On sums of Ramanujan sums

by

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1. Introduction. In this note, we study the moments of the average of the Ramanujan sum. To be precise, we want to evaluate the sums

$$C_k(x,y) = \sum_{n \le y} \left(\sum_{q \le x} c_q(n) \right)^k,$$

where k is a positive integer, x and y are large real numbers, and

$$(1.1) \qquad c_q(n) = \sum_{\substack{1 \le a \le q \\ (a,q)=1}} e(-an/q) = \sum_{d|(q,n)} d\mu(q/d)$$

is the Ramanujan sum. Our interest in this question stems from an old attempt [1] to apply Fourier techniques to Diophantine approximations of reals by sums of rational numbers.

At the time, we were surprised that information on the asymptotic behavior of $C_k(x,y)$ appeared to be missing from the literature. Of course, it is easy to evaluate $C_k(x,y)$ in some cases. For example, elementary arguments based on the second formula in (1.1) yield the asymptotic formulas

$$C_1(x,y) = y + O(x^2)$$

and

$$(1.2) \qquad C_2(x,y) = \frac{yx^2}{2\zeta(2)} + O(x^4 + xy \log x).$$

However, these bounds (as well as other simple things we have tried) are only of interest when $yx^{-2} \to \infty$ as $x \to \infty$. In our application to Diophantine approximations, on the other hand, we were interested in $C_2(x,y)$ when $x^{1+\epsilon} \le y \le x^{2+\epsilon}$ for some fixed $\epsilon > 0$. Some numerical experimentation suggested that in that range the order of $C_2(x,y)$ is still yx^2, though the coefficient $1/(2\zeta(2))$ seemed somewhat off when y is close to x^2. In the present note, we prove, among other things, that those empirical observations are true.

[1]

Our first result is an asymptotic formula for $C_1(x, y)$.

THEOREM 1.1. *Let x be a large real number and $y \geq x$. Then*

$$C_1(x, y) = y - \frac{x^2}{4\zeta(2)} + O\big(xy^{1/3} \log x + x^3 y^{-1}\big).$$

The proof of Theorem 1.1 uses exponential sum estimates and can be easily modified to yield different estimates for the remainder. The remainder claimed in the statement of the theorem is the result of applying one of the simplest exponential sum bounds—the exponential pair $(1/2, 1/2)$ from van der Corput's method [2]. More sophisticated exponential pairs will result in replacing the error term $xy^{1/3}$ by different ones, which will be sharper for certain choices of the relative sizes of x and y. The interested reader will have no trouble obtaining such improvements by appropriate modifications of (2.5) below. Since our main focus is on maximizing the "support" of the result, we prefer the simple version above.

Theorem 1.1 demonstrates that when $x = o(y)$, the Ramanujan sum $c_q(n)$ is $o(1)$ on average over $q \leq x$ and $n \leq y$. Our next theorem shows that this is no longer the case when the average $C_1(x, y)$ is replaced by the mean square sum $C_2(x, y)$.

THEOREM 1.2. *Let x be a large real number, $y \geq x$, and $B > 0$ be fixed.*

(i) *If $y \geq x^2(\log x)^B$, then (1.2) holds.*
(ii) *If $x \leq y \leq x^2(\log x)^B$, then*

$$C_2(x, y) = \frac{yx^2}{2\zeta(2)}(1 + 2\kappa(u)) + O(yx^2(\log x)^{10}(x^{-1/2} + (y/x)^{-1/2})),$$

where $u = \log(yx^{-2})$, and $\kappa(u)$ is defined by (4.19) below and satisfies the inequalities

$$\kappa(u) > -0.4, \qquad \kappa(u) \ll \exp(-|u|^{3/5-\epsilon})$$

for any fixed $\epsilon > 0$. In particular, $\kappa(u) = o(1)$ as $|u| \to \infty$.

The lower bound for $\kappa(u)$ is far from sharp and has been chosen so as to simplify the arguments in §5. In fact, by increasing the amount of numerical calculations in §5 by an order of magnitude or two, it can be shown that $\kappa(u) > -0.3$. We chose not to pursue such a sharper bound here for the sake of clarity. However, for the benefit of the reader, we should mention that the more elaborate numerical computations that we carried out suggest that $\kappa(u)$ is decreasing when $u < u_0$ and increasing when $u > u_0$, where $u_0 \approx 1.63$; its minimum value is approximately $\kappa(1.63) \approx -0.2943$. It also seems that $\kappa(u)$ is always negative.

It is natural to ask whether similar asymptotic formulas hold for $C_k(x, y)$ when $k \geq 3$. It appears that the analytic method in §4 below should yield

some result, at least when $k = 3$. However, since the technical details will most likely be quite ungainly, we leave such considerations to future work.

2. The first moment. By (1.2), we have
$$C_1(x,y) = \sum_{n \leq y} \sum_{q \leq x} \sum_{\substack{d|q \\ d|n}} d\mu(q/d) = \sum_{n \leq y} \sum_{\substack{dk \leq x \\ d|n}} d\mu(k).$$

Interchanging the order of summation, we obtain

$$(2.1) \qquad C_1(x,y) = \sum_{dk \leq x} d\mu(k) \sum_{\substack{n \leq y \\ d|n}} 1 = \sum_{dk \leq x} d\mu(k) \left[\frac{y}{d}\right]$$

$$= y \sum_{dk \leq x} \mu(k) - \frac{1}{2} \sum_{dk \leq x} d\mu(k) - \sum_{dk \leq x} d\mu(k)\psi(y/d)$$

$$= C_{1,1}(x,y) - C_{1,2}(x,y) - C_{1,3}(x,y), \qquad \text{say.}$$

Here, for a real t, $[t]$ is the integral part of t and $\psi(t) = t - [t] - \frac{1}{2}$ is the saw-tooth function. It is easy to see that

$$(2.2) \qquad C_{1,1}(x,y) = y \sum_{dk \leq x} \mu(k) = y \sum_{m \leq x} \sum_{k|m} \mu(k) = y.$$

Further,

$$(2.3) \qquad C_{1,2}(x,y) = \frac{1}{2} \sum_{dk \leq x} d\mu(k) = \frac{1}{2} \sum_{k \leq x} \mu(k) \sum_{d \leq x/k} d$$

$$= \frac{1}{2} \sum_{k \leq x} \mu(k) \left(\frac{x^2}{2k^2} + O\left(\frac{x}{k}\right)\right)$$

$$= \frac{x^2}{4} \sum_{k \leq x} \frac{\mu(k)}{k^2} + O(x \log x) = \frac{x^2}{4\zeta(2)} + O(x \log x).$$

It remains to estimate $C_{1,3}(x,y)$. By partial summation,

$$(2.4) \qquad C_{1,3}(x,y) \ll \sum_{k \leq x} \sum_{j=0}^{\infty} N_j \sup_{\mathbf{I}} \left|\sum_{d \in \mathbf{I}} \psi(y/d)\right|,$$

where $N_j = N_{j,k} = (x/k)2^{-j}$ and the supremum is over all subintervals $\mathbf{I}$ of $(N_j, 2N_j]$. We remark that the sum over j is, in fact, finite and has $O(\log x)$ terms. We now use the following lemma, which is a special case of [2, Lemma 4.3] (see [2] or [4] for the definition of exponential pairs).

LEMMA 2.1. *Suppose that $(\varkappa, \lambda)$ is an exponential pair and that $\mathbf{I}$ is a subinterval of $(N, 2N]$. Then*

$$\sum_{n \in \mathbf{I}} \psi(y/n) \ll y^{\varkappa/(\varkappa+1)} N^{(\lambda-\varkappa)/(\varkappa+1)} + N^2 y^{-1}.$$

We apply Lemma 2.1 with the exponential pair $(\varkappa, \lambda) = (1/2, 1/2)$ to the sum over d on the right side of (2.4). This yields

$$(2.5) \qquad C_{1,3}(x,y) \ll \sum_{k \le x} \sum_{j=0}^{\infty} (N_j y^{1/3} + N_j^3 y^{-1}) \ll \sum_{k \le x} ((x/k)y^{1/3} + (x/k)^3 y^{-1})$$

$$\ll (xy^{1/3} \log x + x^3 y^{-1}).$$

Theorem 1.1 now follows easily from (2.1)–(2.5).

3. The second moment: An elementary approach. In this section, we prove (1.2). Similarly to §2, we obtain

$$C_2(x,y) = \sum_{n \le y} \Big(\sum_{\substack{dk \le x \\ d|n}} d\mu(k) \Big)^2 = \sum_{d_1 k_1 \le x} \sum_{d_2 k_2 \le x} d_1 d_2 \mu(k_1) \mu(k_2) \Big[\frac{y}{[d_1, d_2]} \Big],$$

where $[d_1, d_2]$ denotes the least common multiple of d_1 and d_2. Hence,

$$(3.1) \qquad C_2(x,y) = y \sum_{d_1 k_1 \le x} \sum_{d_2 k_2 \le x} (d_1, d_2) \mu(k_1) \mu(k_2) + O(E),$$

where

$$(3.2) \qquad E = y \sum_{d_1 k_1 \le x} \sum_{d_2 k_2 \le x} d_1 d_2 \ll x^4.$$

Furthermore,

$$(3.3) \qquad \sum_{d_1 k_1 \le x} \sum_{d_2 k_2 \le x} (d_1, d_2) \mu(k_1) \mu(k_2) = \sum_{d \le x} d \sum_{\substack{dl_1 k_1 \le x \\ (l_1, l_2) = 1}} \sum_{dl_2 k_2 \le x} \mu(k_1) \mu(k_2)$$

$$= \sum_{d \le x} d \sum_{dl_1 k_1 \le x} \sum_{dl_2 k_2 \le x} \mu(k_1) \mu(k_2) \sum_{l | (l_1, l_2)} \mu(l)$$

$$= \sum_{dl \le x} d\mu(l) \Big(\sum_{mk \le x/(dl)} \mu(k) \Big)^2 = \sum_{dl \le x} d\mu(l) = \frac{x^2}{2\zeta(2)} + O(x \log x),$$

on using variants of (2.2) and (2.3). The asymptotic formula (1.2) follows from (3.1)–(3.3).

4. The second moment: An analytic approach. In this section, we assume that x and y are as in part (ii) of Theorem 1.2. Recall the identity

$$(4.1) \qquad \sum_{q=1}^{\infty} \frac{c_q(n)}{q^s} = \frac{\sigma_{1-s}(n)}{\zeta(s)} \qquad (\mathrm{Re}(s) > 1),$$

where $\zeta(s)$ denotes the Riemann zeta-function and $\sigma_z(n) = \sum_{d|n} d^z$. We note that without loss of generality, we may assume that $x, y \in \mathbb{Z} + 1/2$.

Then, using (1.2) and the truncated Perron formula [3, Corollary 5.3], we get

$$(4.2) \qquad \sum_{q \leq x} c_q(n) = \frac{1}{2\pi i} \int_{\alpha - iT}^{\alpha + iT} \frac{\sigma_{1-s}(n)}{\zeta(s)} \frac{x^s}{s} \, ds + E_1(x, n).$$

Here, T is a real parameter at our disposal, $\alpha > 1 + (\log x)^{-1}$ is a real number, and the remainder $E_1(x, n)$ satisfies

$$E_1(x, n) \ll \frac{x^\alpha}{T} \sum_{q=1}^\infty \frac{|c_q(n)|}{q^\alpha} + \sum_{x/2 < q < 2x} |c_q(n)| \min\left(1, \frac{x}{T|x-q|}\right).$$

If we assume further that $\alpha < 1 + 3(\log x)^{-1}$, we can show that

$$(4.3) \qquad E_1(x, n) \ll (x/T)\sigma_0(n) \log x.$$

Applying (4.2) and (4.3) with $\alpha = \alpha_j = 1 + j(\log x)^{-1}$, $j = 1, 2$, we obtain

$$(4.4) \quad \left(\sum_{q \leq x} c_q(n)\right)^2 = \frac{1}{(2\pi i)^2} \int_{\alpha_1 - iT}^{\alpha_1 + iT} \int_{\alpha_2 - iT}^{\alpha_2 + iT} F(s_1, s_2, n) \, ds_2 ds_1 + E_2(x, n),$$

where

$$F(s_1, s_2, n) = \frac{\sigma_{1-s_1}(n)\sigma_{1-s_2}(n)}{\zeta(s_1)\zeta(s_2)} \frac{x^{s_1+s_2}}{s_1 s_2}$$

and

$$E_2(x, n) \ll (x^2/T)\sigma_0(n)^2 \log x.$$

Summing (4.4) over n, we deduce

$$(4.5) \quad C_2(x, y)$$
$$= \frac{1}{(2\pi i)^2} \int_{\alpha_1 - iT}^{\alpha_1 + iT} \int_{\alpha_2 - iT}^{\alpha_2 + iT} \frac{G(s_1, s_2; y)}{\zeta(s_1)\zeta(s_2)} \frac{x^{s_1+s_2}}{s_1 s_2} \, ds_2 \, ds_1 + O\left(\frac{yx^2 L^4}{T}\right),$$

where $L = \log(xy)$ and

$$G(s_1, s_2; y) = \sum_{n \leq y} \sigma_{1-s_1}(n)\sigma_{1-s_2}(n).$$

We now recall Ramanujan's identity

$$(4.6) \qquad \sum_{n=1}^\infty \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)},$$

valid for $\mathrm{Re}(s) > \max\{1, 1 + \mathrm{Re}(a), 1 + \mathrm{Re}(b), 1 + \mathrm{Re}(a+b)\}$. From (4.6), by another application of Perron's formula, we get

$$(4.7) \qquad G(s_1, s_2; y) = \frac{1}{2\pi i} \int_{\alpha - 3iT}^{\alpha + 3iT} H(s_1, s_2, w) \frac{y^w}{w} \, dw + O\left(\frac{yL^4}{T}\right),$$

where $\alpha = 1 + 3(\log y)^{-1}$, $y \in \mathbb{Z} + 1/2$, and

$$H(s_1, s_2, w) = \frac{\zeta(w)\zeta(w + s_1 - 1)\zeta(w + s_2 - 1)\zeta(w + s_1 + s_2 - 2)}{\zeta(2w + s_1 + s_2 - 2)}.$$

Let $\Gamma(\alpha, \beta, T)$ denote the contour consisting of the line segments $[\alpha - iT, \beta - iT]$, $[\beta - iT, \beta + iT]$ and $[\beta + iT, \alpha + iT]$. We now move the integration in (4.7) to $\Gamma(\alpha, 1/2, 3T)$; let us denote the respective integrals I_1, I_2 and I_3.

When $\mathrm{Re}(s_j) = \alpha_j$, the integrals I_1 and I_3 over the two horizontal line segments are bounded above by

$$(4.8) \qquad I_1, I_3 \ll \zeta(\alpha_1 + \alpha_2 - 1)L^4 \int_{1/2}^{\alpha} T^{1-2\sigma} y^\sigma d\sigma \ll L^5\left(y^{1/2} + yT^{-1}\right).$$

Here, we have used the standard convexity bound (see [4, eqn. (5.1.4)] for a slightly weaker version)

$$(4.9) \qquad |\zeta(\sigma + it)| \ll (|t| + 2)^{(1-\sigma)/2} \log(|t| + 2) \qquad (1/2 \le \sigma \le 1).$$

Furthermore, by Hölder's inequality,

$$(4.10) \quad I_2 \ll y^{1/2}\zeta(\alpha_1 + \alpha_2 - 1) \prod_{0 \le j,k \le 1} M_4(1/2 + j(\alpha_1 - 1) + k(\alpha_2 - 1), 5T),$$

where for $1/2 \le \sigma \le 1$,

$$M_4^4(\sigma, T) = \int_{-T}^{T} |\zeta(\sigma + it)|^4 \frac{dt}{1 + |t|}.$$

Appealing to the fourth-moment estimates for $\zeta(s)$ [4, §7.5 & §7.6], we deduce from (4.10) that

$$(4.11) \qquad\qquad\qquad I_2 \ll y^{1/2}L^6.$$

Combining (4.7), (4.8) and (4.11), we obtain

$$(4.12) \qquad G(s_1, s_2; y) = \sum_{j=1}^{4} R_j(s_1, s_2; y) + O(L^6(y^{1/2} + yT^{-1})),$$

where $R_j(s_1, s_2; y)$ are the four residues of the integrand in (4.7):

$$R_1(s_1, s_2; y) = \frac{y\zeta(s_1)\zeta(s_2)\zeta(s_1 + s_2 - 1)}{\zeta(s_1 + s_2)},$$

$$R_2(s_1, s_2; y) = \frac{y^{2-s_1}\zeta(2 - s_1)\zeta(1 - s_1 + s_2)\zeta(s_2)}{(2 - s_1)\zeta(2 - s_1 + s_2)},$$

$$R_3(s_1, s_2; y) = \frac{y^{2-s_2}\zeta(2 - s_2)\zeta(1 + s_1 - s_2)\zeta(s_1)}{(2 - s_2)\zeta(2 + s_1 - s_2)},$$

$$R_4(s_1, s_2; y) = \frac{y^{3-s_1-s_2}\zeta(3 - s_1 - s_2)\zeta(2 - s_2)\zeta(2 - s_1)}{(3 - s_1 - s_2)\zeta(4 - s_1 - s_2)}.$$

Substituting (4.12) into the right side of (4.5), we get

$$(4.13) \qquad C_2(x,y) = \sum_{j=1}^{4} C_{2,j}(x,y) + O(yx^2 L^{10}(y^{-1/2} + T^{-1})),$$

where

$$C_{2,j}(x,y) = \frac{1}{(2\pi i)^2} \int\limits_{\alpha_1-iT}^{\alpha_1+iT} \int\limits_{\alpha_2-iT}^{\alpha_2+iT} \frac{R_j(s_1,s_2;y)}{\zeta(s_1)\zeta(s_2)} \frac{x^{s_1+s_2}}{s_1 s_2} \, ds_2 \, ds_1.$$

We proceed to evaluate the integrals $C_{2,j}(x,y)$, starting with $C_{2,1}(x,y)$. We first move the integration over s_2 to the contour $\Gamma(\alpha_2, 1/2, T)$; let us denote the integrals over the three line segments $J_{1,1}$, $J_{1,2}$ and $J_{1,3}$. The contributions from the integrals over the two horizontal line segments are bounded using (4.9):

$$J_{1,1}, J_{1,3} \ll \frac{xyL}{T} \int\limits_{-T}^{T} \frac{dt}{1+|t|} \int\limits_{1/2}^{\alpha_2} T^{(1-\sigma)/2} x^\sigma d\sigma \ll xyL^2(xT^{-1}+1).$$

Furthermore, using estimates for the mean square of $\zeta(s)$ in the critical strip, we find that the contribution from the integral over the line $\mathrm{Re}(s_2) = 1/2$ is

$$J_{1,2} \ll yx^{3/2} \int\limits_{-T}^{T} \int\limits_{-T}^{T} \frac{|\zeta(\alpha_1 - 1/2 + i(t_1+t_2))|}{(|t_1|+1)(|t_2|+1)} \, dt_1 dt_2$$

$$\ll yx^{3/2} \int\limits_{-2T}^{2T} |\zeta(\alpha_1 - 1/2 + iu)| \int\limits_{-T}^{T} \frac{dt \, du}{(|t|+1)(|t-u|+1)}$$

$$\ll yx^{3/2}L \int\limits_{-2T}^{2T} |\zeta(\alpha_1 - 1/2 + iu)| \frac{du}{|u|+1} \ll yx^{3/2}L^3.$$

Hence, accounting for the residue at $s_2 = 2 - s_1$, we have

$$(4.14) \qquad C_{2,1}(x,y) = \frac{yx^2}{\zeta(2)} \frac{1}{2\pi i} \int\limits_{\alpha_1-iT}^{\alpha_1+iT} \frac{ds_1}{s_1(2-s_1)} + O(yx^2 L^3(x^{-1/2} + T^{-1})),$$

$$= \frac{yx^2}{2\zeta(2)} + O(yx^2 L^3(x^{-1/2} + T^{-1})).$$

To estimate $C_{2,4}(x,y)$, we move the integration over s_2 to the contour $\Gamma(\alpha_2, \beta, T)$, where $\beta = 5/2 - \alpha_1$; let us denote the respective integrals $J_{4,1}, J_{4,2}$ and $J_{4,3}$. Similarly to the estimation of $J_{1,j}$, we have

$$J_{4,1}, J_{4,3} \ll yx^2 L^5(T^{-1} + (x/yT)^{1/2}), \qquad J_{4,2} \ll y^{1/2} x^{5/2} L^5.$$

Since the integrand is holomorphic in the region between the two contours, we deduce that

$$(4.15) \qquad C_{2,4}(x,y) \ll yx^2 L^5 ((x/y)^{1/2} + T^{-1}).$$

Similarly, by moving the integration to $\Gamma(\alpha_2, 3/2, T)$, we find that

$$(4.16) \qquad C_{2,3}(x,y) \ll yx^2 L^5 ((x/y)^{1/2} + T^{-1}),$$

and

$$(4.17) \qquad C_{2,2}(x,y) = \frac{yx^2}{\zeta(2)} K(x,y) + O(yx^2 L^5 ((x/y)^{1/2} + T^{-1})),$$

where

$$K(x,y) = \frac{1}{2\pi i} \int\limits_{\alpha_1 - iT}^{\alpha_1 + iT} \frac{\zeta(2-s)}{\zeta(s)} \frac{(yx^{-2})^{1-s}}{s^2(2-s)} \, ds.$$

The first term on the right side of (4.17) arises from the pole of $R_2(s_1, s_2; y)$ at $s_2 = s_1$. Thus, by (4.13)–(4.17),

$$(4.18) \quad C_2(x,y) = \frac{yx^2}{\zeta(2)} \big(1/2 + K(x,y)\big) + O(yx^2 L^{10}(x^{-1/2} + (x/y)^{1/2} + T^{-1})).$$

It remains to evaluate the integral $K(x,y)$.

Recall that $yx^{-2} \leq L^B$. We move the integration to the contour $\Gamma(\alpha_1, 1, T)$ and denote the integrals along the edges $K_1(x,y)$, $K_2(x,y)$ and $K_3(x,y)$. Then, by the bounds for $\zeta(s)$ in [4, §6.19], we have

$$K_1(x,y), K_3(x,y) \ll T^{-2} \quad \text{and} \quad K_2(x,y) = \kappa(u) + O(T^{-2}),$$

where $u = \log(yx^{-2})$ and $\kappa(u)$ is the Fourier integral

$$(4.19) \quad \kappa(u) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} f(it)e^{-itu} \, dt, \qquad f(s) = \frac{\zeta(1-s)}{\zeta(1+s)} \frac{1}{(1+s)^2(1-s)}.$$

We choose $T = x$ and get the asymptotic formula

$$(4.20) \qquad C_2(x,y) = \frac{yx^2}{\zeta(2)} (1/2 + \kappa(u)) + O(yx^2 L^{10}((x/y)^{1/2} + x^{-1/2})).$$

Thus, to complete the proof of part (ii) of the theorem, we need only establish the desired properties of $\kappa(u)$. We examine $\kappa(u)$ in the next section.

5. Estimation of $\kappa(u)$. Suppose first that $|u| \to \infty$. For a fixed $\epsilon > 0$ and a real t, we define

$$\delta(t) = C_\epsilon (\log(|t| + 2))^{-2/3 - \epsilon}.$$

By the standard bounds for $\zeta(s)$ near the edge of the critical strip [4, p. 135],

$$|f(\sigma + it)| \ll \frac{\delta(t)^{-2}}{(1+|t|)^3} \quad \text{when } |\sigma| \leq \delta(t).$$

Applying Cauchy's integral formula for the circle γ_t, $|s - it| = \delta(t)$, we deduce that

$$|f^{(k)}(it)| = \left| \frac{k!}{2\pi i} \int_{\gamma_t} \frac{f(s)}{(s - it)^{k+1}} \, ds \right| \ll \frac{k! \delta(t)^{-k-2}}{(1 + |t|)^3} \quad (k \geq 1).$$

Hence, a k-fold integration by parts gives

$$\kappa(u) \ll \frac{1}{|u|^k} \left| \int_{-\infty}^{\infty} f^{(k)}(it) e^{-itu} \, dt \right| \ll \frac{k! C_\epsilon^{-k-2}}{|u|^k} \int_2^{\infty} (\log t)^{(2/3+\epsilon)(k+2)} t^{-3} \, dt.$$

Using the inequality

$$\int_2^{\infty} (\log t)^\alpha t^{-3} \, dt \leq \int_0^{\infty} z^\alpha e^{-2z} \, dz = \frac{\Gamma(\alpha + 1)}{2^{\alpha+1}},$$

we deduce that

$$\kappa(u) \ll \frac{k! m_k!}{2^{m_k} (C_\epsilon |u|)^k} \ll \frac{k^{(5/3+\epsilon)k}}{(C_\epsilon |u|)^k},$$

where $m_k = \lceil (2/3 + \epsilon)(k + 2) \rceil$. Finally, we set $k = \lfloor C_\epsilon' |u|^{1/(5/3+\epsilon)} \rfloor$ to obtain

$$\kappa(u) \ll \exp(-|u|^{3/5-\epsilon}).$$

Next, we focus on the case when $|u|$ is bounded. The integral $\kappa(u)$ is differentiable, and

$$|\kappa'(u)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|t| \, dt}{(1 + t^2)^{3/2}} \leq \frac{1}{2}.$$

Hence,

(5.1) $$|\kappa(u) - \kappa(v)| \leq \frac{|u - v|}{2}.$$

Since the integrand is bounded above by $|t|^{-3}$, we have

(5.2) $$\kappa(u) = \frac{1}{2\pi} \int_{-60}^{60} f(it) e^{-itu} \, dt + \vartheta_u 10^{-4},$$

where $|\vartheta_u| \leq 1/2$. Let $\kappa_0(u)$ denote the integral on the right side of (5.2). By partial integration,

$$|\kappa_0(u)| \leq \frac{1}{2\pi |u|} \left(|f(60i)| + |f(-60i)| + \int_{-60}^{60} |f'(it)| \, dt \right) \leq \frac{4.2}{2\pi |u|} \leq 0.67 |u|^{-1}.$$

Combining the latter inequality and (5.2), we deduce that $|\kappa(u)| \leq 0.395$ when $|u| \geq 1.7$. It remains to estimate $\kappa(u)$ when $|u| \leq 1.7$.

In view of (5.1) and (5.2), it suffices to tabulate $\kappa_0(u)$ for a set of suitably chosen values $u_1, u_2, \ldots$, with $|u_j| \leq 1.7$. To this end, a quick calculation using *Mathematica* yields the following table:

u_j	$\kappa_0(u_j)$	Covered range
-1	-0.0277	$-1.7 \leq u \leq -0.3$
0.3	-0.0984	$-0.3 \leq u \leq 0.9$
1.1	-0.2586	$0.9 \leq u \leq 1.3$
1.5	-0.2925	$1.3 \leq u \leq 1.7$

Each row of this table lists a value of $\kappa_0(u_j)$ and a range of the variable u, where we can infer the bound $\kappa(u) \geq -0.4$ from (5.1), (5.2) and the listed value of $\kappa_0(u_j)$. For example, when $0.9 \leq u \leq 1.3$, we have

$$\kappa(u) \geq \kappa(1.1) - 0.5|u - 1.1| > \kappa_0(1.1) - 0.1001 > -0.36.$$

Clearly, this completes the proof of part (ii) of Theorem 1.2.

References

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