_1 \in I_1$ we have

$$\int_0^{-\lambda_2 X} \max(0;\eta-|y_1-y_2+u|)\,dy_2 = \int_{-y_1-u}^{-\lambda_2 X-y_1-u} \max(0;\eta-|w|)\,dw = \eta^2,$$

and the total contribution of I_1 to J_0 is therefore $\eta^2(-\lambda_2 X-\eta-u)$. The contribution of I_2 is nonnegative, and it is easily seen that it is $\mathcal{O}(\eta^3)$. Finally,

$$J_0(u,\eta) = -\frac{1}{\lambda_1\lambda_2}\eta^2(-\lambda_2 X-\eta-u)+\mathcal{O}(\eta^3) \ge \frac{1}{\lambda_1}X\eta^2+\mathcal{O}(\epsilon X\eta^2).$$

Arguing as above, it is also easy to see that $J_0(u,\eta)-J(u,\eta) \ll \epsilon X\eta^2$, and (4.3) follows.

5. The trivial arc. Recalling (2.4) and the trivial estimate $|G(\mu_i\alpha)| \le L$, and using the Cauchy–Schwarz inequality, we get

$$|I(X;\mathfrak{t})| \ll L^s\Big(\int_{L^2}^{+\infty} |S(\lambda_1\alpha)|^2 K(\alpha,\eta)\,d\alpha\Big)^{1/2}\Big(\int_{L^2}^{+\infty} |S(\lambda_2\alpha)|^2 K(\alpha,\eta)\,d\alpha\Big)^{1/2}.$$

By (2.2) and making a change of variable, we find, for $i = 1, 2$, that

$$\begin{aligned}\int_{L^2}^{+\infty} |S(\lambda_i\alpha)|^2 K(\alpha,\eta)\,d\alpha &\ll_{\boldsymbol{\lambda}} \int_{\lambda_i L^2}^{+\infty} \frac{|S(\alpha)|^2}{\alpha^2}\,d\alpha \\ &\ll \sum_{n\ge\lambda_i L^2} \frac{1}{(n-1)^2} \int_{n-1}^{n} |S(\alpha)|^2\,d\alpha \\ &\ll_{\boldsymbol{\lambda}} L^{-2}\int_0^1 |S(\alpha)|^2\,d\alpha \ll_{\boldsymbol{\lambda},M,\epsilon} \frac{X}{\log X},\end{aligned}$$

by the Prime Number Theorem, and hence (2.7) holds.

6. The minor arc: $\lambda_1/\lambda_2 \in \mathfrak{R}$. Recalling first

$$I(X;\mathfrak{m}) = \int_{\mathfrak{m}} S(\lambda_1\alpha)S(\lambda_2\alpha)G(\mu_1\alpha)\cdots G(\mu_s\alpha)e(\gamma\alpha)K(\alpha,\eta)\,d\alpha,$$

and letting $c \in (0,1)$ to be chosen later, we first split $\mathfrak{m}$ as $\mathfrak{m}_1 \cup \mathfrak{m}_2$, $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \emptyset$, where $\mathfrak{m}_2$ is the set of $\beta \in \mathfrak{m}$ such that $|G(\beta)| > \nu(c)L$ and $\nu(c)$ is defined in Lemma 5. We will choose c to get $|I(X;\mathfrak{m}_2)| = o(\eta X)$, since, again by Lemma 5, we know that $|\mathfrak{m}_2| \ll_{M,\epsilon} sL^2X^{-c}$.

To this end, we first use the trivial estimates $|G(\mu_i\alpha)| \le L$ and $K(\alpha,\eta) \ll \eta^2$, and the Cauchy–Schwarz inequality, thus obtaining

$$\begin{aligned}(6.1)\qquad &|I(X;\mathfrak{m}_2)| \\ &\le L^s\Big(\int_{\mathfrak{m}_2} |S(\lambda_1\alpha)S(\lambda_2\alpha)|^2 K(\alpha,\eta)\,d\alpha\Big)^{1/2}\Big(\int_{\mathfrak{m}_2} K(\alpha,\eta)\,d\alpha\Big)^{1/2} \\ &\ll \eta L^s|\mathfrak{m}_2|^{1/2}\Big(\int_{\mathfrak{m}_2} |S(\lambda_1\alpha)S(\lambda_2\alpha)|^2 K(\alpha,\eta)\,d\alpha\Big)^{1/2}.\end{aligned}$$

We can now argue as in Section 4 of Brüdern–Cook–Perelli [2] to get

$$\int_{\mathfrak{m}_2} |S(\lambda_1\alpha)S(\lambda_2\alpha)|^2 K(\alpha,\eta)\,d\alpha \ll_\epsilon \eta X^{8/3+\epsilon'}. \tag{6.2}$$

Hence, by (6.2), (6.1) becomes

$$|I(X;\mathfrak{m}_2)| \ll_{M,\epsilon} s^{1/2}\eta^{3/2}X^{4/3+2\epsilon'-c/2}.$$

Taking $c = 2/3 + 10^{-20}$ and using (3.10), we get, for $\nu = 0.83372131685$ and $\epsilon' > 0$ sufficiently small,

$$|I(X;\mathfrak{m}_2)| = o(\eta X). \tag{6.3}$$

Now we evaluate the contribution of $\mathfrak{m}_1$. Using Lemma 4 and the Cauchy–Schwarz inequality, we infer that

$$\begin{aligned} |I(X;\mathfrak{m}_1)| &\le (\nu L)^{s-2}\Big(\int_{\mathfrak{m}} |S(\lambda_1\alpha)G(\mu_1\alpha)|^2 K(\alpha,\eta)\,d\alpha\Big)^{1/2} \\ &\quad \times \Big(\int_{\mathfrak{m}} |S(\lambda_2\alpha)G(\mu_2\alpha)|^2 K(\alpha,\eta)\,d\alpha\Big)^{1/2} \\ &< \nu^{s-2} C(q_1,q_2)\eta X L^s, \end{aligned} \tag{6.4}$$

where, recalling Lemmas 2 and 4, $C(q_1,q_2)$ is defined as in (1.4).

Hence, by (6.3) and (6.4), for X sufficiently large we finally get

$$|I(X;\mathfrak{m})| < (0.83372131685)^{s-2} C(q_1,q_2)\eta X L^s$$

whenever $\lambda_1/\lambda_2 \in \mathfrak{R}$.

This means that (2.8) holds with $c_2(s) = (0.83372131685)^{s-2} C(q_1,q_2)$.

7. The minor arc: $\lambda_1/\lambda_2 \in \mathfrak{R}'$. We act on $\mathfrak{m}_1$ as in (6.4) of the previous section to obtain

$$|I(X;\mathfrak{m}_1)| < \nu^{s-2} C(q_1,q_2)\eta X L^s, \tag{7.1}$$

where $C(q_1,q_2)$ is defined in (1.4).

Now we proceed to estimate $I(X;\mathfrak{m}_2)$. First we argue as in the previous section until (6.1) and then we work as in Section 8 of Cook–Harman [4] and pp. 221–223 of Harman [8] to obtain

$$\int_{\mathfrak{m}_2} |S(\lambda_1\alpha)S(\lambda_2\alpha)|^2 K(\alpha,\eta)\,d\alpha \ll \eta^2 X^{14/5+\epsilon'} + \eta X^{13/5+\epsilon'}.$$

This, using (6.1), leads to

$$|I(X;\mathfrak{m}_2)| \ll_{M,\epsilon} s^{1/2} X^{-c/2}(\eta^2 X^{7/5+\epsilon'} + \eta^{3/2} X^{13/10+\epsilon'}).$$

Taking $c = 4/5 + 10^{-20}$ and using (3.11), we get, for $\nu = 0.91237810306$ and $\epsilon' > 0$ sufficiently small,

$$|I(X;\mathfrak{m}_2)| = o(\eta X). \tag{7.2}$$

Hence, by (7.1) and (7.2), for X sufficiently large we finally get

$$|I(X;\mathfrak{m})| < (0.91237810306)^{s-2} C(q_1,q_2)\eta X L^s$$

whenever $\lambda_1/\lambda_2 \in \mathfrak{R}'$.

This means that (2.8) holds with $c_2(s) = (0.91237810306)^{s-2} C(q_1,q_2)$.

8. Proof of the Theorem. We have to verify whether there exists an $s_0 \in \mathbb{N}$ such that (2.9) holds for X sufficiently large. Combining the inequalities (2.6)–(2.8), where $c_2(s) = (0.83372131685)^{s-2} C(q_1, q_2)$ if $\lambda_1/\lambda_2 \in \mathfrak{R}$, and if $\lambda_1/\lambda_2 \in \mathfrak{R}'$, $c_2(s) = (0.91237810306)^{s-2} C(q_1, q_2)$, we deduce for $s \geq s_0$, with s_0 defined in (1.2)–(1.3), that (2.9) holds in both cases. This completes the proof of the Theorem.

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