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A note on the article by F. Luca "On the system of Diophantine equations $a^2+b^2=(m^2+1)^r$ and $a^x+b^y=(m^2+1)^z$"

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by

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1. Introduction. For positive integers r, m with $r > 1$ and m even, we define integers A, B by $A+B\sqrt{-1} = (m+\sqrt{-1}\,)^r$. Consider the Diophantine equation

$$|A|^x + |B|^y = (m^2+1)^z \tag{1.1}$$

in positive integers x, y and z. In 2012, Luca [Lu] proved that there are only finitely many pairs of (r, m) such that equation (1.1) has a solution $(x, y, z) \neq (2, 2, r)$. This result is effective, namely he showed that there exists an effectively computable constant $c_0 > 0$ such that all such solutions satisfy $\max\{r, m, x, y, z\} \le c_0$. The aim of this article is to show an explicit refinement of that result with some simplifications and improvements. Our main result is as follows.

Theorem 1.1. *If $r > 10^{74}$ or $m > 10^{34}$, then equation* (1.1) *has no solution other than* $(x, y, z) = (2, 2, r)$.

2. Preliminaries. In this section, we list the estimates for linear forms in logarithms that we will need, in both complex and p-adic cases. Let α_1, α_2 be non-zero algebraic numbers. Write $\mathbb{L} = \mathbb{Q}(\alpha_1, \alpha_2)$ and denote by D the degree of $\mathbb{L}$ over $\mathbb{Q}$.

First, we present lower bounds for linear forms in two complex logarithms due to Laurent [La]. Consider the linear form

$$\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2,$$

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where b_1, b_2 are positive integers, and $\log \alpha_1, \log \alpha_2$ are any determinations of the logarithms of α_1, α_2 respectively. We assume $|\alpha_1|$, $|\alpha_2| \geq 1$. Put

$$D' = D/[\mathbb{R}(\alpha_1, \alpha_2) : \mathbb{R}].$$

For any algebraic number α, we define as usual the absolute logarithmic height of α by

$$\mathrm{h}(\alpha) = \frac{1}{d}\Big(\log c_0 + \sum_{i=1}^{d} \log \max\{1, |\alpha^{(i)}|\}\Big),$$

where $c_0 > 0$ is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$, and $\alpha^{(1)}, \ldots, \alpha^{(d)}$ are the conjugates of α in the field of complex numbers.

The following is the main result of [La].

Proposition 2.1 ([La, Theorem 1]). *Let K be an integer ≥ 2, and let L, R_1, R_2, S_1, S_2 be positive integers. Let ρ and μ be real numbers with $\rho > 1$ and $1/3 \leq \mu \leq 1$. Put*

$$R = R_1 + R_2 - 1, \quad S = S_1 + S_2 - 1, \quad N = KL, \quad g = \frac{1}{4} - \frac{N}{12RS},$$
$$\sigma = \frac{1 + 2\mu - \mu^2}{2}, \qquad b = \frac{(R-1)\, b_2 + (S-1)\, b_1}{2}\Big(\prod_{k=1}^{K-1} k!\Big)^{-2/(K^2-K)}.$$

Let H_1, H_2 be positive real numbers such that

$$H_i \geq \rho|\log \alpha_i| - \log|\alpha_i| + 2D'\mathrm{h}(\alpha_i) \quad (i = 1, 2).$$

Suppose

$$\text{(I)} \qquad \begin{cases} \mathrm{Card}\{\alpha_1^r \alpha_2^s : 0 \leq r < R_1,\ 0 \leq s < S_1\} \geq L, \\ \mathrm{Card}\{r b_2 + s b_1 : 0 \leq r < R_2,\ 0 \leq s < S_2\} > (K-1)L \end{cases}$$

and

$$\text{(II)} \qquad K(\sigma L - 1)\log \rho - (D' + 1)\log N - D'(K-1)\log b - gL(RH_1 + SH_2) > \varepsilon(N),$$

where

$$\varepsilon(N) = \frac{2\log\big(N! N^{-N+1}(e^N + (e-1)^N)\big)}{N}.$$

Then

$$|\Lambda'| \geq \rho^{-\mu KL} \quad \text{with} \quad \Lambda' = \Lambda \max\left\{\frac{LSe^{LS|\Lambda|/(2b_2)}}{2b_2}, \frac{LRe^{LR|\Lambda|/(2b_1)}}{2b_1}\right\}.$$

We also rely on the following result of [La].

Proposition 2.2 ([La, Corollary 2, $m = 10$]). *For algebraic numbers α_1, α_2, suppose that $\alpha_1, \alpha_2, \log \alpha_1, \log \alpha_2$ are all real and positive. Assume* *further that α_1, α_2 are multiplicatively independent. Let H_1, H_2 be real numbers such that*

$$H_i \geq \max\bigl\{\mathrm{h}(\alpha_i), (\log \alpha_i)/D, 1/D\bigr\} \quad (i = 1, 2).$$

Put

$$b' = \frac{b_1}{DH_2} + \frac{b_2}{DH_1}.$$

Then

$$\log |\Lambda| \geq -25.2\, D^4 H_1 H_2 \bigl(\max\{\log b' + 0.38,\ 10/D,\ 1\}\bigr)^2.$$

Next, we present lower bounds for linear forms in two p-adic logarithms, due to Bugeaud and Laurent [BL] and Bugeaud [B]. Put

$$\Gamma = \alpha_1^{b_1} \alpha_2^{b_2} - 1,$$

where b_1, b_2 are non-zero rational integers. We assume that α_1, α_2 are multiplicatively independent. Suppose that π is a prime ideal in the ring of integers of $\mathbb{L}$ which does not divide the ideal $(\alpha_1\alpha_2)$. Let f_π be its inertia index. We denote by g the minimal positive integer such that both $\alpha_1^g - 1$ and $\alpha_2^g - 1$ belong to π.

Let H_1, H_2 be real numbers such that

$$H_i \geq \max\{\mathrm{h}(\alpha_i), (\log p)/D_\pi\} \quad (i = 1, 2),$$

where p is the rational prime such that p belongs to π and $D_\pi = D/f_\pi$. Put

$$b' = \frac{|b_1|}{H_2} + \frac{|b_2|}{H_1}.$$

For $\alpha \in \mathbb{L}\backslash\{0\}$, we denote by $\mathrm{ord}_\pi(\alpha)$ the exponent of π in the factorization of the fractional ideal generated by α inside $\mathbb{L}$. The next proposition is proven in [BL].

Proposition 2.3 ([BL, Théorème 3]). *Under the above assumptions,*

$$\mathrm{ord}_\pi(\Gamma) \leq \frac{24pgH_1H_2D_\pi^4}{(p-1)(\log p)^4}\bigl(\max\{\log b' + \log\log p + 0.4,\ (10/D_\pi)\log p,\ 10\}\bigr)^2.$$

Under the hypothesis of Proposition 2.3, we further suppose that both $\alpha_1 = a_1$ and $\alpha_2 = a_2$ are rational integers. Then $\pi = p$. Assume that there exists a real number E such that

$$\frac{1}{p-1} < E \leq \mathrm{ord}_p(a_1^g - 1).$$

Let H_1, H_2 be real numbers such that

$$H_i \geq \max\{\log |a_i|, E \log p\} \quad (i = 1, 2).$$

We put $b' = |b_1|/H_2 + |b_2|/H_1$. The estimate below is obtained in [B].

Proposition 2.4 ([B, Theorem 2]). *Under the above assumptions, if either p is odd, or $p = 2$ and $\mathrm{ord}_2(a_2 - 1) \ge 2$, then*

$$\mathrm{ord}_p(\Gamma) \le \frac{36.1 g H_1 H_2}{E^3 (\log p)^4} \big(\max\{\log b' + \log(E \log p) + 0.4,\, 6E \log p,\, 5\}\big)^2$$

and

$$\mathrm{ord}_p(\Gamma) \le \frac{53.8 g H_1 H_2}{E^3 (\log p)^4} \big(\max\{\log b' + \log(E \log p) + 0.4,\, 4E \log p,\, 5\}\big)^2,$$

If $p = 2$ and $\mathrm{ord}_2(a_2 - 1) < 2$, then

$$\mathrm{ord}_2(\Gamma) \le 208\, H_1 H_2 (\max\{\log b' + 0.04,\, 10\})^2.$$

3. Proof of Theorem 1.1. Let r, m be positive integers with $r > 1$ and m even. Define a, b and c by $a = |A|$, $b = |B|$ and $c = m^2 + 1$. We see that a, b and c are co-prime integers such that $a^2 + b^2 = c^r$ with $\min\{a, b, c\} > 1$. Both the facts that $\gcd(a, b, c) = 1$ and $\min\{a, b, c\} > 1$ are easily shown (cf. [Lu, Lemma 5(i) & (iv)]). Also, A, B satisfy

$$A^2 + B^2 = (A + B\sqrt{-1})(A - B\sqrt{-1}) = (m + \sqrt{-1})^r (m - \sqrt{-1})^r = (m^2 + 1)^r.$$

Our proof is organized in several stages below.

3.1. Elementary estimates for variables

Lemma 3.1. *Let (x, y, z) be a solution to* (1.1). *Put*

$$X := \max\{x, y\}, \quad \Delta := rX - 2z.$$

Then:

(i) $\Delta \ge 0$. *Moreover, if $\Delta = 0$, then $(x, y, z) = (2, 2, r)$.*
(ii) *If $\Delta > 0$, then*

$$\Delta > \frac{\log \min\{a, b\}}{\log c}.$$

Proof. (i) Since $a, b < c^{r/2}$ and $c \ge 5$, we have

$$c^z < 2 \max\{a^x, b^y\} \le 2 \max\{a, b\}^X < 2c^{rX/2},$$

and so $c^{2z} < 4c^{rX} < c^{rX+1}$.

Suppose $\Delta = 0$, that is, $z = rX/2$. Then $X > 1$ by [Lu, Lemma 5(v)]. Since $a^X + b^X \ge a^x + b^y = c^{rX/2} = (a^2 + b^2)^{X/2}$, we find $X = 2$, and $(x, y, z) = (2, 2, r)$.

(ii) Reducing (1.1) modulo a and b, we have $c^{|ry - 2z|} \equiv 1 \pmod{a}$ and $c^{|rx - 2z|} \equiv 1 \pmod{b}$, respectively. These together give the desired inequality. ■

3.2. Upper bound for X in terms of r and c

Lemma 3.2. *Let* (x, y, z) *be a solution to* (1.1)*. Then*

$$X < 50\, r^2 (\log c)^2 (\log(69\, r^2 \log c))^2.$$

Proof. We only consider the case where r is odd (the case where r is even can be dealt with similarly). By the definition of a and b, we easily observe $a \equiv 0 \pmod{m}$ and $b \equiv \pm 1 \pmod{m^2}$; in particular, a is even, $a \geq m$ and $b \geq m^2 - 1$. We will consider the cases $a^x < b^{y/2}$ and $a^x \geq b^{y/2}$ separately.

First, we suppose $a^x < b^{y/2}$. Then

$$x < \frac{\log b}{2 \log a}\, y < \frac{\log(m^2+1)}{4 \log m}\, ry < 0.6\, ry \quad (> y).$$

Put $\Lambda := z \log c - y \log b$ (> 0). Since $\Lambda < \exp(\Lambda) - 1 = a^x b^{-y} < b^{-y/2}$, we have

$$\log \Lambda < -\frac{\log b}{2}\, y.$$

We apply Proposition 2.2 with $(\alpha_1, \alpha_2) = (c, b)$ and $(b_1, b_2) = (z, y)$. Then

$$\log \Lambda \geq -25.2 (\log b)(\log c)(\max\{\log b' + 0.38,\ 10\})^2,$$

where $b' = y/\log c + z/\log b$. It follows that

$$\frac{y}{\log c} < 50.4 (\max\{\log b' + 0.38,\ 10\})^2.$$

This inequality together with $b' < 2y/\log c + 1$ (since $c^z < 2b^y$) implies $y/\log c < 5040$, and so $X < 0.6\, ry < 3024\, r \log c$.

Second, we suppose $a^x \geq b^{y/2}$. Then

$$y \leq \frac{2 \log a}{\log b} x < \frac{2 \log(m^2+1)^{r/2}}{\log(m^2-1)} x < 1.5\, rx.$$

Hence, we may assume $x > 1$. Since $c^z = a^x + b^y < 2a^{2x} < 2c^{rx} < c^{rx+1}$, we find $z \leq rx$. Note that y is even if $b \equiv 3 \pmod 4$ (which can be seen by reducing (1.1) modulo 4). Put $\Gamma := c^z b^{-y} - 1$. We will apply Proposition 2.4 with $(\alpha_1, \alpha_2) = (c, (-1)^{(b-1)/2} b)$, $(b_1, b_2) = (z, -y)$ and $p = 2$. Since $g = 1$, we may take $E = 2$ and $(H_1, H_2) = (\log c, \log(b+1))$. It follows from $3 \leq b < c^{r/2}$ and $\mathrm{ord}_2(\Gamma) \geq x$ that

$$x \leq \frac{36.1\, r (\log c)^2}{8 (\log 2)^3 \log 3} \big(\max\{\log b' + \log(2 \log 2) + 0.4,\ 12 \log 2\}\big)^2,$$

where $b' = y/\log c + z/\log(b+1)$. Observe that

$$b' < \frac{1.5\, rx}{\log c} + \frac{rx}{\log(b+1)} < \frac{2.7\, r}{\log c}\, x.$$

We may assume

$$x \geq \frac{2^{11}}{2.7 (\log 2) \exp(0.4)}\, \frac{\log c}{r}.$$

Write

$$s = \frac{5.4(\log 2)\exp(0.4)\, r}{\log c}\, x.$$

Then $s/(\log s)^2 < 69\, r^2 \log c$ (≥ 444), from which we have

$$s < 276\, r^2 (\log c)(\log(69\, r^2 \log c))^2.$$

Hence, $X < 1.5\, rx < 50\, r^2 (\log c)^2 (\log(69\, r^2 \log c))^2$. ∎

3.3. Lower bounds for X in terms of r and c

LEMMA 3.3. *Let* $(x, y, z) \neq (2, 2, r)$ *be a solution to* (1.1). *Then:*

(i) $X \geq 2\sqrt{c-1}/r^2$. *Moreover, if* $\min\{x, y\} \geq 4$, *then* $X \geq 2(c-1)/r^2$.
(ii) *If* $c > 10^{68}$ *and* $r < c^{1/3}$, *then* $X \geq 2(c - r^3\sqrt{c-1} - 1)/r^2$.

Proof. Clearly, we may assume $m > 2$. The proof proceeds along similar lines to that of [Lu, Lemma 8].

We only consider the case where r is even (the case of r odd can be dealt with similarly). Then

$$A = \frac{(m+\sqrt{-1})^r + (m-\sqrt{-1})^r}{2} = (-1)^{r/2}\left(1 - \binom{r}{2} m^2 + \cdots\right),$$

$$B = \frac{(m+\sqrt{-1})^r - (m-\sqrt{-1})^r}{2\sqrt{-1}} = (-1)^{r/2}\left(rm - \binom{r}{3} m^3 + \cdots\right).$$

Write $a = \epsilon A$ and $b = \eta B$, where $\epsilon, \eta \in \{1, -1\}$. Then, reducing modulo m^4, we find

$$a^x = \epsilon^x (-1)^{rx/2}\left(1 - \binom{r}{2} m^2 + \cdots\right)^x \equiv \epsilon_1\left(1 - \binom{r}{2} m^2 x\right) \pmod{m^4},$$

$$b^y = \eta^y (-1)^{ry/2} m^y \left(r - \binom{r}{3} m^2 + \cdots\right)^y$$

$$\equiv \eta_1\, r^{y-1} m^y \left(r - \binom{r}{3} m^2 y\right) \pmod{m^4},$$

$$c^z = (m^2+1)^z \equiv m^2 z + 1 \pmod{m^4},$$

where $\epsilon_1 = \epsilon^x (-1)^{rx/2}$ and $\eta_1 = \eta^y (-1)^{ry/2}$. It follows from (1.1) that

$$\epsilon_1\left(1 - \binom{r}{2} m^2 x\right) + \eta_1 r^{y-1} m^y \left(r - \binom{r}{3} m^2 y\right) \equiv m^2 z + 1 \pmod{m^4}.$$

This implies $\epsilon_1 \equiv 1 \pmod{m}$, and so $\epsilon_1 = 1$, since $m > 2$. Hence,

$$-\binom{r}{2} m^2 x + \eta_1 r^{y-1} m^y \left(r - \binom{r}{3} m^2 y\right) \equiv m^2 z \pmod{m^4}.$$

This congruence yields:

$$
\text{(3.1)}\qquad \begin{cases} r \equiv 0 \pmod{m} & \text{if } y = 1, \\ z + \dfrac{r(r-1)}{2}x - r^2 \equiv 0 \pmod{m^2} & \text{if } y = 2, \\ z + \dfrac{r(r-1)}{2}x - \eta_1 r^3 m \equiv 0 \pmod{m^2} & \text{if } y = 3, \\ z + \dfrac{r(r-1)}{2}x \equiv 0 \pmod{m^2} & \text{if } y \geq 4. \end{cases}
$$

(i) As in the proof of [Lu, Lemma 8], we can observe that the left-hand side of the congruence in (3.1) for $y = 2$ is non-zero. Hence, congruences (3.1) with Lemma 3.1(i) imply

$$
\frac{r^2 X}{2} \geq z + \frac{r(r-1)}{2}X \geq m \ (= \sqrt{c-1}).
$$

Also, if $y \neq 3$, then we can replace the rightmost side above by m^2 $(= c-1)$.

(ii) Assume $c > 10^{68}$ and $r < c^{1/3}$. It suffices to show that the left-hand side of the congruence in (3.1) for $y = 3$ is non-zero. If $z + \frac{r(r-1)}{2}\, x = r^3 m$, then the proof of (i) and Lemma 3.2 yield

$$
\begin{aligned}
\sqrt{c-1} \leq \frac{X}{2r} &< 25\, r(\log c)^2(\log(69\, r^2 \log c))^2 \\
&< 25\, c^{1/3}(\log c)^2(\log(69\, c^{2/3} \log c))^2,
\end{aligned}
$$

which contradicts the assumption $c > 10^{68}$. ■

3.4. Lower bounds for r in terms of c

Lemma 3.4. *Assume $c > 10^{68}$. Let $(x, y, z) \neq (2, 2, r)$ be a solution to (1.1). Then $r > c^{1/6.01}$. Moreover, if $\min\{x, y\} \geq 4$, then $r > c^{1/4.66}$.*

Proof. We may assume $r < c^{1/3}$. Then Lemmata 3.2 and 3.3(ii) imply

$$
c - r^3\sqrt{c-1} - 1 < 25\, r^4(\log c)^2(\log(69\, r^2 \log c))^2.
$$

Combining this inequality with the assumption $c > 10^{68}$, we have $r > c^{1/6.01}$. Similarly, if $\min\{x, y\} \geq 4$, then $c-1 < 25\, r^4(\log c)^2(\log(69\, r^2 \log c))^2$, which implies $r > c^{1/4.66}$. ■

3.5. Prime factors of c

Lemma 3.5. *Let $(x, y, z) \neq (2, 2, r)$ be a solution to (1.1). Then:*

(i) *$r < 4 \cdot 10^5\, c$. Moreover, if $c > 10^{68}$, then $r < 5341\, c$.*
(ii) *Assume $c > 10^{68}$. Let p be any prime factor of c. If $\min\{x, y\} \geq 4$, then $p > c^{1/4.66}/76000$.*

Proof. By [Lu, Lemma 5(v)], we know $x \neq y$. As in the proof of [Lu, Lemma 7(iii)], we see that

$$
\Gamma := \alpha^{4r|x-y|}\, 2^{-4|x-y|} - 1 \equiv 0 \pmod{\beta^{\lceil r/2 \rceil}},
$$

where $\alpha = m + \sqrt{-1}$ and $\beta = m - \sqrt{-1}$. Let p be any prime factor of $c = m^2 + 1$. Since $p \equiv 1 \pmod 4$, we can write $p = \pi\bar{\pi}$ with $\pi \neq \bar{\pi}$, where π is a prime in $\mathbb{Z}[\sqrt{-1}\,]$, and $\bar{\pi}$ is the complex conjugate of π. We may assume that π divides β. We will apply Proposition 2.3 with $(\alpha_1, \alpha_2) = (\alpha, 2)$ and $(b_1, b_2) = (4r|x-y|, -4|x-y|)$. Observe that $D_\pi = 2$ and g is a divisor of $p-1$. We may take $(H_1, H_2) = ((\log c)/2, (\log p)/2)$. It follows that

$$r \le \frac{192\, p \log c}{(\log p)^3}\big(\max\{\log b' + \log\log p + 0.4,\ 5\log p\}\big)^2,$$

where $b' = 8r|x-y|/\log p + 8|x-y|/\log c$. We may assume $r \ge 5341\, c$. Then, from Lemma 3.2, we see that

$$\begin{aligned} b' \cdot (\log p) \cdot \exp(0.4) &< \frac{8X(r+1)}{\log p} \cdot (\log p) \cdot \exp(0.4) \\ &< 400 \exp(0.4)\, r^2 (r+1)(\log c)^2 (\log(69\, r^2 \log c))^2 < r^5. \end{aligned}$$

Hence,

$$\frac{r}{(\log r)^2} \le \frac{4800\, p \log c}{(\log p)^3}. \tag{3.2}$$

Since $5 \le p \le c$, we have $r/(\log r)^2 < 4800\, c/(\log c)^2$. Write $r = \mathcal{C}c$. Then $\mathcal{C} < 4800(1 + (\log \mathcal{C})/\log c)^2$. Since $c \ge 5$, we have $\mathcal{C} < 4 \cdot 10^5$, which can be replaced by $\mathcal{C} < 5341$ if $c > 10^{68}$.

(ii) By Lemma 3.4, we may assume $p < r$. With the notation in (i), we see from Lemmata 3.2 and 3.4 that $b' \cdot (\log p) \cdot \exp(0.4) < r^5$, and so (3.2) holds. Write $c^{1/4.66} = \mathcal{C}' p$. Then $\mathcal{C}' < 22368(1 + (\log \mathcal{C}')/\log p)^3$. Since $p \ge 5$, we have $\mathcal{C}' < 4 \cdot 10^7$. Hence, $p > c^{1/4.66}/(4 \cdot 10^7) > 10^7$, and so $\mathcal{C}' < 1.2 \cdot 10^5$. Repeating this process twice, we obtain $\mathcal{C}' < 76000$. ∎

3.6. Accurate estimates for $\log a$ and $\log b$

Lemma 3.6. *Assume $c > 10^{68}$. Let $(x, y, z) \neq (2, 2, r)$ be a solution to* (1.1). *Then*

$$\max\left\{\frac{r}{2}\log c - \log a,\ \frac{r}{2}\log c - \log b\right\} < 17.04(\log c)^3.$$

Proof. Write

$$\log a - (r/2)\log c = \log|\Gamma| - \log 2 \ (< 0),$$

where $\Gamma = \gamma^r + 1$ with $\gamma = \frac{m-\sqrt{-1}}{m+\sqrt{-1}}$. We may assume $|\Gamma| < 1/3$. Then there exists a non-negative integer j with $j \le r + 2$ such that $|\Gamma| \ge |\Lambda|/2$ with $\Lambda := r\log\gamma - j\log(-1)$, where the former log denotes the principal determination of the logarithm, and the latter denotes a determination such that $\log(-1) = \pm\pi\sqrt{-1}$. We define $\theta \in [0, \pi/2]$ by $\tan\theta = \frac{2m}{m^2+1}$. If $j = 0$, then $\log|\Lambda| = \log(r\theta) > -0.4\log c$, where the last inequality follows from

Lemma 3.4. Hence, we may assume $j > 0$. We will apply Proposition 2.1 with $(\alpha_1, \alpha_2) = (\gamma, -1)$ and $(b_1, b_2) = (r/d_0,\, j/d_0)$, where $d_0 = \gcd(r, j)$. For this, we choose the parameters as follows:

$$L = \log c,\ \ \rho = 4.07,\ \ \mu = 0.93,\ \ K = \lceil LH_1H_2 \rceil,$$
$$R_1 = \lceil L/2 \rceil,\ \ S_1 = 2,\ \ R_2 = \lceil LH_2 \rceil,\ \ S_2 = \lceil \big(1 + (K-1)L\big)/R_2 \rceil,$$

where we take $(H_1, H_2) = (\rho|\log\gamma| + \log c, \rho\pi)$. Let us check both conditions (I) and (II). The first inequality in (I) clearly holds. Also, using $c > 10^{68}$ and $r < 5341\,c$ by Lemma 3.5(i), we can verify (II). It remains to establish the second inequality in (I). For this, we will show $r/d_0 > R_2$. Suppose $r/d_0 \le R_2 = \lceil LH_2 \rceil$. Then since $(r/d_0)\theta$ is very small, we see from Lemma 3.4 that

$$\begin{aligned}\log|\Lambda| &= \log d_0 + \log|(r/d_0)\log\gamma - (j/d_0)\log(-1)| \\ &\ge \log(r/(\rho\pi\log c)) + \log|(r/d_0)\theta \pm (j/d_0)\pi| > 18 + \log 3,\end{aligned}$$

which is clearly absurd. Hence, $b_1 = r/d_0 > R_2$. Now, we suppose that

$$ub_2 + vb_1 = u'b_2 + v'b_1$$

for some integers u, u', v, v' such that $0 \le u, u' < R_2$ and $0 \le v, v' < S_2$. This implies $b_2(u - u') \equiv 0 \pmod{b_1}$, and so $u - u' \equiv 0 \pmod{b_1}$, as $\gcd(b_1, b_2) = 1$. Since $b_1 > R_2$ and $|u - u'| < R_2$, we find $u = u'$, and $v = v'$. This shows that the second inequality in (I) holds.

Then, we have

$$\begin{aligned}-\mu(\log\rho)KL &\le \log|\Lambda/d_0| + \log\max\left\{\frac{LSe^{LS|\Lambda|/(2j)}}{2j/d_0}, \frac{LRe^{LR|\Lambda|/(2r)}}{2r/d_0}\right\} \\ &\le \log|\Lambda| + \log(LT) + \frac{LT|\Lambda|}{2b_3d_0} - \log(2b_3) \\ &< \log|\Lambda| + \log(LT) + \frac{LT}{3} - \log 2,\end{aligned}$$

where $(T, b_3) \in \{(R, r/d_0), (S, j/d_0)\}$. Since $L > 68\log 10$, we find

$$R = \lceil L/2 \rceil + \lceil LH_2 \rceil - 1 < (1/2 + H_2)L + 1 < 6.3L,$$
$$S = \lceil \big(1 + (K-1)L\big)/R_2 \rceil + 1 < KL/R_2 + 2 < LH_1 + 1/H_2 + 2 < 1.01\,L^2.$$

Hence, $\log|\Lambda|$ is greater than

$$\begin{aligned}&-\mu(\log\rho)\lceil LH_1H_2 \rceil L - \log(1.01\,L^3) - \frac{1.01\,L^3}{3} + \log 2 \\ &\quad > -\left(1.001\,\pi\mu\rho\log(\rho) + \frac{\mu(\log\rho)}{L^2} + \frac{\log(1.01\,L^3)}{L^3} + \frac{1.01}{3} - \frac{\log 2}{L^3}\right)L^3 \\ &\quad > -17.03\,L^3.\end{aligned}$$

Similarly, we have the desired estimate for $\log b$. ∎

3.7. Bounding X and Δ

Lemma 3.7. *Assume $c > 10^{68}$. Let $(x, y, z) \neq (2, 2, r)$ be a solution to* (1.1). *Then:*

(i) $X < 7 \cdot 10^9 \log c$. *Moreover, if* $\min\{x, y\} < 4$, *then* $X < 2522 \log c$.
(ii) $\Delta < 34.2(\log c)^2 X$.

Proof. We only consider the case where $a^x < b^y$ (the remaining case can be dealt with similarly). Since $c^z < 2b^y$, we see $|z \log c - y \log b| < \log 2$. Therefore, Lemma 3.6 gives

$$\left|\left(\frac{ry}{2} - z\right)\log c\right| = \left|\left(\frac{r}{2}\log c - \log b\right)y - (z\log c - y\log b)\right| < 17.1(\log c)^3 y,$$

which together with Lemma 3.4 implies

$$ry < (2 + 10^{-5})z. \tag{3.3}$$

Put $\Lambda := z \log c - y \log b$. Observe $\Lambda \in (0, 1)$. Then, as in the proof of Lemma 3.2, Proposition 2.2 tells us that

$$|\log \Lambda| < 12.6\, r(\log c)^2(\max\{\log b' + 0.38, 10\})^2,$$

where $b' = y/\log c + z/\log b$ $(< (2y + 0.02)/\log c)$. On the other hand, we see from Lemma 3.6 that

$$|\log \Lambda| > y \log b - x \log a = r(\log c)(y - x)/2 + R \quad (> 0),$$

where $|R| < 34.1(\log c)^3 X$. Hence,

$$|x - y| < 25.2(\log c)(\max\{\log s, 10\})^2 + \frac{68.2(\log c)^2}{r} X, \tag{3.4}$$

where

$$s = \frac{2\exp(0.38)}{\log c}(X + 0.01).$$

(i) First, let us consider the case $\min\{x, y\} < 4$. Inequality (3.4) implies

$$\left(1 - \frac{68.2(\log c)^2}{r}\right) X < 3 + 25.2(\log c)(\max\{\log s, 10\})^2.$$

Since $r > c^{1/6.01}$ by Lemma 3.4, we have $s < 73.77(\max\{\log s, 10\})^2$. Hence, $s < 7377$ and $X < 2522 \log c$.

Next, we assume $\min\{x, y\} \geq 4$. Then $r > c^{1/4.66}$ by Lemma 3.4. Let p be any prime factor of c. Put $\Gamma := a^{4x}b^{-4y} - 1$. We apply Proposition 2.4 with $(\alpha_1, \alpha_2) = (a^4, b^4)$ and $(b_1, b_2) = (x, -y)$. Observe that g is a divisor of $|x - y|$ (≥ 1), and set $E = r_1 := \lceil r/2 \rceil$. We may take $H_1 = H_2 = 2r \log c$. It follows from $\mathrm{ord}_p(\Gamma) \geq z$ that

$$z \leq \frac{215.2\, r^2(\log c)^2 g}{r_1^3(\log p)^4}\left(\max\{\log b'' + \log(r_1 \log p) + 0.4,\ 4r_1 \log p\}\right)^2,$$

where $b'' = (x+y)/(2r\log c)$. From Lemmata 3.2 and 3.4, we see that

$$\begin{aligned} b'' \cdot (r_1 \log p) \cdot \exp(0.4) &< \frac{X}{r\log c} \cdot (r_1 \log p) \cdot \exp(0.4) \\ &< 37.5\, r^2 (\log c)^2 \big(\log(69\, r^2 \log c)\big)^2 < 5^{2r}. \end{aligned}$$

Hence, Lemma 3.5(ii) yields

$$z < \frac{6886.4(\log c)^2}{(\log(c^{1/4.66}/76000))^2}\, rg < 3.4 \cdot 10^5 r|x-y|. \tag{3.5}$$

In view of (3.3)–(3.5), we have

$$y < 1.73 \cdot 10^7 (\log c)(\max\{\log s, 10\})^2 + 0.46\, X. \tag{3.6}$$

Since Lemma 3.6 gives

$$x < \frac{\log b}{\log a}\, y < \frac{r/2}{r/2 - 17.03(\log c)^2}\, y < (1 + 3 \cdot 10^{-9})\, y,$$

it follows from (3.6) that $s < 3.3 \cdot 10^7 (\max\{\log s, 10\})^2$. Hence, $s < 1.9 \cdot 10^{10}$ and $X < 7 \cdot 10^9 \log c$.

(ii) The desired estimate for Δ follows easily from $c^z > \min\{a,b\}^X$ together with Lemmata 3.1(i) and 3.6. ■

3.8. The end of the proof. We assume $r > 10^{74}$ or $m > 10^{34}$. Suppose that there exists a solution $(x,y,z) \neq (2,2,r)$ to (1.1). By Lemma 3.5(i), we have $c > 10^{68}$. We will consider the cases $\min\{x,y\} \geq 4$ and $\min\{x,y\} < 4$ separately.

Suppose $\min\{x,y\} \geq 4$. By Lemmata 3.3(i) and 3.7(i), we have

$$c - 1 < 3.5 \cdot 10^9 r^2 \log c.$$

Since $\Delta > 0$ by Lemma 3.1(i), we see that Lemmata 3.1(ii), 3.6 and 3.7 yield

$$\frac{1}{2}\sqrt{\frac{c-1}{3.5 \cdot 10^9 \log c}} - 17.04(\log c)^2 < \Delta < 239.4 \cdot 10^9 (\log c)^3,$$

which gives $c < 10^{48}$, a contradiction.

Suppose $\min\{x,y\} < 4$. By Lemmata 3.3(i) and 3.7(i), we have

$$\sqrt{c-1} < 1261\, r^2 \log c.$$

As in the preceding case, we find

$$\frac{1}{2}\sqrt{\frac{\sqrt{c-1}}{1261 \log c}} - 17.04(\log c)^2 < 86252.4(\log c)^3,$$

which gives $c < 10^{57}$, a contradiction. This completes the proof of Theorem 1.1.

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